XXV Polish – Russian – Slovak Seminar “Theoretical Foundation of Civil Engineering”

Semianalytical structural analysis based on combined application of finite element method and discrete-continual finite element method Part 1: Two-dimensional theory of elasticity

Pavel A. Akimov\textsuperscript{a,h,c,*}, Oleg A. Negrozov\textsuperscript{a,b}

\textsuperscript{a}Moscow State University of Civil Engineering (National Research University), 26 Yaroslavskoye Shosse, Moscow, 129337, Russia 
\textsuperscript{b}Russian Academy of Architecture and Construction Sciences, 24, Ulitsa Bolshaya Dmitrovka, Moscow, 107031, Russia 
\textsuperscript{c}Research & Educational Center “StaDyO”, Office 810, 18, 3-ya Ulitsa Yamskogo Polya, Moscow, 125040, Russia

Abstract

This paper is devoted to so-called semianalytical structural analysis, based on combined application of finite element method (FEM) \cite{1} and discrete-continual finite element method (DCFEM) \cite{2-7}. Boundary problems of two-dimensional theory of elasticity (static analysis of deep beam \cite{1}) are under consideration. In accordance with the method of extended domain, the given domain is embordered by extended one. The field of application of DCFEM comprises structures with regular (constant or piecewise constant) physical and geometrical parameters in some dimension (“basic” dimension). DCFEM presupposes finite element mesh approximation for non-basic dimension of extended domain while in the basic dimension problem remains continual. Corresponding discrete and discrete-continual approximation models for subdomains and coupled multilevel approximation model for extended domain are under consideration. Brief information about software and verification samples are presented as well.

Keywords: discrete-continual finite element method; finite element method; semianalytical structural analysis; two-dimensional theory of elasticity

* Corresponding author. Tel.: +7(495)625-71-63; fax: +7(495)650-27-31.
E-mail address: akimov@raasn.ru
1. Formulation of the problem and notation system

Let’s consider problem of static structural analysis of deep beam loaded by concentrated force with hinged ends (cross-sections) along basic dimension (Fig. 1). Some elements of notation system is presented at Fig. 1 as well.

Fig. 1. Considering structure (deep beam)

Let’s \( \Omega \) be domain occupied by structure, \( \Omega = \Omega_1 \cup \Omega_2 \) and \( \Omega_i = \{(x_1, x_2) : 0 < x_i < l_i, 0 < x_j < l_j \}$, where \( x_i, x_j \) are coordinates (\( x_2 \) corresponds to basic dimension); \( x_{2,i}^k = 0, x_{2,i}^k = l_{2,i}, x_{2,i}^{k+1} = l_{2,i} + 1 \) are coordinates of corresponding boundary points (cross-sections) along basic dimension; \( \Omega_i \) and \( \Omega_j \) are subdomains of \( \Omega \); \( \omega_i \) and \( \omega_j \) are extended subdomains, embordering subdomains \( \Omega_j \subset \omega_i \) and \( \Omega_i \subset \omega_j \); \( \omega = \omega_1 \cup \omega_2 \); \( x_{i,j}^k \), \( i = 1, 2, ..., N_i^k \) and \( x_{i,j}^{k+1} \), \( j = 1, 2, ..., N_j^k \) are coordinates (along \( x_i \) of nodes (nodal lines) of discrete-continual finite elements, which are used for approximation of domain \( \omega_i \); \( N_i^k - 1 \) is the number of discrete-continual finite elements; \( x_{i,j}^k \), \( i = 1, 2, ..., N_i^k \) and \( x_{i,j}^{k+1} \), \( j = 1, 2, ..., N_j^k \) are coordinates (along \( x_i \) and \( x_j \)) of nodes of finite elements, which are used for approximation of domain \( \omega_j \); \( N_i^k - 1 \) and \( N_j^k - 1 \) are numbers of finite elements along coordinates \( x_i \) and \( x_j \).

Two-index notation system is used for numbering of discrete-continual finite elements. Typical number of has the form \( (k,i) \), where \( k \) is the number of subdomain, \( i \) is the number of element (along \( x_i \)). Three-index system is used for numbering of finite elements. Typical number of has the form \( (k,i,j) \), where \( k \) is the number of subdomain, \( i \) and \( j \) are numbers of elements (along \( x_i \) and \( x_j \)). Let’s \( N_i^k = N_i^k \) and \( x_{i,j}^k = x_{i,j}^{k+1} \), \( i = 1, 2, ..., N_i^k \).

2. Discrete-continual approximation model for subdomain

Discrete-continual approximation model is used for two-dimensional problems. It presupposes mesh approximation for non-basic dimension of extended domain (along \( x_i \)) while in the basic dimension (along \( x_j \)) problem remains continual. Thus extended subdomain \( \omega_i \) is divided into discrete-continual finite elements

\[
\omega_i = \bigcup_{j=1}^{N_i^k} \omega_{i,j} ; \quad \omega_{i,j} = \{(x_1, x_2) : x_{i,j} < x_i < x_{i,j+1}, x_{2,i}^b < x_2 < x_{2,i}^b \}.
\]
where $\theta_{ij}$ is the characteristic function of element $\omega_{ij}$.

Basic nodal unknown functions are displacement components $u_i^{(ii)}, u_j^{(ii)}$ and their derivatives $v_i^{(ii)}, v_j^{(ii)}$ with respect to $x_i$ (superscript hereinafter corresponds to the number of considered subdomain i.e. $\omega_1$). Thus for node $(1,1)$ we have the following unknown functions: $u_1^{(1,1)}, u_1^{(1,1)}$ and $v_1^{(1,1)}, v_2^{(1,1)}$.

Linear approximation is used for unknown functions within discrete-continual finite element.

DCFEM is reduced at some stage to the solution of systems of $4N_i$ first-order ordinary differential equations:

$$
\begin{align*}
\bar{U}_i(x_1) &= A_i \bar{U}_i(x_2) + \bar{R}_i(x_2),
\end{align*}
$$

where $\bar{U}_i(x_2)$ is global vector of nodal unknown functions (subscript corresponds to the number of subdomain $\omega_i$).

$$
\bar{U}_i = \bar{U}_i(x_2) = \begin{bmatrix} (\bar{u}_i)^T & (\bar{v}_i)^T \end{bmatrix}^T;
$$

$$
\bar{u}_i = \bar{u}_i(x_2) = \begin{bmatrix} (\bar{u}_i^{(1,i)})^T & (\bar{u}_i^{(2,i)})^T & \cdots & (\bar{u}_i^{(1,2)})^T & (\bar{u}_i^{(2,2)})^T \end{bmatrix}^T;
$$

$$
\bar{v}_i = \bar{v}_i(x_2) = \begin{bmatrix} (\bar{v}_i^{(1,i)})^T & (\bar{v}_i^{(2,i)})^T & \cdots & (\bar{v}_i^{(1,2)})^T & (\bar{v}_i^{(2,2)})^T \end{bmatrix}^T;
$$

$$
A_i \text{ is global matrix of coefficients of order } 4N_i; \bar{R}_i(x_2) \text{ is the right-side vector of order } 4N_i.
$$

Correct analytical solution of (4) is defined by formula

$$
\bar{U}_i(x_2) = E_i(x_2) \bar{C}_i + \bar{S}_i(x_2),
$$

where

$$
E_i(x_2) = \varepsilon_i(x_2 - x_i^0) - \varepsilon_i(x_2 - x_i^0); \quad \bar{S}_i(x_2) = \varepsilon_i(x_2) * \bar{R}_i(x_2);
$$

$\varepsilon_i(x_2)$ is the fundamental matrix-function of system (4), which is constructed in the special form convenient for problems of structural mechanics [2]; * is convolution notation; $\bar{C}_i$ is the vector of constants of order $4N_i$.

### 3. Discrete (finite element) approximation model for subdomain

Discrete (finite element) approximation model for the considering two-dimensional problems presupposes finite element approximation along $x_1$ and $x_2$. Thus extended subdomain $\omega_2$ is divided into finite elements

$$
\omega_2 = \bigcup_{i=1}^{N_{1,i}} \bigcup_{j=1}^{N_{2,1,j}} \omega_{2,i,j}; \quad \omega_{2,i,j} = \{ (x_1, x_2) : x_{i,j} < x_1 < x_{i,j+1}, x_{2,i} < x_2 < x_{2,i+1} \}.
$$

$Lame$ constants for finite element are defined by formulas:

$$
\bar{\lambda}_{2,i,j} = \theta_{2,i,j} \lambda; \quad \bar{\mu}_{2,i,j} = \theta_{2,i,j} \mu,
$$
where $\theta_{2,i,j}$ is the characteristic function of element $\omega_{2,i,j}$,

$$
\theta_{2,i,j} = \begin{cases} 
1, & \omega_{2,i,j} \subset \Omega_2; \\
0, & \omega_{2,i,j} \not\subset \Omega_2.
\end{cases} 
$$

Basic nodal unknowns are displacement components $u_i^{(2)}, u_j^{(2)}$ (superscript hereinafter corresponds to the number of considered subdomain i.e. $\omega_2$). Thus for node $(2,i,j)$ we have the following unknowns: $u_i^{(2,i,j)}, u_j^{(2,i,j)}$.

Bilinear approximation of unknowns is used within finite element (conventional plane rectangular 4-node finite element of two-dimensional problem of elasticity theory).

As known, FEM is reduced to the solution of systems of $2N_iN_j$ linear algebraic equations:

$$
K_jU_j = R_j, 
$$

where $\bar{U}_j$ is global vector of nodal unknowns (subscript corresponds to the number of subdomain $\omega_2$),

$$
\bar{U}_j = [ (u_i^{(2,1,1)})^T \ (u_i^{(2,2,1)})^T \ \ldots \ \ (u_i^{(2,N_i,1)})^T \ \ (u_i^{(2,2,2)})^T \ \ldots \ \ (u_i^{(2,N_i,2)})^T \ \ldots \ \ (u_i^{(2,1,1,N_j)})^T \ \ (u_i^{(2,2,2,N_j)})^T \ \ldots \ \ (u_i^{(2,N_i,2,N_j)})^T ]^T; 
$$

$$
\bar{u}_i^{(2,i,j)} = [ u_i^{(2,i,j)} \ u_j^{(2,i,j)} ]^T, \ i = 1, 2, \ldots, N_i; \ j = 1, 2, \ldots, N_j; 
$$

$K_j$ is global stiffness matrix of order $2N_iN_j$; $\bar{R}_j$ is global right-side vector of order $2N_iN_j$ (global load vector).

4. Multilevel approximation model for domain

System (13) can be rewritten for all nodes with indexes $1 < j < N_2$ (i.e. $x_{2,2}^b < x_2 < x_{2,3}^b$) in the following form (resolving system of $2N_i(N_2 - 2)$ linear algebraic equations):

$$
\tilde{K}_j\tilde{U}_j = \bar{R}_j, 
$$

where $\tilde{K}_j$ is reduced global stiffness matrix of size $[2N_i(N_2 - 2)] \times [2N_iN_j]$; $\bar{R}_j$ is reduced right-side vector of order $2N_i(N_2 - 2)$.

Boundary conditions at section $x_2 = x_{2,j}^b$ (hinged edge) has the form ($2N_i$ equations):

$$
u_i^{(1,i)}(x_{2,j}^b, + 0) = 0, \ i = 1, 2, \ldots, N_i; \ \ u_j^{(1,i)}(x_{2,j}^b, + 0) = 0, \ i = 1, 2, \ldots, N_i. 
$$

Equations (17) can be rewritten in matrix form:

$$
B_i^\dagger \bar{U}_j(x_{2,i}^b, + 0) = \bar{g}_i^+, 
$$

where $B_i^\dagger$ is matrix of boundary conditions of size $2N_i \times 4N_i$, which can be constructed in accordance with algorithm presented at Table 1; $\bar{g}_i^+$ is the zero vector of order $2N_i$ (i.e. $\bar{g}_i^+ = 0$).
<table>
<thead>
<tr>
<th>Numbers (indexes) of elements</th>
<th>Element value</th>
<th>Corresponding boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2i−1, 2i−1), i = 1, 2, ..., N₁</td>
<td>1</td>
<td>The first equation from (17)</td>
</tr>
<tr>
<td>(2i, 2i), i = 1, 2, ..., N₁</td>
<td>1</td>
<td>The second equation from (17)</td>
</tr>
</tbody>
</table>

After substitution of (8) into (18) it can be obtained that

\[ B_{1}^{i} E_{i} (x_{2,i}^{b} + 0) \bar{C}_{i} = \bar{g}_{1} - B_{1}^{i} \bar{S}_{i} (x_{2,i}^{b} + 0) \quad \text{or} \quad Q_{i} \bar{C}_{i} = \bar{G}_{i}, \]  

where \( Q_{i} \) is the matrix of size \( 2N_{i} \times 4N_{i} \); \( \bar{G}_{i} \) is the vector of order \( 2N_{i} \);

\[ Q_{i} = B_{1}^{i} E_{i} (x_{2,i}^{b} + 0); \quad \bar{G}_{i} = \bar{g}_{1} - B_{1}^{i} \bar{S}_{i} (x_{2,i}^{b} + 0). \]  

Boundary conditions at section \( x_{2} = x_{2,2}^{b} \) (perfect contact) has the form \((4N_{i} \text{ equations})\):

\[ u_{i}^{(1,i)} (x_{2,2}^{b} - 0) = u_{i}^{(1,i)}; \quad i = 1, 2, ..., N_{i}, \quad j = 1; \quad \sigma_{1,2}^{(1,i)} (x_{2,2}^{b} - 0) = \sigma_{1,2}^{(1,i)}; \quad i = 1, 2, ..., N_{i}, \quad j = 1; \]  

\[ (21) \]

\[ \sigma_{1,2}^{(2,i)} (x_{2,2}^{b} - 0) = \sigma_{2,2}^{(2,i)}; \quad i = 1, 2, ..., N_{i}, \quad j = 1; \]  

\[ \sigma_{1,2}^{(2,i)} (x_{2,2}^{b} - 0) = \sigma_{2,2}^{(2,i)}; \quad i = 1, 2, ..., N_{i}, \quad j = 1; \]  

\[ (22) \]

where \( \sigma_{1,2}^{(1,i)} (x_{2}) \) and \( \sigma_{2,2}^{(1,i)} (x_{2}) \) are nodal functions (after corresponding averaging) of stress components \( \sigma_{1,2} (x_{2}) \) and \( \sigma_{2,2} (x_{2}) \) for discrete-continual finite element \((1, i)\); \( \sigma_{1,2}^{(2,i)} \) and \( \sigma_{2,2}^{(2,i)} \) are nodal stress components \( \sigma_{1,2} \) and \( \sigma_{2,2} \) (after corresponding averaging) for finite element \((2, i, j)\); \( j = 1 \).

Equations (21) and (22) can be rewritten in matrix form:

\[ B_{2}^{i} U_{i} (x_{2,2}^{b} - 0) = B_{2}^{i} \bar{U}_{2} , \]  

where \( B_{2}^{i} \) is matrix of boundary conditions of size \( 4N_{i} \times 4N_{i} \), which can be constructed in accordance with algorithm presented at Table 2; \( B_{2}^{i} \) is matrix of boundary conditions of size \( 4N_{i} \times 2N_{i}N_{2} \), which can be constructed in accordance with so-called method of basis variations [2-7].

After substitution of (8) into (22) it can be obtained that

\[ B_{2}^{i} E_{i} (x_{2,2}^{b} - 0) \bar{C}_{i} - B_{2}^{i} \bar{U}_{2} = -B_{2}^{i} \bar{S}_{i} (x_{2,2}^{b} - 0) \quad \text{or} \quad Q_{2,i} \bar{C}_{i} + Q_{2,2} \bar{U}_{2} = \bar{G}_{2} , \]  

where \( Q_{2,i} \) is the matrix of size \( 4N_{i} \times 4N_{i} \); \( Q_{2,2} \) is the matrix of size \( 4N_{i} \times 2N_{i}N_{2} \); \( \bar{G}_{2} \) is the vector of order \( 4N_{i} \),

\[ Q_{2,i} = B_{2}^{i} E_{i} (x_{2,2}^{b} - 0); \quad Q_{2,2} = -B_{2}^{i}; \quad \bar{G}_{2} = -B_{2}^{i} \bar{S}_{i} (x_{2,2}^{b} - 0) . \]  

\[ (25) \]

Boundary conditions at section \( x_{2} = x_{2,3}^{b} \) (hinged edge) has the form \((2N_{i} \text{ equations}; \ j = N_{i})\):

\[ u_{i}^{(2,i,j)} (0) = 0, \quad i = 1, 2, ..., N_{i}, \quad j = N_{i}; \quad u_{2}^{(2,i,j)} (0) = 0, \quad i = 1, 2, ..., N_{i}, \quad j = N_{2} . \]  

\[ (26) \]

Equations (26) can be rewritten in matrix form:

\[ B_{3} \bar{U}_{3} = \bar{g}_{3} , \]  

\[ (27) \]
Table 2. Algorithm of construction of matrix $B_i$ (All other elements of matrix $B_i$ are equal to zero).

<table>
<thead>
<tr>
<th>Numbers (indexes) of elements</th>
<th>Element value</th>
<th>Corresponding boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(i, 2i-1)$, $i = 1, 2, ..., N_i$</td>
<td>1</td>
<td>The first equation from (21)</td>
</tr>
<tr>
<td>$(N_i + i, 2i)$, $i = 1, 2, ..., N_i$</td>
<td>1</td>
<td>The second equation from (21)</td>
</tr>
<tr>
<td>$(2N_i + 1, 2)$</td>
<td>$\overline{\mu}_{i,1} \frac{1}{h_i} N'_i(0)$</td>
<td>The first equation from (22), $i = 1$</td>
</tr>
<tr>
<td>$(2N_i + 1, 4)$</td>
<td>$\overline{\mu}_{i,1} \frac{1}{h_i} N'_i(0)$</td>
<td>The first equation from (22), $i = 1$</td>
</tr>
<tr>
<td>$(2N_i + 1, 2N_i + 1)$</td>
<td>$\overline{\mu}_{i,1} \frac{1}{h_i} N'_i(0)$</td>
<td>The first equation from (22), $i = 1$</td>
</tr>
<tr>
<td>$(2N_i + i, 2(i-1))$, $i = 2, 3, ..., N_i - 1$</td>
<td>$\frac{1}{2} \overline{\mu}<em>{i-1,1} \frac{1}{h</em>{i-1}} N'_i(1)$</td>
<td>The first equation from (22), $i = 2, 3, ..., N_i - 1$</td>
</tr>
<tr>
<td>$(2N_i + i, 2i)$, $i = 2, 3, ..., N_i - 1$</td>
<td>$\frac{1}{2} \overline{\mu}<em>{i-1,1} \frac{1}{h</em>{i-1}} N'_i(1)$</td>
<td>The first equation from (22), $i = 2, 3, ..., N_i - 1$</td>
</tr>
<tr>
<td>$(2N_i + i, 2(i+1))$, $i = 2, 3, ..., N_i - 1$</td>
<td>$\frac{1}{2} \overline{\mu}<em>{i-1,1} \frac{1}{h</em>{i-1}} N'_i(1)$</td>
<td>The first equation from (22), $i = 2, 3, ..., N_i - 1$</td>
</tr>
<tr>
<td>$(2N_i + i, 2(N_i + i) - 1)$, $i = 2, 3, ..., N_i - 1$</td>
<td>$\frac{1}{2} \overline{\mu}<em>{i-1,1} \frac{1}{h</em>{i-1}} N'_i(1)$</td>
<td>The first equation from (22), $i = 2, 3, ..., N_i - 1$</td>
</tr>
<tr>
<td>$(3N_i, 2N_i)$</td>
<td>$\overline{\mu}<em>{i,1} \frac{1}{h</em>{i-1}} N'_i(1)$</td>
<td>The second equation from (22), $i = N_i$</td>
</tr>
<tr>
<td>$(3N_i, 4N_i - 1)$</td>
<td>$\overline{\mu}<em>{i,1} \frac{1}{h</em>{i-1}} N'_i(1)$</td>
<td>The second equation from (22), $i = N_i$</td>
</tr>
<tr>
<td>$(3N_i + 1, 1, 1)$</td>
<td>$\overline{\lambda}<em>{i,1} \frac{1}{h</em>{i-1}} N'_i(0)$</td>
<td>The second equation from (22), $i = 1$</td>
</tr>
<tr>
<td>$(3N_i + 1, 3)$</td>
<td>$\overline{\lambda}<em>{i,1} \frac{1}{h</em>{i-1}} N'_i(0)$</td>
<td>The second equation from (22), $i = 1$</td>
</tr>
<tr>
<td>$(3N_i + 1, 2N_i + 1)$</td>
<td>$\overline{\lambda}<em>{i,1} + 2 \overline{\mu}</em>{i,1}$</td>
<td>The second equation from (22), $i = 1$</td>
</tr>
<tr>
<td>$(3N_i + i, 2i - 3)$, $i = 2, 3, ..., N_i - 1$</td>
<td>$\frac{1}{2} \overline{\lambda}<em>{i-1,1} \frac{1}{h</em>{i-1}} N'_i(1)$</td>
<td>The second equation from (22), $i = 2, 3, ..., N_i - 1$</td>
</tr>
<tr>
<td>$(3N_i + i, 2i)$, $i = 2, 3, ..., N_i - 1$</td>
<td>$\frac{1}{2} \overline{\lambda}<em>{i-1,1} \frac{1}{h</em>{i-1}} N'_i(1)$</td>
<td>The second equation from (22), $i = 2, 3, ..., N_i - 1$</td>
</tr>
<tr>
<td>$(3N_i + i, 2(N_i + i) - 1)$, $i = 2, 3, ..., N_i - 1$</td>
<td>$\frac{1}{2} \overline{\lambda}<em>{i-1,1} \frac{1}{h</em>{i-1}} N'_i(1)$</td>
<td>The second equation from (22), $i = 2, 3, ..., N_i - 1$</td>
</tr>
<tr>
<td>$(3N_i + i, 2i + 1)$, $i = 2, 3, ..., N_i - 1$</td>
<td>$\frac{1}{2} \overline{\lambda}<em>{i-1,1} \frac{1}{h</em>{i-1}} N'_i(1)$</td>
<td>The second equation from (22), $i = 2, 3, ..., N_i - 1$</td>
</tr>
<tr>
<td>$(3N_i + i, 2(N_i + i))$, $i = 2, 3, ..., N_i - 1$</td>
<td>$\frac{1}{2} \overline{\lambda}<em>{i-1,1} \frac{1}{h</em>{i-1}} N'_i(1)$</td>
<td>The second equation from (22), $i = 2, 3, ..., N_i - 1$</td>
</tr>
<tr>
<td>$(4N_i, 2N_i - 3)$</td>
<td>$\overline{\lambda}<em>{i,1} \frac{1}{h</em>{i-1}} N'_i(1)$</td>
<td>The second equation from (22), $i = N_i$</td>
</tr>
<tr>
<td>$(4N_i, 2N_i - 1)$</td>
<td>$\overline{\lambda}<em>{i,1} \frac{1}{h</em>{i-1}} N'_i(1)$</td>
<td>The second equation from (22), $i = N_i$</td>
</tr>
<tr>
<td>$(4N_i, 4N_i)$</td>
<td>$\overline{\lambda}<em>{i,1} + 2 \overline{\mu}</em>{i,1}$</td>
<td>The second equation from (22), $i = N_i$</td>
</tr>
</tbody>
</table>
where $B_1^-$ is matrix of boundary conditions of size $2N_i \times 2N_i N_2$, which can be constructed in accordance with algorithm presented at Table 3; $\bar{g}_i^-$ is the zero vector of order $2N_i$ (i.e. $\bar{g}_i^- = 0$).

Table 3. Algorithm of construction of matrix $B_1^-$. (All other elements of matrix $B_1^-$ are equal to zero).

<table>
<thead>
<tr>
<th>Numbers (indexes) of elements</th>
<th>Element value</th>
<th>Corresponding boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2i - 1, 2N_i(N_2 - 1) + 2i - 1)$, $i = 1, 2, ..., N_i$</td>
<td>1</td>
<td>The first equation from (26)</td>
</tr>
<tr>
<td>$(2i, 2N_i(N_2 - 1) + 2i)$, $i = 1, 2, ..., N_i$</td>
<td>1</td>
<td>The second equation from (26)</td>
</tr>
</tbody>
</table>

Thus, the total number of equation is equal to $2N_i(N_2 - 2) + 2N_i + 4N_i + 2N_i = 2N_i N_2 + 4N_i$ ($4N_i$ components of vector $\bar{C}_i$ and $2N_i N_2$ components of nodal displacements $u_i^{(2,j,i)}$, $u_i^{(2,j,i)}$, $i = 1, 2, ..., N_i$, $j = 1, 2, ..., N_2$). Corresponding coupled system of $2N_i N_2 + 4N_i$ linear algebraic equations with $2N_i N_2 + 4N_i$ unknowns has the form:

$$
\begin{bmatrix}
Q_1 & 0 \\
Q_{2,1} & Q_{2,2} \\
0 & K_i \\
0 & B_1^-
\end{bmatrix}
\begin{bmatrix}
\bar{C}_i \\
\bar{U}_i \\
\bar{R}_i \\
\bar{g}_i^-
\end{bmatrix}
=
\begin{bmatrix}
G_i \\
G_i \\
R_i \\
G_i
\end{bmatrix}.
$$

(28)

It should be noted that boundary conditions (27) can be taken into account automatically within construction of global stiffness matrix and global right-side vector corresponding to subdomain $\omega_2$. Then we get (instead of (28)):

$$
\begin{bmatrix}
Q_1 & 0 \\
Q_{2,1} & Q_{2,2} \\
0 & \tilde{K}_i \\
0 & \tilde{B}_1^-
\end{bmatrix}
\begin{bmatrix}
\bar{C}_i \\
\bar{U}_i \\
\bar{R}_i \\
\bar{g}_i^-
\end{bmatrix}
=
\begin{bmatrix}
G_i \\
G_i \\
\bar{R}_i \\
G_i
\end{bmatrix}.
$$

(29)

where $\tilde{K}_i$ is corresponding reduced global stiffness matrix of size $[2N_i(N_2 - 1)] \times [2N_i N_2]$; $\bar{R}_i$ is corresponding reduced global right-side vector of order $2N_i(N_2 - 1)$.

Strain and stress components are computed according to well-known formulas after solving of system (29).

5. Software and verification samples

We should stress that all methods and algorithms considered in this paper have been realized in software. The main purpose of Analysis system CSASA2D (DCFEM + FEM) is semianalytical structural analysis (static structural analysis of deep beam within two-dimensional theory of elasticity), based on combined application of FEM and DCFEM. Programming environment is Microsoft Visual Studio 2013 Community and Intel Parallel Studio 2015XE with Intel MKL Library [8]. Software is designed for Microsoft Windows 8.1/10.

Corresponding verification samples (ANSYS Mechanical 15.0 [6,7] was used for verification purposes) proved that DCFEM is more effective in the most critical, vital, potentially dangerous areas of structure in terms of fracture (areas of the so-called edge effects), where some components of solution are rapidly changing functions and their rate of change in many cases can’t be adequately taken into account by the standard FEM [1].

Acknowledgements

The Reported study was Funded by Government Program of the Russian Federation “Development of science and technology” (2013-2020) within Program of Fundamental Researches of Ministry of Construction, Housing and
Utilities of the Russian Federation and Russian Academy of Architecture and Construction Sciences, the Research Project 7.1.1”.

References