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Free topological groups of spaces and their subspaces

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Abstract

We prove that if X is a Tychonoff topological space, Y a subspace of X, and every bounded continuous pseudometric on Y can be extended to a continuous pseudometric on X, then the free topological group $F_M(Y)$ coincides with the topological subgroup of $F_M(X)$ generated by Y. For this purpose, a new description for the topology of a free topological group in terms of continuous pseudometrics and group seminorms is given. It follows from what has been shown by Uspenskiĭ that this result implies the Weil completeness of $F_M(X)$ for any Dieudonné complete X. It is also proved that if $\dim X = 0$, then $\operatorname{ind} F_M(X) = 0$. © 2000 Elsevier Science B.V. All rights reserved.

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The object we study in this paper is the free topological group in the sense of Markov, introduced by Markov in [2]. The *free topological group* $F_M(X)$ of a Tychonoff space X is the free algebraic group of the set X with the strongest group topology that induces the original topology on X, or, equivalently, such that any continuous mapping of X to an arbitrary topological group G can be extended to a continuous homomorphism of $F_M(X)$ to G. The reason why these groups are important is that any topological group G algebraically generated by its subspace homeomorphic to X is a continuous homomorphic image of the free topological group of X; moreover, if X is a continuous image of Y, then G is a continuous homomorphic image of $F_M(Y)$.

Let X be a Tychonoff space, Y a subspace of X, $F_M(X)$ the free topological group of X, and $F_M(Y|X)$ the topological subgroup of $F_M(X)$ generated by Y. This paper is concerned with one of the most fundamental problems in the theory of free topological groups:

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When does the topology of $F_M(Y|X)$ coincide with the topology of the free group $F_M(Y)$? Apparently, the problem was first tackled in 1948 by Samuel [6]; it has been extensively studied since then (see, e.g., [1,4,8]). Samuel proved that if X is a Tychonoff space and μX its Dieudonné completion, then $F_M(X|\mu X) = F_M(X)$. An essential advancement was made by Pestov [5]. First, he proved that if $Y \subset X$ and $F_M(Y|X)$ is the free topological group of Y, then the restriction of the universal uniformity of X to Y is the universal uniformity of Y, or equivalently, every bounded continuous pseudometric on Y can be extended to a continuous pseudometric on X. Secondly, he showed that for Y dense in X the converse is true. The latter result has naturally brought up the question if the condition of density of Y in X is necessary. This work answers the question in the negative. Thus, a complete description of all subspaces Y of a space X such that $F_M(Y|X)$ coincides with $F_M(Y)$ ensues. The description is:

> Let X be a completely regular T_1 space and $Y \subset X$. The free topological group $F_M(Y)$ coincides with $F_M(Y|X)$ if and only if every bounded continuous pseudometric on Y can be extended to a continuous pseudometric on X.

The scheme of the proof is as follows. First, we define a family $\mathfrak{N}$ of continuous seminorms on $F_M(X)$ using a series of auxiliary constructions. Next, we prove that this family generates the topology of $F_M(X)$, i.e., for every open neighborhood U of the identity in $F_M(X)$ there exist a seminorm $\|\cdot\|$ in $\mathfrak{N}$ and $a > 0$ such that

$$\{g \in F_M(X)\colon\ \|g\| < a\} \subset U.$$

Finally, for an arbitrary bounded continuous seminorm $\|\cdot\|_Y$ on $F_M(Y)$, we construct a continuous seminorm $\|\cdot\| \in \mathfrak{N}$ (on $F_M(X)$) such that $\|h\|_Y \leqslant \|h\|$ for each h in $F_M(Y)$. This gives the desired statement, because the family of all continuous seminorms generates the topology of $F_M(Y)$.

0. Terminology and notation

Let X be a Tychonoff space, one and the same throughout the paper.

The letters x, y, and z refer to elements of X; k, l, m, n, r, s, and t denote nonnegative integers; ε and δ take values 1 and -1; $\mathbb{N}^+$ stands for the set of all positive integers, and $\mathbb{N}$ for the set of all nonnegative integers.

For a pseudometric p on X, $a > 0$, and $x \in X$,

$$B_p(x, a) = \{y \in X\colon\ p(x, y) < a\}$$

is the ball of radius a with the center at x relative to p.

The *support* of a function f on X is the set $\operatorname{supp} f = \{x \in X\colon\ f(x) \neq 0\}$.

The semigroup of all (reduced and nonreduced) words in the alphabet $X \oplus X^{-1}$ (X^{-1} is a homeomorphic copy of X) is denoted as $S(X)$, and

$$S^*(X) = \left\{x_1^{\varepsilon_1} \ldots x_{2n}^{\varepsilon_{2n}} \in S(X)\colon\ n \in \mathbb{N},\ \sum_{i=1}^{2n} \varepsilon_i = 0\right\}.$$

The free algebraic group of X, i.e., the set of all irreducible words from $S(X)$, is denoted by $F(X)$, and

$$F^*(X)=\left\{x_1^{\varepsilon_1}\dots x_{2n}^{\varepsilon_{2n}}\in F(X)\colon\ n\in\mathbb{N},\ \sum_{i=1}^{2n}\varepsilon_i=0\right\};$$

$F_M(X)$ is the free topological group of X in the sense of Markov.

The symbol e stands for the empty word, which is the identity element of $S(X)$ (and $F(X)$).

For $g,h\in S(X)$, $g\equiv h$ means that the words g and h are equal as elements of the semigroup $S(X)$, i.e., they consist of the same number of letters and their corresponding letters coincide. By $g=h$ we denote the equality of the reduced forms of these words. When g and h are treated as elements of the semigroup $S(X)$ or its subsemigroup $S^*(X)$, gh denotes the semigroup product of g and h, i.e., the word obtained by successively writing g and h. When we speak about (irreducible) words g and h as elements of $F(X)$ or its subgroup $F^*(X)$, the same combination denotes the usual group product of g and h. Thus, when we write $x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n}\in F(X)$, we mean the reduced form of the word $x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n}$, and when we write $x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n}\in S(X)$, we mean the sequence of letters $x_i^{\varepsilon_i}$. For $g\equiv x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n}\in S(X)$, g^{-1} stands for the word $x_n^{-\varepsilon_n}\dots x_1^{-\varepsilon_1}$.

Let $g\equiv x_1^{\varepsilon_1}\dots x_n^{\varepsilon_n}\in S(X)$. The number n is the *length* $l(g)$ of the word g. We use the standard notation $F_n(X)$ for the set of all words in $F(X)$ whose length does not exceed n.

1. Schemes of words

Let $g\equiv x_1^{\varepsilon_1}\dots x_{2n}^{\varepsilon_{2n}}\in S^*(X)$, and let

$$\langle i_1,j_1\rangle,\dots,\langle i_n,j_n\rangle$$

be a partition of the set $\{1,\dots,2n\}$ into pairs such that $i_s<j_s$, $\varepsilon_{i_s}=-\varepsilon_{j_s}$, and for all $s,t\leqslant n$, either the segments $[i_s,j_s]$, $[i_t,j_t]$ are disjoint, or one of them contains the other. We say that the set

$$\boldsymbol{\sigma}=\left\{\langle i_s,j_s\rangle\colon\ 1\leqslant s\leqslant n\right\}$$

is a *scheme* for g. The word g together with a fixed scheme $\boldsymbol{\sigma}$ is denoted as $[g,\boldsymbol{\sigma}]$ or simply $[g]$. The empty word e admits only one scheme, the empty set.

Put

$$\left[S^*(X)\right]=\left\{[g,\boldsymbol{\sigma}]\colon\ g\in S^*(X),\ \boldsymbol{\sigma}\text{ is a scheme for }g\right\}.$$

We retain the term “words” for elements of $[S^*(X)]$ as well as $S^*(X)$.

The symbol $\boldsymbol{\sigma}_g$ always denotes a scheme for g, and it is always implied that $[g]=[g,\boldsymbol{\sigma}_g]$.

Let $[a],[b]\in[S^*(X)]$ and $l(a)=n$. Put

$$\boldsymbol{\sigma}_{ab}=\boldsymbol{\sigma}_a\cup\left\{\langle i+n,j+n\rangle\colon\ \langle i,j\rangle\in\boldsymbol{\sigma}_b\right\}.$$

Then $\boldsymbol{\sigma}_{ab}$ is a scheme for the word ab. We write $[g]=[a][b]$ when $g\equiv ab$ and the scheme $\boldsymbol{\sigma}_g$ coincides with $\boldsymbol{\sigma}_{ab}$.

Let $[g] \in [S^*(X)]$ and $l(g) = n$. Put

$$\boldsymbol{\sigma}_{g^{-1}} = \bigl\{\langle n-j+1, n-i+1\rangle\colon \langle i,j\rangle \in \boldsymbol{\sigma}_g\bigr\}.$$

Then $\boldsymbol{\sigma}_{g^{-1}}$ is a scheme for g^{-1}. We write $[g^{-1}]$ to denote the word g^{-1} with the scheme $\boldsymbol{\sigma}_{g^{-1}}$.

Let $g \in [S^*(X)]$, $l(g) = n$, and $\boldsymbol{\sigma}_g$ be a scheme for g. We call the word $[g, \boldsymbol{\sigma}_g]$ *nonfactorable* if g is nonempty (i.e., $n \geqslant 2$) and $\langle 1, n\rangle \in \boldsymbol{\sigma}_g$. For $[g], [\tilde{g}] \in [S^*(X)]$, the relation $[g] = [x^\varepsilon[\tilde{g}]y^{-\varepsilon}]$ means that $g \equiv x^\varepsilon \tilde{g} y^{-\varepsilon}$ and

$$\boldsymbol{\sigma}_g = \bigl\{\langle 1, l(g)\rangle\bigr\} \cup \bigl\{\langle i+1, j+1\rangle\colon (i,j) \in \boldsymbol{\sigma}_{\tilde{g}}\bigr\}.$$

Clearly, a word is nonfactorable if and only if it has the form $[x^\varepsilon[\tilde{g}]y^{-\varepsilon}]$.

Remark 1. Every nonempty $[g] \in [S^*(X)]$ can be represented as a product $[g_1][g_2]$, where g_1 is an arbitrary (possibly, empty) and $[g_2]$ a nonfactorable word from $[S^*(X)]$, and this representation is unique. Indeed, for $g \equiv x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n}$, find the pair $\langle k, n\rangle \in \boldsymbol{\sigma}_g$ that contains n and put

$$g_1 \equiv x_1^{\varepsilon_1} \ldots x_{k-1}^{\varepsilon_{k-1}}, \qquad g_2 \equiv x_k^{\varepsilon_k} \ldots x_n^{\varepsilon_n},$$
$$\boldsymbol{\sigma}_{g_1} = \bigl\{\langle i,j\rangle \in \boldsymbol{\sigma}_g\colon\ j < k\bigr\}, \quad \boldsymbol{\sigma}_{g_2} = \bigl\{\langle i-k+1, j-k+1\rangle\colon \langle i,j\rangle \in \boldsymbol{\sigma}_g,\ i \geqslant k\bigr\}.$$

Let $h \equiv x_1^{\varepsilon_1} \ldots x_n^{\varepsilon_n} \in S(X)$ and $[g], [\tilde{g}] \in [S^*(X)]$. We write $[g] = [h[\tilde{g}]h^{-1}]$ if

$$[g] = \bigl[x_1^{\varepsilon_1}\bigl[x_2^{\varepsilon_2}\bigl[\ldots[x_n^{\varepsilon_n}[\tilde{g}]x_n^{-\varepsilon_n}]\ldots\bigr]x_2^{-\varepsilon_2}\bigr]x_1^{-\varepsilon_1}\bigr].$$

We call a word $[g]$ *factorable* if it is nonempty and not nonfactorable. Clearly, $[g]$ is factorable if and only if there exist $n \geqslant 2$ and nonfactorable words $[g_i]$, $i = 1, \ldots, n$ such that $[g] = [g_1] \ldots [g_n]$, and this representation of $[g]$ is unique.

Let $[g] \in [S^*(X)]$, $g \equiv a x^\varepsilon x^{-\varepsilon} b$ for some $a, b \in S(X)$, $\hat{g} \equiv ab$, and $l(a) = k-1$. Clearly, $\hat{g} \in S^*(X)$. Put

$$\begin{aligned}\boldsymbol{\sigma}_{\hat{g}} = {} & \bigl\{\langle i,j\rangle \in \boldsymbol{\sigma}_g\colon\ j < k\bigr\}\\ & \cup \bigl\{\langle i, j-2\rangle\colon \langle i,j\rangle \in \boldsymbol{\sigma}_g,\ i < k,\ j > k+1\bigr\}\\ & \cup \bigl\{\langle i-2, j-2\rangle\colon \langle i,j\rangle \in \boldsymbol{\sigma}_g,\ i > k+1\bigr\}\\ & \cup \bigl\{\langle i, j-2\rangle\colon \langle i,k\rangle \in \boldsymbol{\sigma}_g,\ \langle k+1, j\rangle \in \boldsymbol{\sigma}_g\bigr\}.\end{aligned}$$

Note that if $\langle k, k+1\rangle \in \boldsymbol{\sigma}_g$, then the last term in the union is empty.

It is readily verified that $\boldsymbol{\sigma}_{\hat{g}}$ is a scheme for the word $\hat{g}$. We write $[\hat{g}]$ to denote $\hat{g}$ with the scheme $\boldsymbol{\sigma}_{\hat{g}}$.

2. Definition of family $\mathfrak{S}$

Let $\langle \boldsymbol{P}, \leqslant\rangle$ be a partially ordered set.

Define a relation $\lhd$ on the family of all nonempty subsets in $\boldsymbol{P}$ by the rule:

$A \lhd B$ if for every $\alpha \in A$ there exists a $\beta \in B$ such that $\alpha \leqslant \beta$.

Obviously, $\lhd$ is transitive.

For $\alpha \in \boldsymbol{P}$ and $B \subset \boldsymbol{P}$, we put

$$B(\alpha) = \{\beta \in B\colon\ \alpha \leqslant \beta\}.$$

Remark 2. If A is a nonempty antichain in $\boldsymbol{P}$ and $B \subset \boldsymbol{P}$, then the family $\{B(\alpha)\colon\ \alpha \in A\}$ is disjoint.

Fix a partially ordered set $\langle \boldsymbol{P}, \leqslant\rangle$.

Let $\mathcal{A}$ be a collection of nonempty subsets of $\boldsymbol{P}$ labeled by nonnegative integers:

$$\mathcal{A} = \{A_k\colon\ k \in \mathbb{N}\}.$$

Consider a set $\mathfrak{S} = \mathfrak{S}(\boldsymbol{P})$ of triples $\mathfrak{s} = \langle \mathcal{A}, \mathcal{F}, \mathcal{D}\rangle$ satisfying the following conditions:

0°. (a)

$$\mathcal{A} = \{A_k\colon\ k \in \mathbb{N}\},$$

where A_k are disjoint nonempty antichains in $\boldsymbol{P}$;

(b)

$$\mathcal{F} = \{F_k\colon\ k \in \mathbb{N}\}$$

is a collection of families

$$F_k = \{f_\alpha\colon\ \alpha \in A_k\}$$

of continuous nonnegative-valued functions on X such that for every $x \in X$ and $k \in \mathbb{N}$, the set $\{\alpha \in A_k\colon\ f_\alpha(x) \neq 0\}$ is finite;

(c)

$$\mathcal{D} = \{d_k\colon\ k \in \mathbb{N}\}$$

is a family of continuous pseudometrics on X.

When we refer to an element $\mathfrak{s}$ of the family $\mathfrak{S}$, we always imply that $\mathfrak{s} = \langle \mathcal{A}, \mathcal{F}, \mathcal{D}\rangle$ and the sets $\mathcal{A}$, $\mathcal{F}$, and $\mathcal{D}$ have the form specified in condition 0°. Primed, indexed, or otherwise marked $\mathcal{A}$, $\mathcal{F}$, $\mathcal{D}$, A, F, f, and d correspond to the similarly marked $\mathfrak{s}$. For example, $\mathfrak{s}' = \langle \mathcal{A}', \mathcal{F}', \mathcal{D}'\rangle$, $\mathcal{A}' = \{A'_k\colon\ k \in \mathbb{N}\}$, etc.

1°. If $k < m$, then

(a) $A_k \lhd A_m$;

(b) for any $x \in X$ and $\alpha \in A_k$,

$$f_\alpha(x) \leqslant \sum_{\beta \in A_m(\alpha)} f_\beta(x);$$

(c) for any $x, y \in X$,

$$2 \cdot d_k(x, y) \leqslant d_m(x, y).$$

2°. For all x, y, and k,

(a)

$$\sum_{\alpha \in A_k} f_\alpha(x) \geqslant 1;$$

(b)

$$2 \cdot \sum_{\alpha \in A_k} \left| f_\alpha(x) - f_\alpha(y) \right| \leqslant d_k(x, y).$$

To formulate the last condition on the family $\mathfrak{S}$, we need to order its elements. Let $\mathfrak{s}, \mathfrak{s}' \in \mathfrak{S}$. We write $\mathfrak{s} < \mathfrak{s}'$ if for any $k \in \mathbb{N}$, the following relations hold:

(1)

$$A_k \lhd A'_k;$$

(2) for any $x \in X$ and $\alpha \in A_k$,

$$f_\alpha(x) \leqslant \sum_{\beta \in A'_k(\alpha)} f'_\beta(x);$$

(3) for any $x, y \in X$,

$$2 \cdot d_k(x, y) \leqslant d'_k(x, y).$$

3°. To every $\mathfrak{s} = \langle \mathcal{A}, \mathcal{F}, \mathcal{D} \rangle$, there is assigned a family

$$\left\{ \mathfrak{s}_\alpha = \langle \mathcal{A}_\alpha, \mathcal{F}_\alpha, \mathcal{D}_\alpha \rangle \in \mathfrak{S} \colon \alpha \in \bigcup \mathcal{A} = \bigcup_{k \in \mathbb{N}} A_k \right\}$$

such that $\mathfrak{s}_\alpha > \mathfrak{s}$ for all $\alpha \in \bigcup \mathcal{A}$ and if $\mathfrak{s}, \mathfrak{s}' \in \mathfrak{S},\ \alpha \in \bigcup \mathcal{A},\ \alpha' \in \bigcup \mathcal{A}',\ \mathfrak{s} \leqslant \mathfrak{s}'$, and $\alpha \leqslant \alpha'$, then $\mathfrak{s}_\alpha < \mathfrak{s}'_{\alpha'}$.

Note that condition 3° implies the presence of a complex structure on $\mathfrak{S}$: since the triples $\mathfrak{s}_\alpha$ assigned to $\mathfrak{s}$ belong to $\mathfrak{S}$, they are also assigned certain triples from $\mathfrak{S}$, and so on. This structure is discussed in more detail in the proof of Principal Statement 2; now we only need the formal definition given above. Note also that not all partially ordered sets $\boldsymbol{P}$ admit a nonempty family $\mathfrak{S}$ with the properties 0°–3°: for example, 0°(a) implies that $\boldsymbol{P}$ should be infinite and 3° that $\boldsymbol{P}(\alpha)$ should be infinite for infinitely many $\alpha \in \boldsymbol{P}$; moreover, 3° implies that $\boldsymbol{P}$ should contain an infinite number of infinite chains. In Sections 3–6, we assume that $\mathfrak{S}$ is a fixed nonempty family defined for a suitable ordered set $\boldsymbol{P}$ and satisfying conditions 0°–3°.

3. Definition of functions N and $\overline{N}$

Take $\mathfrak{s} \in \mathfrak{S}$. Let us construct functions $N_\mathfrak{s}$ and $\overline{N}_\mathfrak{s}$ on the set $[S^*(X)]$, i.e., define numbers $N_\mathfrak{s}([g])$ and $\overline{N}_\mathfrak{s}([g])$ for each $[g]$ from $[S^*(X)]$. The functions will be constructed by induction on the length of g.

Put $N_\mathfrak{s}([e]) = \overline{N}_\mathfrak{s}([e]) = 0$ for all $\mathfrak{s} \in \mathfrak{S}$.

Let $\mathfrak{s} \in \mathfrak{S}$ and $[g] \in [S^*(X)],\ l(g) > 0$. Let us assume that for all $\mathfrak{s}' \in \mathfrak{S}$ and $[h] \in [S^*(X)]$ with $l(h) < l(g)$, the numbers $N_{\mathfrak{s}'}([h])$ and $\overline{N}_{\mathfrak{s}'}([h])$ are already defined. There are two possibilities:

(A) The word $[g]$ is factorable, i.e., $[g] = [g_1] \ldots [g_n]$, where $n \geqslant 2$ and all $[g_i]$ are nonfactorable; clearly, $l(g_i) < l(g)$ for all $i \leqslant n$. Define

$$N_{\mathfrak{s}}\big([g]\big)=\sum_{i\leqslant n}\overline{N}_{\mathfrak{s}}\big([g_i]\big)\quad\text{and}$$
$$\overline{N}_{\mathfrak{s}}\big([g]\big)=\min\big\{N_{\mathfrak{s}}([g]),1\big\}.$$

(B) The word $[g]$ is nonfactorable, i.e., $[g]=[x^{\varepsilon}[\tilde{g}]y^{-\varepsilon}]$ for some x, y, ε and $\tilde{g}$. Put

$$^{k}N_{\mathfrak{s}}\big([g]\big)=2^{k}\cdot\sum_{\alpha\in A_k}\min\big\{f_\alpha(x),\,f_\alpha(y)\big\}\cdot\overline{N}_{\mathfrak{s}_\alpha}\big([\tilde{g}]\big)+\frac{1}{2^{k}}+2^{k}\cdot d_k(x,y)\quad\text{and}$$
$$N_{\mathfrak{s}}\big([g]\big)=\inf_{k\in\mathbb{N}}\big\{{}^{k}N_{\mathfrak{s}}([g])\big\}.$$

Finally, define

$$^{k}\overline{N}_{\mathfrak{s}}\big([g]\big)=\min\big\{{}^{k}N_{\mathfrak{s}}([g]),1\big\}\quad\text{and}$$
$$\overline{N}_{\mathfrak{s}}\big([g]\big)=\inf_{k\in\mathbb{N}}\big\{{}^{k}\overline{N}_{\mathfrak{s}}([g])\big\}=\min\big\{N_{\mathfrak{s}}([g]),1\big\}.$$

The functions $N_{\mathfrak{s}}$ and $\overline{N}_{\mathfrak{s}}$ are defined.

Let us introduce one more notation: put

$$^{k}B_{\mathfrak{s}}\big(x,y,[h]\big)=\sum_{\alpha\in A_k}\min\big\{f_\alpha(x),\,f_\alpha(y)\big\}\cdot\overline{N}_{\mathfrak{s}_\alpha}\big([h]\big)$$

for $\mathfrak{s}\in\mathfrak{S}$, $[h]\in[S^*(X)]$, $x,y\in X$, and $k\in\mathbb{N}$. Then

$$^{k}N_{\mathfrak{s}}\big([g]\big)=2^{k}\cdot{}^{k}B_{\mathfrak{s}}\big(x,y,[\tilde{g}]\big)+\frac{1}{2k}+2^{k}\cdot d_k(x,y).$$

The functions ${}^{k}N_{\mathfrak{s}}$, ${}^{k}\overline{N}_{\mathfrak{s}}$, and ${}^{k}B_{\mathfrak{s}}$ will be used below.

The subscript $\mathfrak{s}$ will often be omitted. The functions N, $\overline{N}$, ${}^{k}N$, ${}^{k}\overline{N}$, and ${}^{k}B$ are then assumed to correspond to the triple $\mathfrak{s}$. Marked N and B correspond to the similarly marked $\mathfrak{s}$. For example, the functions N_α, $\overline{N}_\alpha$, ${}^{k}N_\alpha$, ${}^{k}\overline{N}_\alpha$ and ${}^{k}B_\alpha$ correspond to $\mathfrak{s}_\alpha$, and the functions N', $\overline{N}'$, ${}^{k}N'$, ${}^{k}\overline{N}'$ and ${}^{k}B'$ to $\mathfrak{s}'$.

Remark 3. If $\mathfrak{s}\in\mathfrak{S}$ and $[g]=[a][b]\in[S^*(X)]$, then

$$\overline{N}\big([g]\big)\leqslant\overline{N}\big([a]\big)+\overline{N}\big([b]\big)\leqslant N\big([g]\big),$$

and if $\overline{N}([a])+\overline{N}([b])\leqslant 1$ then

$$\overline{N}\big([g]\big)=\overline{N}\big([a]\big)+\overline{N}\big([b]\big)=N\big([g]\big).$$

4. Lemmas

Everywhere below, letters denote inequalities and digits the last links in chains of inequalities.

Lemma 1. *Suppose that f is a function on X, $[g]\in[S^*(X)]$, and $\mathfrak{s}\in\mathfrak{S}$. Then for any x and y,*

$$f(x)\cdot\overline{N}\big([g]\big)\leqslant f(y)\cdot\overline{N}\big([g]\big)+\big|f(x)-f(y)\big|$$

and, therefore,

$$f(x)\cdot\overline{N}\big([g]\big)\leqslant\min\big\{f(x),\,f(y)\big\}\cdot\overline{N}\big([g]\big)+\big|f(x)-f(y)\big|.$$

Proof. It is sufficient to apply the inequalities $0\leqslant\overline{N}([g])\leqslant 1$. □

Lemma 2. *Suppose that* $\mathfrak{s},\mathfrak{s}'\in\mathfrak{S}$, $\mathfrak{s}<\mathfrak{s}'$, *and* $[g]\in[S^*(X)]$. *Then* $\overline{N}([g])\leqslant\overline{N}'([g])$.

Proof. Let us apply induction on $l(g)$. For $g\equiv e$, the assertion of Lemma 2 is trivial. Assume that $l(g)>0$ and the statement is already proved for words of smaller lengths. There are two possibilities:

(A) The word $[g]$ is factorable. Then $[g]=[g_1]\dots[g_n]$, where $n\geqslant 2$ and all $[g_i]$ are nonfactorable. Since $l(g_i)<l(g)$, we can apply the induction hypothesis and obtain

$$N\big([g]\big)=\sum_{i\leqslant n}\overline{N}\big([g_i]\big)\leqslant\sum_{i\leqslant n}\overline{N}'\big([g_i]\big)=N'\big([g]\big).$$

(B) The word $[g]$ is nonfactorable, i.e., $[g]=[x^{\varepsilon}[\tilde g]y^{-\varepsilon}]$. Let us prove that ${}^kN([g])\leqslant{}^kN'([g])$ for all k. To do this, it suffices to show that

$${}^kB\big(x,y,[\tilde g]\big)+d_k(x,y)\leqslant{}^kB'\big(x,y,[\tilde g]\big)+d'_k(x,y). \tag{a}$$

We have

$${}^kB\big(x,y,[\tilde g]\big)=\sum_{\alpha\in A_k}\min\big\{f_\alpha(x),\,f_\alpha(y)\big\}\cdot\overline{N}_\alpha\big([\tilde g]\big)\leqslant\sum_{\alpha\in A_k}f_\alpha(x)\cdot\overline{N}_\alpha\big([\tilde g]\big).$$

Take $\alpha\in A_k$. According to condition (2) from the definition of the relation $<$ on $\mathfrak{S}$,

$$f_\alpha(x)\leqslant\sum_{\beta\in A'_k(\alpha)}f'_\beta(x).$$

For every $\beta\in A'_k(\alpha)$, we have $\mathfrak{s}_\alpha<\mathfrak{s}'_\beta$ (by condition 3° from the definition of $\mathfrak{S}$) and hence $\overline{N}'_\beta([g])\geqslant\overline{N}_\alpha([\tilde g])$ (by the induction hypothesis). Therefore,

$$\begin{aligned}\sum_{\alpha\in A_k}f_\alpha(x)\cdot\overline{N}_\alpha\big([\tilde g]\big)&\leqslant\sum_{\alpha\in A_k}\Bigg(\sum_{\beta\in A'_k(\alpha)}f'_\beta(x)\Bigg)\cdot\overline{N}_\alpha\big([\tilde g]\big)\\&\leqslant\sum_{\alpha\in A_k}\Bigg(\sum_{\beta\in A'_k(\alpha)}f'_\beta(x)\cdot\overline{N}'_\beta\big([\tilde g]\big)\Bigg)\\&=\sum_{\beta\in\bigcup\{A'_k(\alpha):\,\alpha\in A_k\}}f'_\beta(x)\cdot\overline{N}_\beta\big([\tilde g]\big)\\&\leqslant\sum_{\beta\in A'_k}f'_\beta(x)\cdot\overline{N}'_\beta\big([\tilde g]\big).\end{aligned}\tag{1}$$

By Lemma 1,

$$(1)\leqslant\sum_{\beta\in A'_k}\min\big\{f'_\beta(x),\,f'_\beta(y)\big\}\cdot\overline{N}'_\beta\big([\tilde g]\big)+\sum_{\beta\in A'_k}\big|f'_\beta(x)-f'_\beta(y)\big|.$$

Condition 2°(b) from the definition of $\mathfrak{S}$ implies that

$$\sum_{\beta\in A'_k} \left| f'_\beta(x) - f'_\beta(y) \right| \leqslant \frac{d'_k(x,y)}{2};$$

therefore,

$${}^k B\big(x, y, [\tilde{g}]\big) \leqslant {}^k B'\big(x, y, [\tilde{g}]\big) + \frac{d'_k(x,y)}{2}.$$

Finally, condition (3) in the definition of $<$ yields (a).

Thus, ${}^k N([g]) \leqslant {}^k N'([g])$ for all k. Therefore, $N([g]) \leqslant N'([g])$.

We showed that $N([g]) \leqslant N'([g])$ in both cases (A) and (B). This immediately implies the desired inequality $\overline{N}([g]) \leqslant \overline{N}'([g])$. □

Lemma 3. *Suppose that* $[h] \in [S^*(X)]$, $\mathfrak{s} \in \mathfrak{S}$, $x, y, z \in X$, *and* $k, m \in \mathbb{N}$, $k \leqslant m$. *Then*

(i) ${}^k B(x, y, [h]) \leqslant {}^m B(x, z, [h]) + d_m(x, z)/2$;

(ii) ${}^k B(x, y, [h]) + d_k(x, y) \leqslant {}^m B(x, y, [h]) + d_m(x, y)$;

(iii) ${}^k B(y, z, [h]) \leqslant {}^k B(x, y, [h]) + d_k(x, z)/2$.

Proof. (i) By definition,

$$\begin{aligned} {}^k B\big(x, y, [h]\big) &= \sum_{\alpha\in A_k} \min\big\{ f_\alpha(x), f_\alpha(y) \big\} \cdot \overline{N}_\alpha\big([h]\big) \\ &\leqslant \sum_{\alpha\in A_k} f_\alpha(x) \cdot \overline{N}_\alpha\big([h]\big). \end{aligned} \tag{2}$$

Suppose $k < m$. Take $\alpha \in A_k$. By condition 1°(b), we have

$$f_\alpha(x) \leqslant \sum_{\beta\in A_m(\alpha)} f_\beta(x).$$

For any $\beta \in A_m(\alpha)$, we have $\mathfrak{s}_\alpha < \mathfrak{s}_\beta$ (by condition 3°) and hence $\overline{N}_\beta([h]) \geqslant \overline{N}_\alpha([h])$ (by Lemma 2). Therefore,

$$\begin{aligned} (2) &\leqslant \sum_{\alpha\in A_k} \Bigg(\sum_{\beta\in A_m(\alpha)} f_\beta(x) \Bigg) \cdot \overline{N}_\alpha\big([h]\big) \\ &\leqslant \sum_{\alpha\in A_k} \Bigg(\sum_{\beta\in A_m(\alpha)} f_\beta(x) \cdot \overline{N}_\beta\big([h]\big) \Bigg) \leqslant \sum_{\beta\in A_m} f_\beta(x) \cdot \overline{N}_\beta\big([h]\big). \end{aligned} \tag{3}$$

We showed that $(2) \leqslant (3)$ for $k < m$; obviously, this inequality also holds for $k = m$. By Lemma 1 and condition 2°(b),

$$\begin{aligned} (3) &\leqslant \sum_{\beta\in A_m} \min\big\{ f_\beta(x), f_\beta(z) \big\} \cdot \overline{N}_\beta\big([h]\big) + \sum_{\beta\in A_m} \left| f_\beta(x) - f_\beta(z) \right| \\ &\leqslant {}^m B\big(x, z, [h]\big) + d_m(x, z)/2. \end{aligned}$$

Therefore, ${}^k B(x, y, [h]) \leqslant {}^m B(x, z, [h]) + d_m(x, z)/2$, as required.

(ii) The case $k = m$ does not need proving. For $k < m$, it is sufficient to apply (i) and the relation $d_k(x, y) \leqslant d_m(x, y)/2$, which is implied by condition 1°(c) from the definition of $\mathfrak{S}$.

(iii) By definition,

$$^kB\bigl(y, z, [h]\bigr) = \sum_{\alpha \in A_k} \min\bigl\{f_\alpha(y), f_\alpha(z)\bigr\} \cdot \overline{N}_\alpha\bigl([h]\bigr).$$

Take $\alpha \in A_k$. If $\min\{f_\alpha(x), f_\alpha(y), f_\alpha(z)\} \neq f_\alpha(x)$, then

$$\min\bigl\{f_\alpha(y), f_\alpha(z)\bigr\} \cdot \overline{N}_\alpha\bigl([h]\bigr) \leqslant \min\bigl\{f_\alpha(x), f_\alpha(y)\bigr\} \cdot \overline{N}_\alpha\bigl([h]\bigr).$$

If $\min\{f_\alpha(x), f_\alpha(y), f_\alpha(z)\} = f_\alpha(x)$, then

$$\begin{aligned}\min\bigl\{f_\alpha(y), f_\alpha(z)\bigr\} \cdot \overline{N}_\alpha\bigl([h]\bigr) &\leqslant f_\alpha(z) \cdot \overline{N}_\alpha\bigl([h]\bigr)\\ &\leqslant f_\alpha(x) \cdot \overline{N}_\alpha\bigl([h]\bigr) + \bigl|f_\alpha(x) - f_\alpha(z)\bigr|\end{aligned}$$

(by Lemma 1), and the last sum is equal to

$$\min\bigl\{f_\alpha(x), f_\alpha(y)\bigr\} \cdot \overline{N}_\alpha\bigl([h]\bigr) + \bigl|f_\alpha(x) - f_\alpha(z)\bigr|.$$

Therefore,

$$\min\bigl\{f_\alpha(y), f_\alpha(z)\bigr\} \cdot \overline{N}_\alpha\bigl([h]\bigr) \leqslant \min\bigl\{f_\alpha(x), f_\alpha(y)\bigr\} \cdot \overline{N}_\alpha\bigl([h]\bigr) + \bigl|f_\alpha(x) - f_\alpha(z)\bigr|$$

for any $\alpha \in A_k$, whence

$$^kB\bigl(y, z, [h]\bigr) \leqslant {}^kB\bigl(x, y, [h]\bigr) + \sum_{\alpha \in A_k} \bigl|f_\alpha(x) - f_\alpha(z)\bigr|.$$

The required inequality follows from this and the relation

$$\sum_{\alpha \in A_k} \bigl|f_\alpha(x) - f_\alpha(z)\bigr| \leqslant \frac{d_k(x, z)}{2},$$

which is implied by condition 2°(b). □

Lemma 4. *Suppose that* $\mathfrak{s} \in \mathfrak{S}$, $k \in \mathbb{N}$, $[a], [b] \in [S^*(X)]$, *and* $x, y, z \in X$. *Then*

$$^kB\bigl(y, z, [a][b]\bigr) \leqslant {}^kB\bigl(x, y, [a]\bigr) + {}^kB\bigl(x, z, [b]\bigr) + \frac{d_k(x, z)}{2}.$$

Proof. First, we show that for any $\alpha \in A_k$,

$$\begin{aligned}&\min\bigl\{f_\alpha(y), f_\alpha(z)\bigr\} \cdot \overline{N}_\alpha\bigl([a][b]\bigr)\\ &\quad\leqslant \min\bigl\{f_\alpha(x), f_\alpha(y)\bigr\} \cdot \overline{N}_\alpha\bigl([a]\bigr) + \min\bigl\{f_\alpha(x), f_\alpha(z)\bigr\} \cdot \overline{N}_\alpha\bigl([b]\bigr)\\ &\qquad+ \bigl|f_\alpha(x) - f_\alpha(z)\bigr|.\end{aligned} \tag{b}$$

Let α belong to A_k. By definition, $\overline{N}_\alpha([a][b]) \leqslant \overline{N}_\alpha([a]) + \overline{N}_\alpha([b])$.

If $\min\{f_\alpha(x), f_\alpha(y), f_\alpha(z)\} \neq f_\alpha(x)$, then

$$\begin{aligned}&\min\bigl\{f_\alpha(y), f_\alpha(z)\bigr\} \cdot \overline{N}_\alpha\bigl([a][b]\bigr)\\ &\quad\leqslant \min\bigl\{f_\alpha(y), f_\alpha(z)\bigr\} \cdot \overline{N}_\alpha\bigl([a]\bigr) + \min\bigl\{f_\alpha(y), f_\alpha(z)\bigr\} \cdot \overline{N}_\alpha\bigl([b]\bigr)\\ &\quad\leqslant \min\bigl\{f_\alpha(x), f_\alpha(y)\bigr\} \cdot \overline{N}_\alpha\bigl([a]\bigr) + \min\bigl\{f_\alpha(x), f_\alpha(z)\bigr\} \cdot \overline{N}_\alpha\bigl([b]\bigr),\end{aligned}$$

which immediately implies (b).

Suppose that $\min\{f_\alpha(x), f_\alpha(y), f_\alpha(z)\} \neq f_\alpha(x)$. Then

$$\min\{f_\alpha(y), f_\alpha(z)\} \cdot \overline{N}_\alpha([a][b]) \leqslant f_\alpha(z) \cdot \overline{N}_\alpha([a][b]). \tag{4}$$

By Lemma 1,

$$\begin{aligned}(4) &\leqslant f_\alpha(x) \cdot \overline{N}_\alpha([a][b]) + |f_\alpha(x) - f_\alpha(z)| \\ &\leqslant f_\alpha(x) \cdot (\overline{N}_\alpha([a]) + \overline{N}_\alpha([b])) + |f_\alpha(x) - f_\alpha(z)| \\ &= \min\{f_\alpha(x), f_\alpha(y)\} \cdot \overline{N}_\alpha([a]) \\ &\quad + \min\{f_\alpha(x), f_\alpha(z)\} \cdot \overline{N}_\alpha([b]) + |f_\alpha(x) - f_\alpha(z)|.\end{aligned}$$

Thus, (b) holds for all α from A_k; therefore

$${}^kB(y, z, [a][b]) \leqslant {}^kB(x, y, [a]) + {}^kB(x, z, [b]) + \sum_{\alpha \in A_k} |f_\alpha(x) - f_\alpha(z)|.$$

This and 2°(b) imply the required inequality. □

Lemma 5. *If* $\mathfrak{s} \in \mathfrak{S}$, $[\tilde{g}_1], [\tilde{g}_2] \in [S^*(X)]$, $k \in \mathbb{N}$, $x, y \in X$, $\varepsilon \in \{-1, 1\}$, *and*

$${}^kN([x^\varepsilon[\tilde{g}_1][\tilde{g}_2]y^{-\varepsilon}]) < 1,$$

then

$$2^k \cdot {}^kB(x, y, [\tilde{g}_1][\tilde{g}_2]) \geqslant 2^k \cdot {}^kB(x, y, [\tilde{g}_1]) + \overline{N}([\tilde{g}_2]).$$

Proof. By definition,

$${}^kB(x, y, [\tilde{g}_1][\tilde{g}_2]) = \sum_{\alpha \in A_k} \min\{f_\alpha(x), f_\alpha(y)\} \cdot \overline{N}_\alpha([\tilde{g}_1][\tilde{g}_2]).$$

Take $\alpha \in A_k$. If $\overline{N}_\alpha([\tilde{g}_1]) + \overline{N}_\alpha([\tilde{g}_2]) < 1$, then by Remark 3, the relation $\mathfrak{s} < \mathfrak{s}_\alpha$ (implied by 3°) and Lemma 2 yield

$$\overline{N}_\alpha([\tilde{g}_1][\tilde{g}_2]) = \overline{N}_\alpha([\tilde{g}_1]) + \overline{N}_\alpha([\tilde{g}_2]) \geqslant \overline{N}_\alpha([\tilde{g}_1]) + \overline{N}([\tilde{g}_2]).$$

Note that in this case, $\overline{N}_\alpha([\tilde{g}_1][\tilde{g}_2]) < 1$. If $\overline{N}_\alpha([\tilde{g}_1]) + \overline{N}_\alpha([\tilde{g}_2]) \geqslant 1$, then

$$\overline{N}_\alpha([\tilde{g}_1][\tilde{g}_2]) = 1 \geqslant \overline{N}_\alpha([\tilde{g}_1]) = \overline{N}_\alpha([\tilde{g}_1]) + \overline{N}([\tilde{g}_2]) - \overline{N}([\tilde{g}_2]).$$

Thus,

$$\begin{aligned}&\sum_{\alpha \in A_k} \min\{f_\alpha(x), f_\alpha(y)\} \cdot \overline{N}_\alpha([\tilde{g}_1][\tilde{g}_2]) \\ &\quad \geqslant \sum_{\alpha \in A_k} \min\{f_\alpha(x), f_\alpha(y)\} \cdot (\overline{N}_\alpha([\tilde{g}_1]) + \overline{N}([\tilde{g}_2])) \\ &\qquad - \sum_{\substack{\alpha \in A_k: \\ \overline{N}_\alpha([\tilde{g}_1][\tilde{g}_2]) = 1}} \min\{f_\alpha(x), f_\alpha(y)\} \cdot \overline{N}([\tilde{g}_2]).\end{aligned} \tag{c}$$

Let us show that

$$\sum_{\alpha\in A_k}\min\{f_\alpha(x),\ f_\alpha(y)\}\geqslant 1-\frac{1}{2^{k+1}}+\frac{1}{2^{2k+1}}. \tag{d}$$

Obviously,

$$\sum_{\alpha\in A_k}\min\{f_\alpha(x),\ f_\alpha(y)\}\geqslant \sum_{\alpha\in A_k} f_\alpha(x)-\sum_{\alpha\in A_k}\left|f_\alpha(x)-f_\alpha(y)\right|. \tag{5}$$

It follows from 2°(a) and (b) that (5) $\geqslant 1-d_k(x,y)/2$. By assumption, $1>{}^kN([x^\varepsilon[\tilde g_1][\tilde g_2]\cdot y^{-\varepsilon}])$, and by the definition of ${}^kN$,

$${}^kN\big(\big[x^\varepsilon[\tilde g_1][\tilde g_2]y^{-\varepsilon}\big]\big)\geqslant 1/2^k+2^k\cdot d_k(x,y);$$

therefore, $d_k(x,y)<1/2^k-1/2^{2k}$ and (5) $\geqslant 1-1/2^{k+1}+1/2^{2k+1}$, which implies (d).

Let us show that

$$2^k\cdot\sum_{\substack{\alpha\in A_k:\\ \overline{N}_\alpha([\tilde g_1][\tilde g_2])=1}}\min\{f_\alpha(x),\ f_\alpha(y)\}\leqslant 1-\frac{1}{2^k}. \tag{e}$$

We have

$$\begin{aligned}
&2^k\cdot\sum_{\substack{\alpha\in A_k:\\ \overline{N}_\alpha([\tilde g_1][\tilde g_2])=1}}\min\{f_\alpha(x),\ f_\alpha(y)\}\\
&\quad=2^k\cdot\sum_{\substack{\alpha\in A_k:\\ \overline{N}_\alpha([\tilde g_1][\tilde g_2])=1}}\min\{f_\alpha(x),\ f_\alpha(y)\}\cdot\overline{N}_\alpha\big([\tilde g_1][\tilde g_2]\big)\\
&\quad\leqslant 2^k\cdot{}^kB\big(x,y,[\tilde g_1][\tilde g_2]\big).
\end{aligned}$$

By condition and the definition of ${}^kN$,

$$1>{}^kN\big(\big[x^\varepsilon[\tilde g_1][\tilde g_2]y^{-\varepsilon}\big]\big)\geqslant 2^k\cdot{}^kB\big(x,y,[\tilde g_1][\tilde g_2]\big)+1/2^k,$$

whence

$$2^k\cdot{}^kB\big(x,y,[\tilde g_1][\tilde g_2]\big)<1-1/2^k,$$

which gives (e).

Inequalities (c), (d), and (e) give

$$\begin{aligned}
&2^k\cdot\sum_{\alpha\in A_k}\min\{f_\alpha(x),\ f_\alpha(y)\}\cdot\overline{N}_\alpha\big([\tilde g_1][\tilde g_2]\big)\\
&\quad\geqslant 2^k\cdot\sum_{\alpha\in A_k}\min\{f_\alpha(x),\ f_\alpha(y)\}\cdot\overline{N}_\alpha\big([\tilde g_1]\big)\\
&\qquad+2^k\cdot\big(1-1/2^{k+1}+1/2^{2k+1}\big)\cdot\overline{N}\big([\tilde g_2]\big)-\big(1-1/2^k\big)\cdot\overline{N}\big([\tilde g_2]\big),
\end{aligned}$$

whence

$$\begin{aligned}
&2^k\cdot{}^kB\big(x,y,[\tilde g_1][\tilde g_2]\big)\\
&\quad\geqslant 2^k\cdot{}^kB\big(x,y,[\tilde g_1]\big)+\big(2^k-1-1/2+1/2^k+1/2^{k+1}\big)\cdot\overline{N}\big([\tilde g_2]\big).
\end{aligned}$$

Direct evaluation shows that $2^k - 1 - 1/2 + 1/2^k + 1/2^{k+1} \geqslant 1$ for each k, which completes the proof of Lemma 5. □

Lemma 6. *If* ${}^kN([x^\varepsilon[\tilde{g}]y^{-\varepsilon}]) < 1$, *then* $\overline{N}([\tilde{g}]) < 1/2^k$.

Proof. By the condition and the definition of ${}^kN$, we have

$$1 > 2^k \cdot {}^kB\bigl(x, y, [\tilde{g}]\bigr) + 1/2^k + 2^k \cdot d_k(x, y).$$

By Lemma 3(i),

$${}^kB\bigl(x, y, [\tilde{g}]\bigr) + d_k(x, y) \geqslant {}^kB\bigl(x, x, [\tilde{g}]\bigr);$$

therefore

$$1 > 2^k \cdot {}^kB\bigl(x, x, [\tilde{g}]\bigr) = 2^k \cdot \sum_{\alpha \in A_k} f_\alpha(x) \cdot \overline{N}_\alpha\bigl([\tilde{g}]\bigr).$$

Since $\mathfrak{s} < \mathfrak{s}_\alpha$ (see 3°), Lemma 2 implies that $\overline{N}_\alpha([\tilde{g}]) \geqslant \overline{N}([\tilde{g}])$ for all $\alpha \in A_k$; by condition 2°(a), we have $1 \leqslant \sum_{\alpha \in A_k} f_\alpha(x)$. Hence, $1 > 2^k \cdot \overline{N}([\tilde{g}])$, as required. □

Lemma 7. *If* $d_m(x, z) \leqslant 1$ *and* $m > 0$, *then*

$$2^m \cdot {}^mB\bigl(x, z, [h]\bigr) \geqslant \overline{N}\bigl([h]\bigr).$$

Proof. By definition and because $m > 0$, we have

$$2^m \cdot {}^mB\bigl(x, z, [h]\bigr) \geqslant 2 \cdot {}^mB\bigl(x, z, [h]\bigr) = 2 \cdot \sum_{\alpha \in A_k} \min\bigl\{f_\alpha(x), f_\alpha(z)\bigr\} \cdot \overline{N}_\alpha\bigl([h]\bigr).$$

Since $\mathfrak{s} < \mathfrak{s}_\alpha$ for all $\alpha \in \bigcup \mathcal{A} = \bigcup_{k \in \mathbb{N}} A_k$ (see 3°), Lemma 2 implies that

$$\overline{N}_\alpha\bigl([h]\bigr) \geqslant \overline{N}\bigl([h]\bigr) \quad \text{for all } \alpha \in A_m.$$

This and 2°(a) and (b) imply that

$$\begin{aligned} &2^m \cdot {}^mB\bigl(x, z, [h]\bigr) \\ &\quad \geqslant 2 \cdot \left(\sum_{\alpha \in A_m} \min\bigl\{f_\alpha(x), f_\alpha(z)\bigr\}\right) \cdot \overline{N}\bigl([h]\bigr) \\ &\quad \geqslant 2 \cdot \overline{N}\bigl([h]\bigr) \cdot \left(\sum_{\alpha \in A_m} f_\alpha(x) - \sum_{\alpha \in A_m} \bigl|f_\alpha(x) - f_\alpha(z)\bigr|\right) \\ &\quad \geqslant 2 \cdot \overline{N}\bigl([h]\bigr) \cdot \bigl(1 - d_m(x, z)/2\bigr). \end{aligned}$$

By condition, $d_m(x, z) \leqslant 1$, whence

$$2 \cdot \bigl(1 - d_m(x, z)/2\bigr) \geqslant 2 \cdot (1 - 1/2) = 1 \quad \text{and}$$
$$2^m \cdot {}^mB\bigl(x, z, [h]\bigr) \geqslant \overline{N}\bigl([h]\bigr). \qquad \square$$

Lemma 8. *Suppose that* $\mathfrak{s} \in \mathfrak{S}$ *and* $[g] = [x^\varepsilon[\tilde{g}]y^{-\varepsilon}] \in [S^*(X)]$. *Then either*

(a) $N([g]) = {}^kN([g])$ (*and* $N([g]) \geqslant 1/2^k$) *for some* k, *or*

(b) ${}^kB(x, y, [\tilde{g}]) = d_k(x, y) = 0$ (*and* $N([g]) = 0$) *for all* k.

Proof. For $k \in \mathbb{N}$, put

$$a_k = {}^kB\big(x, y, [\tilde{g}]\big) + d_k(x, y).$$

We have

$${}^kN\big([g]\big) = 2^k \cdot a_k + 1/2^k \quad \text{and} \quad N\big([g]\big) = \inf_k\big\{2^k \cdot a_k + 1/2^k\big\}.$$

Lemma 3(ii) implies that $a_k \leqslant a_m$ for $k \leqslant m$. Clearly, if $a_{k_0} \neq 0$ for some k_0, then the sequence $\{2^k \cdot a_k + 1/2^k\}_{k=0}^{\infty}$ has a minimal element, i.e., (a) holds. Otherwise (when $a_k = 0$ for all $k \in \mathbb{N}$) (b) holds. □

5. Statements

As previously, we omit the subscript $\mathfrak{s}$ at N, $\overline{N}$, and B.

Statement 1. *Suppose that $\mathfrak{s} \in \mathfrak{S}$, $a, b \in S(X)$, $ab \in S^*(X)$, $g \equiv ax^{\varepsilon}x^{-\varepsilon}b, [g] \in [S^*(X)]$, and $[\hat{g}] = [ab]$ has the scheme $\boldsymbol{\sigma}_{\hat{g}}$ defined at the end of Section* 1. *Then*

$$\overline{N}\big([\hat{g}]\big) \leqslant \overline{N}\big([g]\big).$$

Proof. Apply induction on $l(g)$. If $g \equiv x^{\varepsilon}x^{-\varepsilon}$, then the assertion is obvious. Suppose that $l(g) > 2$ and the required inequality holds for shorter words of the specified form. Consider all possible cases.

(1) $a, b \not\equiv e$.

(1.1) $[g]$ *is nonfactorable*, i.e., $a \equiv y^{\delta}\tilde{a}$, $b \equiv \tilde{b}z^{-\delta}$, $[g] = [y^{\delta}[\tilde{a}x^{\varepsilon}x^{-\varepsilon}\tilde{b}]z^{-\delta}]$, and

$$N\big([g]\big) = \inf_k\left\{2^k \cdot \sum_{\alpha\in A_k} \min\big\{f_\alpha(y), f_\alpha(z)\big\} \cdot \overline{N}_\alpha\big([\tilde{a}x^{\varepsilon}x^{-\varepsilon}\tilde{b}]\big) + \frac{1}{2^k} + 2^k \cdot d_k(y, z)\right\}.$$

Clearly, $[\hat{g}] = [y^{\delta}[\tilde{a}\tilde{b}]z^{-\delta}]$, where $[\tilde{a}\tilde{b}] = [\widehat{\tilde{a}x^{\varepsilon}x^{-\varepsilon}\tilde{b}}]$. By the induction hypothesis,

$$\overline{N}'\big([\tilde{a}\tilde{b}]\big) \leqslant \overline{N}'\big([\tilde{a}x^{\varepsilon}x^{-\varepsilon}\tilde{b}]\big)$$

for any $\mathfrak{s}' \in \mathfrak{S}$; therefore,

$$\begin{aligned} N\big([g]\big) &\geqslant \inf_k\left\{2^k \cdot \sum_{\alpha\in A_k} \min\big\{f_\alpha(z), f_\alpha(y)\big\} \cdot \overline{N}_\alpha\big([\tilde{a}\tilde{b}]\big) + \frac{1}{2^k} + 2^k \cdot d_k(y, z)\right\} \\ &= N\big([\hat{g}]\big), \end{aligned}$$

whence $\overline{N}([g]) \geqslant \overline{N}([\hat{g}])$.

(1.2) $[g]$ *is factorable*, i.e., $[g] = [g_1]\ldots[g_n]$, where $n \geqslant 2$ and all $[g_i]$ are nonfactorable.

(1.2.1a) $l(g_1) \leqslant l(a)$, i.e., $a \equiv g_1\tilde{a}$ for some $\tilde{a} \in S^*(X)$ and $g_2 \ldots g_n \equiv \tilde{a}x^{\varepsilon}x^{-\varepsilon}b$. Endow $\tilde{a}x^{\varepsilon}x^{-\varepsilon}b$ with the scheme such that $[\tilde{a}x^{\varepsilon}x^{-\varepsilon}b] = [g_2]\ldots[g_n]$. We have

$$\begin{aligned} N\big([g]\big) &= \sum_{i=1}^{n} \overline{N}\big([g_i]\big) = \overline{N}\big([g_1]\big) + \sum_{i=2}^{n} \overline{N}\big([g_i]\big) \\ &\geqslant \overline{N}\big([g_1]\big) + \overline{N}\big([g_2]\ldots[g_n]\big) = \overline{N}\big([g_1]\big) + \overline{N}\big([\tilde{a}x^{\varepsilon}x^{-\varepsilon}b]\big). \end{aligned}$$

Let $[\tilde{a}b] = [\tilde{a}\widehat{x^{\varepsilon}x^{-\varepsilon}}b]$ (this, of course, refers to the choice of a scheme for $\tilde{a}b$). Clearly, $[\hat{g}] = [g_1][\tilde{a}b]$. By the induction hypothesis, $\overline{N}([\tilde{a}x^{\varepsilon}x^{-\varepsilon}b]) \geqslant \overline{N}([\tilde{a}b])$, hence,

$$N([g]) \geqslant \overline{N}([g_1]) + \overline{N}([\tilde{a}b]) \geqslant \overline{N}([g_1][\tilde{a}b]) = \overline{N}([\hat{g}]).$$

Thus, $N([g]) \geqslant \overline{N}([\hat{g}])$, and $\overline{N}([g]) \geqslant \overline{N}([\hat{g}])$.

(1.2.1b) $l(g_n) \leqslant l(b)$. This is considered similarly to (1.2.1a).

(1.2.2) $n = 2,\ g_1 \equiv ax^{\varepsilon},\ g_2 \equiv x^{-\varepsilon}b$. Because the words $[g_1]$ and $[g_2]$ are nonfactorable, they can be represented as

$$[g_1] = [y^{-\varepsilon}[\tilde{a}]x^{\varepsilon}], \quad [g_2] = [x^{-\varepsilon}[\tilde{b}]z^{\varepsilon}].$$

Clearly, $[\hat{g}] = [y^{-\varepsilon}[\tilde{a}][\tilde{b}]z^{\varepsilon}]$. We have

$$N([g_1]) = \inf_k \{{}^kN([y^{-\varepsilon}[\tilde{a}]x^{\varepsilon}])\},$$
$$N([g_2]) = \inf_m \{{}^mN([x^{-\varepsilon}[\tilde{b}]z^{\varepsilon}])\},$$
$$N([\hat{g}]) = \inf_l \{{}^lN([y^{-\varepsilon}[\tilde{a}\tilde{b}]z^{\varepsilon}])\}.$$

Let us show that for any k and m not both equal to zero, there exists l such that

$${}^lN([y^{-\varepsilon}[\tilde{a}][\tilde{b}]z^{\varepsilon}]) \leqslant {}^kN([y^{-\varepsilon}[\tilde{a}]x^{\varepsilon}]) + {}^mN([x^{-\varepsilon}[\tilde{b}]z^{\varepsilon}]). \tag{a}$$

For this purpose, we have to consider further subcases.

(1.2.2.1a) $k < m$. Put $l = k$. By Lemma 4,

$${}^kB(y, z, [\tilde{a}][\tilde{b}]) \leqslant {}^kB(x, y, [\tilde{a}]) + {}^kB(x, z, [\tilde{b}]) + d_k(x, z)/2.$$

By Lemma 3(i),

$${}^kB(x, z, [\tilde{b}]) \leqslant {}^mB(x, z, [\tilde{b}]) + d_m(x, z)/2.$$

By condition 1°(c) from the definition of $\mathfrak{S}$, $d_k(x, z) \leqslant d_m(x, z)$. Thus, we have

$${}^kB(y, z, [\tilde{a}][\tilde{b}]) \leqslant {}^kB(x, y, [\tilde{a}]) + {}^mB(x, z, [\tilde{b}]) + d_m(x, z),$$

and

$$\begin{aligned} &{}^kN([y^{-\varepsilon}[\tilde{a}][\tilde{b}]z^{\varepsilon}]) \\ &\quad = 2^k \cdot {}^kB(y, z, [\tilde{a}][\tilde{b}]) + 1/2^k + 2^k \cdot d_k(y, z) \\ &\quad \leqslant 2^k \cdot {}^kB(x, y, [\tilde{a}]) + 2^k \cdot {}^mB(x, z, [\tilde{b}]) + 2^k \cdot d_m(x, z) \\ &\qquad + 1/2^k + 2^k \cdot d_k(x, y) + 2^k \cdot d_k(x, z). \end{aligned} \tag{1}$$

Condition 1°(c) implies that $d_k(x, z) \leqslant d_m(x, z)$; therefore,

$$2^k \cdot d_m(x, z) + 2^k \cdot d_k(x, z) \leqslant 2^{k+1} \cdot d_m(x, z) \leqslant 2^m \cdot d_m(x, z).$$

This proves inequality (a) for $l = k$.

(1.2.2.1b) $m < k$. This case is considered similarly to (1.2.2.1a).

(1.2.2.2) $m = k > 0$. Put $l = k - 1$. We have

$$\begin{aligned} {}^{k-1}B\big(y,z,[\tilde a][\tilde b]\big) &\leqslant {}^{k-1}B\big(y,z,[\tilde a]\big)+{}^{k-1}B\big(y,z,[\tilde b]\big)\\ &\leqslant {}^{k-1}B\big(y,y,[\tilde a]\big)+{}^{k-1}B\big(z,z,[\tilde b]\big) \end{aligned} \tag{2}$$

(this follows from the definition of ${}^{k-1}B$). By Lemma 3(i),

$$(2)\leqslant {}^{k}B\big(x,y,[\tilde a]\big)+d_k(x,y)+{}^{k}B\big(x,z,[\tilde b]\big)+d_k(x,z);$$

therefore,

$$\begin{aligned} &{}^{k-1}N\big(\big[y^{-\varepsilon}[\tilde a][\tilde b]z^{\varepsilon}\big]\big)\\ &\quad= 2^{k-1}\cdot {}^{k-1}B\big(y,z,[\tilde a][\tilde b]\big)+1/2^{k-1}+2^{k-1}\cdot d_{k-1}(y,z)\\ &\quad\leqslant 2^{k-1}\cdot {}^{k}B\big(x,y,[\tilde a]\big)+2^{k-1}\cdot d_k(x,y)+2^{k-1}\cdot {}^{k}B\big(x,z,[\tilde b]\big)+2^{k-1}\cdot d_k(x,z)\\ &\qquad+1/2^k+1/2^k+2^{k-1}\cdot d_{k-1}(x,y)+2^{k-1}\cdot d_{k-1}(x,z)\\ &\quad\leqslant {}^{k}N\big(\big[y^{-\varepsilon}[\tilde a]x^{\varepsilon}\big]\big)+{}^{k}N\big(\big[x^{-\varepsilon}[\tilde b]z^{\varepsilon}\big]\big) \end{aligned}$$

(we applied 1°(c)). This proves (a) for $k=m=l+1$.

Thus, for any k and m not both equal to zero,

(i) there exists l satisfying (a), hence,

(ii) $N([\hat g])\leqslant {}^{k}N([g_1])+{}^{m}N([g_2])$, and, therefore,

(iii) $\overline{N}([\hat g])\leqslant {}^{k}N([g_1])+{}^{m}N([g_2])$.

Obviously, the last inequality also holds for $k=m=0$. We have

$$\overline{N}\big([\hat g]\big)\leqslant N\big([g_1]\big)+N\big([g_2]\big)$$

and, finally, $\overline{N}([\hat g])\leqslant \overline{N}([g])$.

(2) *$a\not\equiv e$ and $b\equiv e$*, i.e., $g\equiv ax^{\varepsilon}x^{-\varepsilon}$.

(2.1) $[g]$ *is nonfactorable*, i.e., $[g]=[y^{\varepsilon}[\tilde g]x^{-\varepsilon}]$. According to Remark 1, there exists a (unique) representation $[\tilde g]=[\tilde g_1][\tilde g_2]$ with nonfactorable $[\tilde g_2]$. Let $[\tilde g_2]=[z^{-\varepsilon}[\tilde{\tilde g}_2]x^{\varepsilon}]$. It is directly verified that

$$[\hat g]=\big[y^{\varepsilon}[\tilde g_1]z^{-\varepsilon}\big]\big[\tilde{\tilde g}_2\big].$$

We have to prove that $\overline{N}([\hat g])\leqslant \overline{N}([g])$. To this end, it suffices to show that

$${}^{k}N\big(\big[y^{\varepsilon}[\tilde g_1]z^{-\varepsilon}\big]\big)+\overline{N}\big(\big[\tilde{\tilde g}_2\big]\big)\leqslant {}^{k}N\big([g]\big)$$

for all k such that ${}^{k}N([g])<1$. Note that all these k are positive and meet the condition $\overline{N}([\tilde g])<1/2^k$ (Lemma 6), which implies that $\overline{N}([\tilde g_2])<1/2^k$.

Thus, take k such that ${}^{k}N([g])<1$. Let $m>k$ and ${}^{m}N([\tilde g_2])\leqslant 1$. By 1°(c), $d_k(x,z)\leqslant d_m(x,z)$; therefore,

$$2^{k+1}\cdot d_k(x,z)\leqslant 2^m\cdot d_m(x,z)$$

and

$$2^{k+1}\cdot d_k(x,z)+\overline{N}\big(\big[\tilde{\tilde g}_2\big]\big)\leqslant 2^m\cdot d_m(x,z)+\overline{N}\big(\big[\tilde{\tilde g}_2\big]\big).$$

It follows from ${}^{m}N([\tilde g_2])\leqslant 1$ that $d_m(x,z)\leqslant 1$. By Lemma 7,

$$2^{k+1}\cdot d_k(x,z)+\overline{N}\big(\big[\tilde{\tilde g}_2\big]\big)\leqslant 2^m\cdot {}^{m}B\big(x,z,\big[\tilde{\tilde g}_2\big]\big)+1/2^m+2^m\cdot d_m(x,z) \tag{b}$$

for all $m>k$.

As mentioned, $\overline{N}([\tilde{g}_2]) < 1/2^k$. Because ${}^mN([\tilde{g}_2]) \geqslant 1/2^m$ by the definition of ${}^mN$, this implies that

$$N([\tilde{g}_2]) = \overline{N}([\tilde{g}_2]) = \inf\{{}^mN([\tilde{g}_2]):\ m > k,\ {}^mN([\tilde{g}_2]) \leqslant 1\}.$$

Inequality (b) implies that

$$2^{k+1} \cdot d_k(x,z) + \overline{N}([\tilde{\tilde{g}}_2]) \leqslant \overline{N}([\tilde{g}_2]).$$

Applying Lemma 5 yields

$$2^k \cdot {}^kB(x, y, [\tilde{g}_1]) + 2^{k+1} \cdot d_k(x,z) + \overline{N}([\tilde{\tilde{g}}_2]) \leqslant 2^k \cdot {}^kB(x, y, [\tilde{g}_1][\tilde{g}_2]).$$

By Lemma 3(iii),

$$2^k \cdot {}^kB(y, z, [\tilde{g}_1]) \leqslant 2^k \cdot {}^kB(x, y, [\tilde{g}_1]) + 2^k \cdot d_k(x,z),$$

hence,

$$2^k \cdot {}^kB(y, z, [\tilde{g}_1]) + 2^k \cdot d_k(x,z) + \overline{N}([\tilde{\tilde{g}}_2]) \leqslant 2^k \cdot {}^kB(x, y, [\tilde{g}_1][\tilde{g}_2]).$$

Finally, it follows from $d_k(y,z) \leqslant d_k(x,z) + d_k(y,z)$ that

$$2^k \cdot {}^kB(y, z, [\tilde{g}_1]) + 2^k \cdot d_k(y,z) + \overline{N}([\tilde{\tilde{g}}_2]) \leqslant 2^k \cdot {}^kB(x, y, [\tilde{g}_1][\tilde{g}_2]) + 2^k \cdot d_k(x,y)$$

and

$${}^kN([y^\varepsilon[\tilde{g}_1]z^{-\varepsilon}]) + \overline{N}([\tilde{\tilde{g}}_2]) \leqslant {}^kN([y^\varepsilon[\tilde{g}_1][\tilde{g}_2]x^{-\varepsilon}]) = {}^kN([g]),$$

as required.

(2.2) $[g]$ *is factorable*, i.e., $[g] = [g_1] \ldots [g_n]$, where $n \geqslant 2$ and all $[g_i]$ are nonfactorable. We have

$$N([g]) = \sum_{i \leqslant n} \overline{N}([g_i]) = \sum_{i<n} \overline{N}([g_i]) + \overline{N}([g_n]).$$

The word g_n has the form $\tilde{g}_n x^\varepsilon x^{-\varepsilon}$. Let us endow $\tilde{g}_n$ with the scheme such that $[\tilde{g}_n] = [\hat{g}_n]$ (i.e., $[\tilde{g}_n]$ is obtained from $[g_n]$ by deleting the pair $x^\varepsilon x^{-\varepsilon}$ in the manner described in Section 2). Obviously, $[\hat{g}] = [g_1] \ldots [g_{n-1}][\tilde{g}_n]$. By the induction hypothesis,

$$\overline{N}([\hat{g}_n]) = \overline{N}([\tilde{g}_n]) \leqslant \overline{N}([g_n]);$$

therefore,

$$N([\hat{g}]) = \sum_{i<n} \overline{N}([g_i]) + \overline{N}([\tilde{g}_n]) \leqslant \sum_{i \leqslant n} \overline{N}([g_i]) = N([g]),$$

which proves that $\overline{N}([\hat{g}]) \leqslant \overline{N}([g])$.

(3) $a \equiv e,\ b \not\equiv e$. Argument is similar to that in case (2). □

Statement 2. *Suppose that* $\mathfrak{s} \in \mathfrak{S}$ *and* $[g] \in [S^*(X)]$. *Then* $\overline{N}([g]) = \overline{N}([g^{-1}])$.

Proof. Let us apply induction on $l(g)$. If $g \equiv e$, then the assertion is obvious. Suppose that $l(g) > 0$ and the statement is valid for shorter words. There are two possibilities:

(A) The word $[g]$ is factorable, i.e., $[g]=[g_1]\dots[g_n]$, where $n\geqslant 2$ and all $[g_i]$ are nonfactorable. Obviously, $[g^{-1}]=[g_n^{-1}]\dots[g_1^{-1}]$ and $l(g_i)<l(g)$ for $i\leqslant n$. By the induction hypothesis, $\overline{N}([g_i^{-1}])=\overline{N}([g_i])$ for $i\leqslant n$; therefore,

$$N\big([g]\big)=\sum_{i\leqslant n}\overline{N}\big([g_i]\big)=\sum_{i\leqslant n}\overline{N}\big([g_i^{-1}]\big)=N\big([g^{-1}]\big),$$

whence $\overline{N}([g])=\overline{N}([g^{-1}])$.

(B) The word $[g]$ is nonfactorable, i.e., $[g]=[x^{\varepsilon}[\tilde g]y^{-\varepsilon}]$. Clearly, $[g^{-1}]=[y^{\varepsilon}[\tilde g^{-1}]x^{-\varepsilon}]$. We have

$${}^kN\big([g]\big)=2^k\cdot{}^kB\big(x,y,[\tilde g]\big)+1/2^k+2^k\cdot d_k(x,y)$$

for all k. By the induction hypothesis, $\overline{N}_\alpha([\tilde g])=\overline{N}_\alpha([\tilde g^{-1}])$ for all $\alpha\in A_k$, hence, ${}^kB(x,y,[\tilde g])={}^kB(x,y,[\tilde g^{-1}])$. Thus,

$$\begin{aligned}{}^kN\big([g]\big)&=2^k\cdot{}^kB\big(x,y,[\tilde g^{-1}]\big)+1/2^k+2^k\cdot d_k(x,y)\\&={}^kN\big(\big[y^{\varepsilon}[\tilde g^{-1}]x^{-\varepsilon}\big]\big)={}^kN\big([g^{-1}]\big)\end{aligned}$$

for all k. By definition, $N([g])=N([g^{-1}])$ and $\overline{N}([g])=\overline{N}([g^{-1}])$. □

Statement 3. *Suppose that $h\in S(X)$, $\mathfrak{s}\in\mathfrak{S}$, and $a>0$. Then there exist $r\in\mathbb{N}^+$, $\mathfrak{s}_1,\dots,\mathfrak{s}_r\in\mathfrak{S}$, and $b>0$ such that if $[g]\in[S^*(X)]$ and $\overline{N}_i([g])<b$ for all $i\leqslant r$, then $\overline{N}([h[g]h^{-1}])<a$.*

Proof. Let us apply induction on $l(h)$. For $h\equiv e$, the assertion is trivially true. Suppose that $l(h)>0$ and the statement is valid for shorter words.

Let $h\equiv x^{\varepsilon}\tilde h$. For each $[g]\in[S^*(X)]$, put $[\tilde g]=[\tilde h[g]\tilde h^{-1}]$. Then for any $[g]$, we have

$$\big[h[g]h^{-1}\big]=\big[x^{\varepsilon}[\tilde g]x^{-\varepsilon}\big]$$

and

$$\overline{N}\big[h[g]h^{-1}\big]=\inf_k\big\{{}^k\overline{N}\big(\big[h[g]h^{-1}\big]\big)\big\}.$$

Note that

$${}^k\overline{N}\big[h[g]h^{-1}\big]=2^k\cdot{}^kB\big(x,x,[\tilde g]\big)+1/2^k,$$

because $d_k(x,x)=0$. Take a positive integer k_0 such that $1/2^{k_0-1}<a$. We have

$$\overline{N}\big[h[g]h^{-1}\big]\leqslant 2^{k_0}\cdot{}^{k_0}B\big(x,x,[\tilde g]\big)+1/2^{k_0}$$

for any $[g]$ from $[S^*(X)]$; therefore, to prove the statement, it suffices to find $r\in\mathbb{N}^+$, $\mathfrak{s}_1,\dots,\mathfrak{s}_r\in\mathfrak{S}$, and $b>0$ such that if $[g]\in[S^*(X)]$ and $\overline{N}_i([g])<b$ for all $i\leqslant r$, then ${}^{k_0}B(x,x,[\tilde g])<1/2^{2k_0}$.

For any $[g]\in[S^*(X)]$, we have

$${}^{k_0}B\big(x,x,[\tilde g]\big)=\sum_{\alpha\in A_{k_0}}f_\alpha(x)\cdot\overline{N}_\alpha\big([\tilde g]\big).$$

Consider

$$\{\alpha \in A_{k_0}\colon\ f_\alpha(x) \neq 0\} = \{\alpha_1, \ldots, \alpha_s\}$$

(this set is finite by condition 0°(b) from the definition of $\mathfrak{S}$). Since $l(\tilde{h}) < l(h)$, the induction hypothesis implies that for each $j \leqslant s$, there exist $r_j \in \mathbb{N}^+$, $\mathfrak{s}_{j1}, \ldots, \mathfrak{s}_{jr_j} \in \mathfrak{S}$, and $b_j > 0$ such that if $[g] \in \mathfrak{S}$ and $\overline{N}_{ji}([g]) < b_j$ for all $i \leqslant r_j$, then $\overline{N}_{\alpha_j}([\tilde{g}]) < 1/(s \cdot 2^{2k_0} \cdot f_{\alpha_j}(x))$.

Put

$$\{\mathfrak{s}_1, \ldots, \mathfrak{s}_r\} = \bigcup_{j \leqslant s} \{\mathfrak{s}_{ji}\colon\ i \leqslant r_j\} \quad \text{and} \quad b = \min_{j \leqslant s} b_j.$$

For each $[g] \in [S^*(X)]$ such that $\overline{N}_i([g]) < b$ for $i \leqslant r$, we have

$$\begin{aligned} {}^{k_0}B\big(x, x, [\tilde{g}]\big) &= \sum_{\alpha \in A_{k_0}} f_\alpha(x) \cdot \overline{N}_\alpha\big([\tilde{g}]\big) = \sum_{j \leqslant s} f_{\alpha_j}(x) \cdot \overline{N}_{\alpha_j}\big([\tilde{g}]\big) \\ &< \sum_{j \leqslant s} f_{\alpha_j}(x) \cdot \frac{1}{s \cdot 2^{2k_0} \cdot f_{\alpha_j}(x)} = s \cdot \frac{1}{s \cdot 2^{2k_0}} = \frac{1}{2^{2k_0}}, \end{aligned}$$

as required. □

Before formulating the next statement, let us mention that each word of length 2 from $S^*(X)$ admits the unique scheme $\{\langle 1, 2\rangle\}$.

Statement 4. *The set*

$$U = \big\{y \in X\colon\ \overline{N}\big([x_0^{-1} y]\big) < a\big\}$$

is open in X for any $x_0 \in X$, $\mathfrak{s} \in \mathfrak{S}$, and $a \leqslant 1$.

Proof. Note that if $\overline{N}([x_0^{-1} y]) < a$, then $\overline{N}([x_0^{-1} y]) < 1$ and

$$\begin{aligned} \overline{N}\big([x_0^{-1} y]\big) = N\big([x_0^{-1} y]\big) &= \inf_k \big\{{}^k N\big([x_0^{-1} y]\big)\big\} \\ &= \inf_k \big\{1/2^k + 2^k \cdot d_k(x, y)\big\}. \end{aligned}$$

Take $y_0 \in U$. We must show that U contains an open neighborhood V of y_0 in X. Since $\overline{N}([x_0^{-1} y_0]) < a < 1$ and $\overline{N}([x_0^{-1} y_0]) \geqslant 0$, there exists k_0 such that $1/2^{k_0} + 2^{k_0} \times d_{k_0}(x_0, y_0) < a$, i.e.,

$$d_{k_0}(x_0, y_0) < (a - 2^{-k_0})/2^{k_0}.$$

Find $b > 0$ for which

$$d_{k_0}(x_0, y_0) < (a - 2^{-k_0})/2^{k_0} - b$$

and put

$$V = \big\{y \in X\colon\ d_{k_0}(y_0, y) < b\big\}.$$

By condition 0°(c) from the definition of $\mathfrak{S}$, the pseudometric d_{k_0} is continuous on X; therefore, V is open. Clearly, $y_0 \in V$. For all $y \in V$, we have $d_{k_0}(x_0, y) < (a - 2^{-k_0})/2^{k_0}$, whence

$$1/2^{k_0} + 2^{k_0} \cdot d_{k_0}(x_0, y) = {}^{k_0}N\big([x_0^{-1}y]\big) < a \quad \text{and}$$
$$\overline{N}\big([x_0^{-1}y]\big) \leqslant {}^{k_0}N\big([x_0^{-1}y]\big) < a,$$

as required. □

6. Definition and properties of seminorms $\|\cdot\|_K$

Let K be a nonempty finite subset of the family $\mathfrak{S}$ and $K = \{\mathfrak{s}_1, \ldots, \mathfrak{s}_n\}$. For each $g \in F(X)$, put

$$\|g\|_K = \begin{cases} \min\big\{\sum_{i \leqslant n} \overline{N}_i\big([g, \boldsymbol{\sigma}_g]\big)\colon \boldsymbol{\sigma}_g \text{ is a scheme for } g\big\} & \text{if } g \in S^*(X), \\ n & \text{otherwise.} \end{cases}$$

Let us note some properties of the function $\|\cdot\|_K$.

(1) Obviously, $\|e\|_K = 0$.

(2) If $a, b \in F(X)$ and $g = ab \in F(X)$ (i.e., g is irreducible and obtained from ab by successively deleting all pairs of letters of the form $x^\varepsilon x^{-\varepsilon}$), then $\|g\|_K \leqslant \|a\|_K + \|b\|_K$.

Indeed, if a or b does not belong to $S^*(X)$, then $\|a\|_K + \|b\|_K \geqslant n$. On the other hand, $\|g\|_K \leqslant n$, because $\overline{N}([h])$ is never greater than 1; therefore, $\|g\|_K \leqslant \|a\|_K + \|b\|_K$. Suppose that $a, b \in S^*(X)$. Then, clearly, $g \in S^*(X)$. Let $\boldsymbol{\sigma}_a$ and $\boldsymbol{\sigma}_b$ be the schemes for a and b, respectively, such that

$$\|a\|_K = \sum_{i \leqslant n} \overline{N}_i\big([a, \boldsymbol{\sigma}_a]\big), \qquad \|b\|_K = \sum_{i \leqslant n} \overline{N}_i\big([b, \boldsymbol{\sigma}_b]\big).$$

For each $i \leqslant n$, we have

$$\overline{N}_i\big([ab, \boldsymbol{\sigma}_{ab}]\big) \leqslant \overline{N}_i\big([a, \boldsymbol{\sigma}_a]\big) + \overline{N}_i\big([b, \boldsymbol{\sigma}_b]\big),$$

hence,

$$\sum_{i \leqslant n} \overline{N}_i\big([ab, \boldsymbol{\sigma}_{ab}]\big) \leqslant \sum_{i \leqslant n} \overline{N}_i\big([a, \boldsymbol{\sigma}_a]\big) + \sum_{i \leqslant n} \overline{N}_i\big([b, \boldsymbol{\sigma}_b]\big) = \|a\|_K + \|b\|_K.$$

Since g is obtained from ab by successively deleting pairs of the form $x^\varepsilon x^{-\varepsilon}$, it follows from Statement 1 that there exists a scheme $\boldsymbol{\sigma}_g$ for g such that $\overline{N}_i([g, \boldsymbol{\sigma}_g]) \leqslant \overline{N}_i([ab, \boldsymbol{\sigma}_{ab}])$; this scheme is uniquely determined by the scheme $\boldsymbol{\sigma}_{ab}$ and the order of deleting the pairs $x^\varepsilon x^{-\varepsilon}$. Therefore,

$$\|g\|_K \leqslant \sum_{i \leqslant n} \overline{N}_i\big([g, \boldsymbol{\sigma}_g]\big) \leqslant \sum_{i \leqslant n} \overline{N}_i\big([ab, \boldsymbol{\sigma}_{ab}]\big) \leqslant \|a\|_K + \|b\|_K.$$

(3) If $g \in F(X)$, then $\|g\|_K = \|g^{-1}\|_K$.

This follows from Statement 2 for $g \in S^*(X)$ and is obvious for $g \notin S^*(X)$.

(4) For any $h \in F(X)$ and $a > 0$, there exist finite $L \subset \mathfrak{S}$ and $b > 0$ such that if $g \in F(X)$, $\|g\|_L < b$, and $u = hgh^{-1} \in F(X)$, then $\|u\|_K < a$.

Indeed, by Statement 3, there exist $L = \{\mathfrak{s}'_1, \dots \mathfrak{s}'_r\} \subset \mathfrak{S}$ and $b > 0$ such that if $[g] \in [S^*(X)]$ and $\overline{N}'_i([g]) \leqslant b$ for $i \leqslant r$, then $\overline{N}_i([h[g]h^{-1}]) < a/n$ for $i \leqslant n$. Consider these L and b. Without loss of generality, we will assume that $b < 1 \leqslant n$. Take $g \in F(X)$ with $\|g\|_L < b$. We have $g \in S^*(X)$, because otherwise $\|g\|_L \geqslant 1 > b$. Fix a scheme $\boldsymbol{\sigma}_g$ for g such that

$$\|g\|_L = \sum_{i \leqslant r} \overline{N}'_i\bigl([g, \boldsymbol{\sigma}_g]\bigr);$$

clearly, $\overline{N}'_i([g, \boldsymbol{\sigma}_g]) < b$ for $i \leqslant r$. Statement 1 implies that there exists a scheme $\boldsymbol{\sigma}_u$ for $u = hgh^{-1}$ for which

$$\overline{N}_i\bigl([u, \boldsymbol{\sigma}_u]\bigr) \leqslant \overline{N}_i\bigl(\bigl[h[g, \boldsymbol{\sigma}_g]h^{-1}\bigr]\bigr).$$

Since $\overline{N}'_i([g, \boldsymbol{\sigma}_g]) < b$ for $i \leqslant r$, we have $\overline{N}_i([h[g, \boldsymbol{\sigma}_g]h^{-1}]) < a/n$ and $\overline{N}_i([u, \boldsymbol{\sigma}_u]) < a/n$ for $i \leqslant n$. Therefore,

$$\|u\|_K \leqslant \sum_{i \leqslant n} \overline{N}_i\bigl([u, \boldsymbol{\sigma}_u]\bigr) < n \cdot \frac{a}{n} = a.$$

Recall that a real-valued function $\|\cdot\|$ on an arbitrary group G is called a *seminorm* if it satisfies conditions (1)–(3) with $\|\cdot\|$ instead of $\|\cdot\|_K$ and G instead of $F(X)$. Seminorms were introduced by Markov [3] (he called them norms). Thus,

$$\mathcal{N} = \bigl\{\|\cdot\|_K\colon\ K \text{ is a finite subset of } \mathfrak{S}\bigr\}$$

is a family of seminorms on $F(X)$.

Using (1)–(4), we can easily verify that the family $\mathcal{N}$ generates a group topology on $F(X)$; i.e., the family

$$\mathcal{B} = \bigl\{U_K(a)\colon\ K \text{ is a finite subset of } \mathfrak{S},\ a > 0\bigr\},$$

where

$$U_K(a) = \bigl\{g \in F(X)\colon\ \|g\|_K < a\bigr\},$$

satisfies the axioms of an open neighborhood base at the identity element. Let us show, for example, that for any $K_1, K_2 \in [\mathfrak{S}]^{<\aleph_0}$ and $a_1, a_2 > 0$, there exist $L \in [\mathfrak{S}]^{<\aleph_0}$ and $b > 0$ such that

$$U_L(b) \subset U_{K_1}(a_1) \cap U_{K_2}(a_2).$$

Clearly,

$$\sum_{\mathfrak{s} \in K_1 \cup K_2} \overline{N}_{\mathfrak{s}}\bigl([g]\bigr) \geqslant \sum_{\mathfrak{s} \in K_1} \overline{N}_{\mathfrak{s}}\bigl([g]\bigr) \quad \text{and}$$

$$\sum_{\mathfrak{s} \in K_1 \cup K_2} \overline{N}_{\mathfrak{s}}\bigl([g]\bigr) \geqslant \sum_{\mathfrak{s} \in K_2} \overline{N}_{\mathfrak{s}}\bigl([g]\bigr)$$

for any $[g] \in [S^*(X)]$; therefore, $\|g\|_{K_1 \cup K_2} \geqslant \|g\|_{K_1}$ and $\|g\|_{K_1 \cup K_2} \geqslant \|g\|_{K_2}$ for every $g \in S(X)$. Because the cardinality of $K_1 \cup K_2$ is not less than each of the cardinalities of K_1 and K_2, this inequality is also valid for $g \in F(X) \setminus S(X)$. Therefore, $L = K_1 \cup K_2$ and $b = \min\{a_1, a_2\}$ meet the requirement.

Thus, the family $\mathcal{N}$ generates a group topology on $F(X)$. Each word from $[S^*(X)]$ of length 2 admits only one scheme $\{\langle 1, 2\rangle\}$; therefore, for all finite $K \subset \mathfrak{S}$ and $g \in F_2(X)$, we have

$$\|g\|_K = \sum_{\mathfrak{s} \in K} \overline{N}_{\mathfrak{s}}\bigl(\bigl[g, \{\langle 1, 2\rangle\}\bigr]\bigr),$$

and Statement 4 implies that the topologies generated by the seminorms $\|\cdot\|_K$ on X are coarser than the original topology of X.

7. Principal statements

The last paragraph of the preceding section implies our first principal statement.

Principal Statement 1. *The family of seminorms*

$$\mathfrak{N} = \bigcup \bigl\{\{\|\cdot\|_K\colon K \text{ is a finite subset of } \mathfrak{S}(\boldsymbol{P})\}\colon \boldsymbol{P} \text{ is a partially ordered set and } \mathfrak{S}(\boldsymbol{P}) \text{ is a family satisfying conditions } 0°\text{–}3°\bigr\}$$

generates a group topology $\mathcal{T}$ on $F(X)$ that is coarser than the topology of $F_M(X)$.

Principal Statement 2 implies that $\mathcal{T}$ coincides with the topology of $F_M(X)$.

Principal Statement 2. *Let Y be a nonempty subspace of X such that any continuous bounded pseudometric on Y can be extended to a continuous pseudometric on X, and $\|\cdot\|_Y$ be a continuous seminorm on $F_M(Y)$ with an upper bound of $1/8$. Then there exist a partially ordered set $\boldsymbol{P}$, a family $\mathfrak{S}$ satisfying the conditions $0°$–$3°$, and an $\mathfrak{s} \in \mathfrak{S}$ such that $\|g\|_Y \leqslant \|g\|_{\{\mathfrak{s}\}}$ for all $g \in F(Y) \subset F(X)$.*

Proof. As mentioned, by condition 3°, the sought family $\mathfrak{S}$ (and the underlying ordered set $\boldsymbol{P}$) should have a fairly complex structure: to every $\mathfrak{s} = \langle \mathcal{A}, \mathcal{F}, \mathcal{D}\rangle \in \mathfrak{S}$ we must assign triples

$$\mathfrak{s}_\alpha = \langle \mathcal{A}_\alpha, \mathcal{F}_\alpha, \mathcal{D}_\alpha\rangle \in \mathfrak{S} \quad \text{for all } \alpha \in \bigcup \mathcal{A} = \bigcup_{k \in \mathbb{N}} A_k,$$

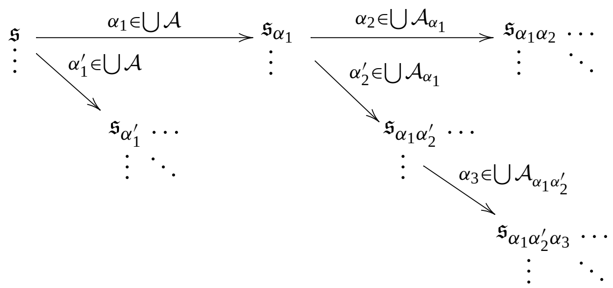
to every $\mathfrak{s}_\alpha$ (as it belongs to $\mathfrak{S}$ and hence satisfies 3°), triples

$$s_{\alpha\beta} = \langle \mathcal{A}_{\alpha\beta}, \mathcal{F}_{\alpha\beta}, \mathcal{D}_{\alpha\beta}\rangle \in \mathfrak{S} \quad \text{for all } \beta \in \bigcup \mathcal{A}_\alpha = \bigcup_{k \in \mathbb{N}} A_{\alpha k},$$

etc. Thus, the sought triple $\mathfrak{s}$ from $\mathfrak{S}$ draws chains of other triples according to the scheme

$$\begin{aligned}\mathfrak{s} = \langle \mathcal{A}, \mathcal{F}, \mathcal{D}\rangle &\xrightarrow{\alpha_1 \in \bigcup \mathcal{A}} \mathfrak{s}_{\alpha_1} = \langle \mathcal{A}_{\alpha_1}, \mathcal{F}_{\alpha_1}, \mathcal{D}_{\alpha_1}\rangle \xrightarrow{\alpha_2 \in \bigcup \mathcal{A}_{\alpha_1}} \mathfrak{s}_{\alpha_1\alpha_2} \\ &= \langle \mathcal{A}_{\alpha_1\alpha_2}, \mathcal{F}_{\alpha_1\alpha_2}, \mathcal{D}_{\alpha_1\alpha_2}\rangle \xrightarrow{\alpha_3 \in \bigcup \mathcal{A}_{\alpha_1\alpha_2}} \cdots \xrightarrow{\alpha_n \in \bigcup \mathcal{A}_{\alpha_1\alpha_2\ldots\alpha_{n-1}}} \mathfrak{s}_{\alpha_1\alpha_2\ldots\alpha_n} \\ &= \langle \mathcal{A}_{\alpha_1\alpha_2\ldots\alpha_n}, \mathcal{F}_{\alpha_1\alpha_2\ldots\alpha_n}, \mathcal{D}_{\alpha_1\alpha_2\ldots\alpha_n}\rangle \xrightarrow{\alpha_{n+1} \in \bigcup \mathcal{A}_{\alpha_1\alpha_2\ldots\alpha_n}} \cdots .\end{aligned}$$

This scheme shows only one chain drawn by $\mathfrak{s}$; in reality, each triple draws a tree of other triples:



It is natural to label the triples (and their elements) by multiindices that indicate their positions in the trees. For example, the multiindex of $\mathfrak{s}$ is empty and has zero length; the triples $\mathfrak{s}_\alpha$ with $\alpha \in \bigcup \mathcal{A}$ that are assigned to $\mathfrak{s}$ $(= \langle \mathcal{A}, \mathcal{F}, \mathcal{D}\rangle)$ have multiindices α of length one; for every $\alpha_1 \in \bigcup \mathcal{A}$, the triples $\mathfrak{s}_{\alpha_1\alpha}$ with $\alpha \in \bigcup \mathcal{A}_{\alpha_1}$ that are assigned to $\mathfrak{s}_{\alpha_1}$ $(= \langle \mathcal{A}_{\alpha_1}, \mathcal{F}_{\alpha_1}, \mathcal{D}_{\alpha_1}\rangle)$ have multiindices $\alpha_1\alpha$ of length two; the triples $\mathfrak{s}_{\alpha_1\alpha_2\alpha}$ with $\alpha \in \bigcup \mathcal{A}_{\alpha_1\alpha_2}$ assigned to $\mathfrak{s}_{\alpha_1\alpha_2}$, where $\alpha_1 \in \bigcup \mathcal{A}$ and $\alpha_2 \in \bigcup \mathcal{A}_{\alpha_1}$, have multiindices $\alpha_1\alpha_2\alpha$ of length three; etc. Thus, the multiindices of the triples drawn by $\mathfrak{s}$ have the form $\alpha_1\alpha_2\ldots\alpha_n$, where $n \in \mathbb{N}$ and

$$\begin{aligned}&\alpha_1 \in \bigcup \mathcal{A} = \bigcup_{k\in\mathbb{N}} A_k, \\ &\alpha_2 \in \bigcup \mathcal{A}_{\alpha_1} = \bigcup_{k\in\mathbb{N}} A_{\alpha_1 k}, \\ &\vdots \\ &\alpha_n \in \bigcup \mathcal{A}_{\alpha_1\alpha_2\ldots\alpha_{n-1}} = \bigcup_{k\in\mathbb{N}} A_{\alpha_1\alpha_2\ldots\alpha_{n-1}k},\end{aligned}$$

and can be treated as points in $\bigcup_{k\in\mathbb{N}} \boldsymbol{P}^k$ (i.e., k-tuples of elements of $\boldsymbol{P}$ with variable length k).

We will construct a family $\mathfrak{S}$ whose all elements (triples) are determined by the sought triple $\mathfrak{s}$ according to condition 3° as described above. The underlying partially ordered set $\boldsymbol{P}$ and the set $\mathcal{C}$ of multiindices (identified with tuples from $\bigcup_{k\in\mathbb{N}} \boldsymbol{P}^k$) will be constructed by induction as the unions of certain sets $\boldsymbol{P}_{k,l}$ and $\mathcal{C}_{k,l}$, respectively, over all $k, l \in \mathbb{N}$ in such a way that $\boldsymbol{P}_{k',l'} \subset \boldsymbol{P}_{k,l}$ and $\mathcal{C}_{k',l'} \subset \mathcal{C}_{k,l}$ for $k' \leqslant k$ and $l' \leqslant l$. Simultaneously with constructing $\boldsymbol{P}_{k,l}$ and $\mathcal{C}_{k,l}$, we will introduce partial orders on these sets such that the order on $\boldsymbol{P}_{k,l}$ $(\mathcal{C}_{k,l})$ is an extension of that on $\boldsymbol{P}_{k',l'}$ $(\mathcal{C}_{k',l'})$ whenever $\boldsymbol{P}_{k',l'} \subset \boldsymbol{P}_{k,l}$ $(\mathcal{C}_{k',l'} \subset \mathcal{C}_{k,l})$.

Bearing this in mind, we will denote the orders on all $\boldsymbol{P}_{k,l}$ by the same symbol $\leqslant$ and the orders on $\mathcal{C}_{k,l}$ by $\preccurlyeq$. The order $\leqslant$ will have the following special features, which are important for our inductive construction:

$$\text{if } \beta \in \boldsymbol{P}_{k,l} \text{ and } \alpha \leqslant \beta, \text{ then } \alpha \in \boldsymbol{P}_{k,l} \tag{$\star$}$$

(this allows us to extend $\leqslant$ from smaller sets to larger ones) and

$$\text{for every } \alpha \in \boldsymbol{P}, \text{ the set of } \beta \in \boldsymbol{P} \text{ such that } \beta \leqslant \alpha \text{ is finite.} \tag{$\star\star$}$$

The order on $\mathcal{C} \subset \bigcup_{n\in\mathbb{N}} \boldsymbol{P}^n$ will be induced by the following natural order $\preccurlyeq$ on $\bigcup_{n\in\mathbb{N}} \boldsymbol{P}^n$. For $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in \boldsymbol{P}$, we define

$$\langle \alpha_1, \ldots, \alpha_m \rangle \preccurlyeq \langle \beta_1, \ldots, \beta_n \rangle$$

if there exists a strictly increasing function $\iota : \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that $\alpha_k \leqslant \beta_{\iota(k)}$ for all $k \in \{1, \ldots, m\}$ (this, in particular, implies that $m \leqslant n$).

We also define

$$\langle \alpha_1, \ldots, \alpha_m, k \rangle \preccurlyeq \langle \beta_1, \ldots, \beta_n, l \rangle$$

if $k \leqslant l$ and $\langle \alpha_1, \ldots, \alpha_m \rangle \preccurlyeq \langle \beta_1, \ldots, \beta_n \rangle$.

We write

$$\langle \alpha_1, \ldots, \alpha_m, k \rangle \prec \langle \beta_1, \ldots, \beta_n, l \rangle$$

$$\text{if } \langle \alpha_1, \ldots, \alpha_m, k \rangle \preccurlyeq \langle \beta_1, \ldots, \beta_n, l \rangle \quad \text{and} \quad \langle \alpha_1, \ldots, \alpha_m, k \rangle \neq \langle \beta_1, \ldots, \beta_n, l \rangle;$$

the relation $\langle \alpha_1, \ldots, \alpha_m \rangle \prec \langle \beta_1, \ldots, \beta_n \rangle$ is defined similarly.

Note that if $\boldsymbol{P}$ satisfies condition ($\star\star$), then the set of $\preccurlyeq$-predecessors of any $\langle \alpha_1, \ldots, \alpha_m, k \rangle \in \bigcup_{n\in\mathbb{N}} \boldsymbol{P}^n \times \mathbb{N}$ is finite.

Simultaneously with constructing $\boldsymbol{P}_{k,l}$ and $\mathcal{C}_{k,l}$, we will construct families $\mathcal{A}$, $\mathcal{F}$, and $\mathcal{D}$ labeled by multiindices from $\mathcal{C}_{k,l}$ and some auxiliary families. Elements of $\mathcal{A}$ will be related to $\boldsymbol{P}_{k,l}$ and $\mathcal{C}_{k,l}$ by

$$\mathcal{C}_{k,l} = \bigcup \Bigl\{ \{\langle \alpha_1, \ldots, \alpha_n \rangle \colon \alpha_1 \in A_{k_1}, \alpha_2 \in A_{\alpha_1 k_2}, \ldots, \alpha_n \in A_{\alpha_1 \ldots \alpha_{n-1} k_n}\} \colon n \leqslant l,\ k_1, \ldots, k_n \leqslant k \Bigr\},$$

or equivalently,

$$\mathcal{C}_{k,l} = \bigcup \Bigl\{ \{\langle \alpha_1, \ldots, \alpha_n \rangle \colon \langle \alpha_1, \ldots, \alpha_{n-1} \rangle \in \mathcal{C}_{k,l-1},\ \alpha_n \in A_{\alpha_1 \ldots \alpha_{n-1} m}\} \colon n \leqslant l,\ m \leqslant k \Bigr\},$$

and

$$\boldsymbol{P}_{k,l} = \bigcup \Bigl\{ A_{\alpha_1 \ldots \alpha_n m} \colon \langle \alpha_1, \ldots, \alpha_n \rangle \in \mathcal{C}_{k,l},\ m \leqslant k \Bigr\}.$$

Since $\mathcal{C}_{k,l} \subset \bigcup_{n=0}^{l} (\boldsymbol{P}_{k,l-1})^n$, the order $\leqslant$ on $P_{k,l-1}$ determines the order $\preccurlyeq$ on $\mathcal{C}_{k,l}$.

The construction involves induction on k and l: first, we define $\boldsymbol{P}_{0,0}$, $\mathcal{C}_{0,1}$, $\mathcal{C}_{n,0}$, and $\boldsymbol{P}_{n,-1}$ for $n \in \mathbb{N}$ and then construct $\boldsymbol{P}_{k,l}$ and $\mathcal{C}_{k,l+1}$ for $\langle k, l \rangle \neq \langle 0, 0 \rangle$ assuming that $\mathcal{C}_{k,l}$ and $\boldsymbol{P}_{k',l'}$ for $k' \leqslant k,\ l' < l$ are defined. Obviously, such induction is valid.

Let us proceed to the construction.

Put $\boldsymbol{P}_{0,0} = \{0\}$, $\mathcal{C}_{0,1} = \{\langle 0\rangle\}$, $\mathcal{C}_{n,0} = \{\emptyset\}$, and $\boldsymbol{P}_{n,-1} = \emptyset$ for all $n \in \mathbb{N}$.

Define a (continuous) pseudometric ρ^Y on Y by

$$\rho^Y(y_1, y_2) = \max\bigl\{4 \cdot \|y_1^{\varepsilon} y_2^{-\varepsilon}\|_Y\colon \varepsilon = \pm 1\bigr\} \quad \text{for } y_1, y_2 \in Y.$$

Since $\|\cdot\|_Y$ is bounded by $1/8$, the pseudometric ρ^Y is bounded by $1/2$. Take a continuous pseudometric ρ on X that extends ρ^Y and is bounded by $1/2$.

Choose an arbitrary point $x_0 \in Y$. Put $U_0 = X$, $A_0 = \{0\}$, $d_0 \equiv 0$ on X^2, $\gamma_0 = \{U_0\}$, $M_0 = \{x_0\}$, $f_0 \equiv 1$ on X, and $F_0 = \{f_0\}$. Note that since ρ is bounded by $1/2$, the cover γ_0 is a refinement of the cover $\{B_\rho(x, 1)\colon x \in X\}$.

Suppose that $k, l \in \mathbb{N}$, $\langle k, l\rangle \neq \langle 0, 0\rangle$, $\mathcal{C}_{k,l}$ with the order $\preccurlyeq$ is defined, and $\boldsymbol{P}_{k',l'}$ with the order $\leqslant$ are defined for all pairs $\langle k', l'\rangle \in \mathbb{N} \times (\mathbb{N} \cup \{-1\})$ such that $k' \leqslant k$ and $l' < l$ (in particular, $\boldsymbol{P}_{k,l-1}$ is defined). Suppose also that every $\alpha \in \boldsymbol{P}_{k,l-1}$ has a finite number of $\leqslant$-predecessors; then every element in $\mathcal{C}_{k,l} \times \mathbb{N}$ has a finite number of $\preccurlyeq$-predecessors. Take $\langle \alpha_1, \ldots, \alpha_n\rangle \in \mathcal{C}_{k,l}$ and $m \leqslant k$. If $\langle \alpha_1, \ldots, \alpha_n, m\rangle$ has no predecessor with respect to $\preccurlyeq$, then n and m are necessarily zero, i.e., $\langle \alpha_1, \ldots, \alpha_n, m\rangle = \langle 0\rangle = \langle \emptyset, 0\rangle$; we have already defined the objects ρ, A_0, d_0, γ_0, M_0 and F_0 that correspond to this $(n+1)$-tuple. Let $\langle \alpha_1, \ldots, \alpha_n, m\rangle$ have precisely r predecessors, where $r > 0$. Suppose that for all $\langle \beta_1, \ldots, \beta_s, t\rangle \in \mathcal{C}_{k,l} \times \{0, \ldots, k\}$ with less than r predecessors, we have already defined the objects $\rho_{\beta_1 \ldots \beta_s}$, $A_{\beta_1 \ldots \beta_s t}$ (along with the extension of $\leqslant$ to this set), $d_{\beta_1 \ldots \beta_s t}$, $\gamma_{\beta_1 \ldots \beta_s t}$, $M_{\beta_1 \ldots \beta_s t}$, and $F_{\beta_1 \ldots \beta_s t}$ satisfying the following conditions:

$0^{\circ\circ}$ (1) $\rho_{\beta_1 \ldots \beta_s}$ is a continuous pseudometric on X bounded by $1/2$;

(2) $A_{\beta_1 \ldots \beta_s t}$ is a nonempty set, and every its element has a finite number of $\leqslant$-predecessors;

(3) $d_{\beta_1 \ldots \beta_s t}$ is a continuous pseudometric on X;

(4) $\gamma_{\beta_1 \ldots \beta_s t} = \{U_\beta\colon \beta \in A_{\beta_1 \ldots \beta_s t}\}$ is a cover of X that is open and locally finite with respect to the topology generated by $d_{\beta_1 \ldots \beta_s t}$ and indexed by the elements of $A_{\beta_1 \ldots \beta_s t}$ (this means, in particular, that if $\alpha \neq \beta$, then U_α and U_β are different elements of γ even if they coincide as sets);

(5) $M_{\beta_1 \ldots \beta_s t} = \{x_\beta\colon \beta \in A_{\beta_1 \ldots \beta_s t}\}$ is a subset of X such that $x_\beta \in U_\beta$ for any β and $x_\beta \in Y$ whenever $U_\beta \cap Y \neq \emptyset$;

(6) $F_{\beta_1 \ldots \beta_s t} = \{f_\beta\colon \beta \in A_{\beta_1 \ldots \beta_s t}\}$ is a family of continuous nonnegative-valued functions on X such that $\operatorname{supp} f_\beta = U_\beta$ for each β.

$1^{\circ\circ}$ If $\langle \theta_1, \ldots, \theta_p, q\rangle$ is an immediate predecessor of $\langle \beta_1, \ldots, \beta_s, t\rangle$ in $\mathcal{C}_{k,l} \times \mathbb{N}$ with respect to the order $\preccurlyeq$, then

(1)

$$A_{\theta_1 \ldots \theta_p q} \lhd A_{\beta_1 \ldots \beta_s t};$$

(2) for any x from X and θ from $A_{\theta_1 \ldots \theta_p q}$,

$$f_\theta(x) = \sum_{\beta \in A_{\beta_1 \ldots \beta_s t}(\theta)} f_\beta(x)$$

(we remind the reader that $A(\theta)$ stands for $\{\alpha \in A\colon \theta \leqslant \alpha\}$);

(3) for any x and y from X,

$$2 \cdot d_{\theta_1\ldots\theta_p q}(x, y) \leqslant d_{\beta_1\ldots\beta_s t}(x, y);$$

(4) for any $\theta \in A_{\theta_1\ldots\theta_p q}$,

$$\bigcup \bigl\{U_\beta\colon\ \beta \in A_{\beta_1\ldots\beta_s t}(\theta)\bigr\} = U_\theta.$$

$2^{\circ\circ}$ (1) If $\{x_{\beta_1}, \ldots, x_{\beta_s}\} \subset Y$, then the restriction of $\rho_{\beta_1\ldots\beta_s}$ to Y^2 is

$$\begin{aligned}&\rho^Y_{\beta_1\ldots\beta_s}(y_1, y_2)\\ &\quad = \max\bigl\{4 \cdot \|x^{\varepsilon_1}_{\beta_1} \ldots x^{\varepsilon_s}_{\beta_s} y^{\varepsilon}_1 y^{-\varepsilon}_2 x^{-\varepsilon_s}_{\beta_s} \ldots x^{-\varepsilon_1}_{\beta_1}\|_Y\colon\ \varepsilon, \varepsilon_i = \pm 1\bigr\};\end{aligned}$$

otherwise, $\rho_{\beta_1\ldots\beta_s} \equiv 0$ on X^2;

(2) for any $x \in X$,

$$\sum_{\beta \in A_{\beta_1\ldots\beta_s t}} f_\beta(x) \geqslant 1;$$

(3) for any $x, y \in X$,

$$2 \cdot \sum_{\beta \in A_{\beta_1\ldots\beta_s t}} \bigl|f_\beta(x) - f_\beta(y)\bigr| \leqslant d_{\beta_1\ldots\beta_s t}(x, y);$$

(4) $\gamma_{\beta_1\ldots\beta_s t}$ refines the cover

$$\bigl\{B_{\rho_{\beta_1\ldots\beta_s}}(x, 1/2^t)\colon\ x \in X\bigr\}.$$

$3^{\circ\circ}$ If $\langle\theta_1, \ldots, \theta_p, q\rangle \in \mathcal{C}_{k,l}$, $q \leqslant k$, $\langle\theta_1, \ldots, \theta_p, q\rangle$ has less than r predecessors in $\mathcal{C}_{k,l} \times \mathbb{N}$ with respect to $\preccurlyeq$, and $\langle\theta_1, \ldots, \theta_p, q\rangle \neq \langle\beta_1, \ldots, \beta_s, t\rangle$, then $A_{\theta_1\ldots\theta_p q} \cap A_{\beta_1\ldots\beta_s t} = \emptyset$; if in addition, there exist $\theta \in A_{\theta_1\ldots\theta_p q}$ and $\beta \in A_{\beta_1\ldots\beta_s t}$ such that $\theta \leqslant \beta$, then $\langle\theta_1, \ldots, \theta_p, q\rangle \prec \langle\beta_1, \ldots, \beta_s, t\rangle$.

Let us define similar objects for $\langle\beta_1, \ldots, \beta_s, t\rangle = \langle\alpha_1, \ldots, \alpha_n, m\rangle$ in such a way that conditions $0^{\circ\circ}$–$2^{\circ\circ}$ be fulfilled.

We start with introducing one more notation: put

$$\begin{aligned}&\operatorname{Pred}\langle\alpha_1, \ldots, \alpha_n, m\rangle\\ &\quad = \bigl\{\langle\beta_1, \ldots, \beta_s, t\rangle \in \mathcal{C}_{k,l} \times \mathbb{N}\colon\\ &\qquad \langle\beta_1, \ldots, \beta_s, t\rangle \text{ is an immediate predecessor of } \langle\alpha_1, \ldots, \alpha_n, m\rangle\\ &\qquad \text{in } \mathcal{C}_{k,l} \times \mathbb{N} \text{ with respect to } \preccurlyeq\bigr\}.\end{aligned}$$

Choose a continuous pseudometric $\rho_{\alpha_1\ldots\alpha_n}$ on X satisfying condition $2^{\circ\circ}(1)$ and bounded by $1/2$. Refine the cover

$$\mu = \bigl\{B_{\rho_{\alpha_1\ldots\alpha_n}}(x, 1/2^m)\colon\ x \in X\bigr\}$$

of X to a cover ν open and locally finite with respect to the topology generated by $\rho_{\alpha_1\ldots\alpha_n}$. Let us index ν using an arbitrary set A: $\nu = \{V_a\colon\ a \in A\}$.

Each $\langle\beta_1, \ldots, \beta_s, t\rangle \in \operatorname{Pred}\langle\alpha_1, \ldots, \alpha_n, m\rangle$ has no more than $r - 1$ predecessors and belongs to $\mathcal{C}_{k,l} \times \{0, \ldots, k\}$; for all these sets the required objects are already defined. Take $\langle\beta_1, \ldots, \beta_s, t\rangle \in \operatorname{Pred}\langle\alpha_1, \ldots, \alpha_n, m\rangle$ and $\beta \in A_{\beta_1\ldots\beta_s t}$ and put

$$A_\beta = \{a \in A\colon\ V_a \cap U_\beta \neq \emptyset\} \quad \text{and}$$

$$A_{\alpha_1\ldots\alpha_n m}[\beta] = \{(a,\beta), \langle \alpha_1,\ldots,\alpha_n,m\rangle\colon\ a \in A_\beta\} \subset A \times \{\beta\} \times \{\langle \alpha_1,\ldots,\alpha_n,m\rangle\}.$$

For any $\alpha = (a,\beta), \langle \alpha_1,\ldots,\alpha_n,m\rangle \in A_{\alpha_1\ldots\alpha_n m}[\beta]$, put $U_\alpha = V_a \cap U_\beta$. The family

$$\gamma_{\alpha_1\ldots\alpha_n m}[\beta] = \{U_\alpha\colon\ \alpha \in A_{\alpha_1\ldots\alpha_n m}[\beta]\}$$

forms a cover of the subspace U_β of X, consists of sets open with respect to the topology $\mathcal{T}'$ generated on X by the pseudometric $\max(d_{\beta_1\ldots\beta_s t}, \rho_{\alpha_1\ldots\alpha_n})$, and is locally finite with respect to the same topology (this follows from the definition of ν and condition $0^{\circ\circ}(4)$).

Take a partition of unity on U_β subordinated to $\gamma_{\alpha_1\ldots\alpha_n m}[\beta]$, i.e., a family

$$\{g_\alpha\colon\ \alpha \in A_{\alpha_1\ldots\alpha_n m}[\beta]\}$$

of nonnegative-valued functions on U_β continuous with respect to $\mathcal{T}' \restriction U_\beta$ and such that $\operatorname{supp} g_\alpha = U_\alpha$ for $\alpha \in A_{\alpha_1\ldots\alpha_n m}[\beta]$ and

$$\sum_{\alpha \in A_{\alpha_1\ldots\alpha_n m}[\beta]} g_\alpha(x) = 1$$

for each $x \in U_\beta$ (the sum is defined, because $\gamma_{\alpha_1\ldots\alpha_n m}[\beta]$ is locally finite). Such a family can be constructed, for example, by setting $g_\alpha(x) = \bar{g}_\alpha(x)/\sum \bar{g}_\alpha(x)$, where $\bar{g}_\alpha(x)$ is the distance between x and $X \setminus U_\alpha$ with respect to the pseudometric $\max(d_{\beta_1\ldots\beta_s t}, \rho_{\alpha_1\ldots\alpha_n})$. For each $\alpha \in A_{\alpha_1\ldots\alpha_n m}[\beta]$ and $x \in X$, put

$$f_\alpha(x) = \begin{cases} 0 & \text{if } x \notin U_\beta, \\ g_\alpha(x) \cdot f_\beta(x) & \text{if } x \in U_\beta. \end{cases}$$

We have

$$\sum_{\alpha \in A_{\alpha_1\ldots\alpha_n m}[\beta]} f_\alpha(x) = \sum_{\alpha \in A_{\alpha_1\ldots\alpha_n m}[\beta]} g_\alpha(x) \cdot f_\beta(x) = f_\beta(x)$$

for all x from X.

Put

$$\begin{aligned} A_{\alpha_1\ldots\alpha_n m} &= \bigcup \{\{A_{\alpha_1\ldots\alpha_n m}[\beta]\colon\ \beta \in A_{\beta_1\ldots\beta_s t}\}\colon \\ &\qquad \langle \beta_1,\ldots,\beta_s,t\rangle \in \operatorname{Pred}\langle \alpha_1,\ldots,\alpha_n,m\rangle\}, \\ \gamma_{\alpha_1\ldots\alpha_n m} &= \bigcup \{\{\gamma_{\alpha_1\ldots\alpha_n m}[\beta]\colon\ \beta \in A_{\beta_1\ldots\beta_s t}\}\colon \\ &\qquad \langle \beta_1,\ldots,\beta_s,t\rangle \in \operatorname{Pred}\langle \alpha_1,\ldots,\alpha_n,m\rangle\} \\ &= \{U_\alpha\colon\ \alpha \in A_{\alpha_1\ldots\alpha_n m}\}, \\ F_{\alpha_1\ldots\alpha_n m} &= \{f_\alpha\colon\ \alpha \in A_{\alpha_1\ldots\alpha_n m}\}. \end{aligned}$$

For each $\alpha \in A_{\alpha_1\ldots\alpha_n m}$, fix $x_\alpha \in U_\alpha$ such that $x_\alpha \in Y$ whenever U_α intersects Y and put

$$M_{\alpha_1\ldots\alpha_n m} = \{x_\alpha\colon\ \alpha \in A_{\alpha_1\ldots\alpha_n m}\}.$$

Finally, put

$$d_{\alpha_1\ldots\alpha_n m}(x,y) = \max\Big\{\rho_{\alpha_1\ldots\alpha_n}(x,y),\ 2\cdot\sum_{\alpha\in A_{\alpha_1\ldots\alpha_n m}}\big|f_\alpha(x)-f_\alpha(y)\big|,$$
$$\max\big\{2\cdot d_{\beta_1\ldots\beta_s t}\colon\ \langle\beta_1,\ldots,\beta_s,t\rangle\in \operatorname{Pred}\langle\alpha_1,\ldots,\alpha_n,m\rangle\big\}\Big\}$$

for all $x,y\in X$.

The desired objects are constructed. It remains to extend the relation $\leqslant$ over $A_{\alpha_1\ldots\alpha_n m}$. Let $\langle\beta_1,\ldots,\beta_s,t\rangle\in\mathcal{C}_{k,l},\ t\leqslant k,\ \langle\beta_1,\ldots,\beta_s,t\rangle$ have no more than r predecessors, the set $A_{\beta_1\ldots\beta_s t}$ be already defined, $\alpha\in A_{\alpha_1\ldots\alpha_n m}$, and $\beta\in A_{\beta_1\ldots\beta_s t}$. We set

(i) $\beta\leqslant\alpha$ if and only if either $\beta=\alpha$ or there exist $\langle\theta_1,\ldots,\theta_p,q\rangle\in\operatorname{Pred}\langle\alpha_1,\ldots,\alpha_n,m\rangle$ and $\theta\in A_{\theta_1\ldots\theta_p q}$ such that $\beta\leqslant\theta$ and $\alpha\in A_{\alpha_1\ldots\alpha_n m}[\theta]$;

(ii) $\alpha\leqslant\beta$ if and only if $\alpha=\beta$.

Note that by construction, the sets $A_{\alpha_1\ldots\alpha_n m}[\theta']$ and $A_{\alpha_1\ldots\alpha_n m}[\theta'']$ are disjoint if $\theta'\neq\theta''$. Therefore, for every $\alpha\in A_{\alpha_1\ldots\alpha_n m}$, there exists exactly one θ such that $\alpha\in A_{\alpha_1\ldots\alpha_n m}[\theta]$; this θ belongs to some $A_{\theta_1\ldots\theta_p q}$, where $\langle\theta_1,\ldots,\theta_p,q\rangle\in\operatorname{Pred}\langle\alpha_1,\ldots,\alpha_n,m\rangle$. Because $\langle\theta_1,\ldots,\theta_p,q\rangle$ has less than r $\preccurlyeq$-predecessors, by the induction hypothesis (condition $0^{\circ\circ}$(2)), the number of $\leqslant$-predecessors of θ is finite; therefore, the number of $\leqslant$-predecessors of α is also finite.

The construction immediately implies the fulfillment of conditions $0^{\circ\circ}$–$2^{\circ\circ}$ with $\langle\beta_1,\ldots,\beta_s,t\rangle=\langle\alpha_1,\ldots,\alpha_n,m\rangle$. It directly follows from the definition of $\leqslant$ on the sets $A_{\alpha_1\ldots\alpha_n m}$ that after we construct $A_{\alpha_1\ldots\alpha_n m}$ for all $\langle\alpha_1,\ldots,\alpha_n,m\rangle\in\mathcal{C}_{k,l}\times\mathbb{N}$ with no more than r predecessors, condition $3^{\circ\circ}$ with $\langle\alpha_1,\ldots,\alpha_n,m\rangle$ instead of $\langle\beta_1,\ldots,\beta_s,t\rangle$ and $r+1$ instead of r will also be fulfilled.

After $A_{\alpha_1\ldots\alpha_n m}$ are constructed for all $\langle\alpha_1,\ldots,\alpha_n\rangle\in\mathcal{C}_{k,l}$ and $m\leqslant k$, put

$$\boldsymbol{P}_{k,l}=\bigcup\big\{A_{\alpha_1\ldots\alpha_n m}\colon\ \langle\alpha_1,\ldots,\alpha_n\rangle\in\mathcal{C}_{k,l},\ m\leqslant k\big\}$$

and

$$\mathcal{C}_{k,l+1}=\bigcup\big\{\{\langle\alpha_1,\ldots,\alpha_{l+1}\rangle\colon\ \langle\alpha_1,\ldots,\alpha_l\rangle\in\mathcal{C}_{k,l},\ \alpha_{l+1}\in A_{\alpha_1\ldots\alpha_l m}\}\colon\ m\leqslant k\big\}\cup\mathcal{C}_{k,l}.$$

The construction is completed.

Put $\boldsymbol{P}=\bigcup_{k,l}\boldsymbol{P}_{k,l}$ and $\mathcal{C}=\bigcup_{k,l}\mathcal{C}_{k,l}$. The partially ordered sets $\boldsymbol{P}_{k,l}$ satisfy condition ($\star$) by construction; their orders $\leqslant$ extend each other, and $\boldsymbol{P}$ is also a partially ordered set. Put $\mathfrak{S}=\{\mathfrak{s}_{\alpha_1\ldots\alpha_n}\colon\ \langle\alpha_1,\ldots,\alpha_n\rangle\in\mathcal{C}\}$. Conditions $0^{\circ\circ}$–$3^{\circ\circ}$ and the transitivity of the relations $\preccurlyeq$ and $\triangleleft$ ensure the fulfillment of conditions 0°–2° from Section 2. Note that $\mathfrak{s}_{\beta_1\ldots\beta_s}<\mathfrak{s}_{\alpha_1\ldots\alpha_n}$ if and only if $\langle\beta_1,\ldots,\beta_s\rangle\prec\langle\alpha_1,\ldots,\alpha_n\rangle$. Thus, 3° also holds. Applying the following lemma completes the proof of Principal Statement 2.

Lemma. *If $n\in\mathbb{N}$, $\langle\alpha_1,\ldots,\alpha_n\rangle\in\mathcal{C}$ is such that $x_{\alpha_1},\ldots,x_{\alpha_n}$ belong to Y, $\varepsilon_1,\ldots,\varepsilon_n=\pm1$, and $g\in F^*(Y)$, then*

$$\big\|x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n}g x_{\alpha_n}^{-\varepsilon_n}\ldots x_{\alpha_1}^{-\varepsilon_1}\big\|_Y\leqslant\|g\|_{\{\mathfrak{s}_{\alpha_1,\ldots,\alpha_n}\}}.$$

Proof. Let us apply induction on $l(g)$. For $g = e$, the assertion of the lemma is obvious. Suppose that $l(g) > 0$ and the lemma is valid for shorter words. Let $[g]$ be g endowed with a scheme such that $\|g\|_{\{\mathfrak{s}_{\alpha_1\ldots\alpha_n}\}} = \overline{N}_{\alpha_1\ldots\alpha_n}([g])$. There are two possibilities:

(A) The word $[g]$ is factorable, i.e., $[g] = [g_1]\ldots[g_k]$, where $k \geqslant 2$ and all $[g_i]$ are nonfactorable. Since g is irreducible and $g \equiv g_1 \ldots g_k$, all g_i are also irreducible and, therefore, $g_i \in F^*(Y)$. In addition, $l(g_i) < l(g)$. The induction hypothesis can be applied. We have

$$\begin{aligned}
&\big\|x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n} g x_{\alpha_n}^{-\varepsilon_n}\ldots x_{\alpha_1}^{-\varepsilon_1}\big\|_Y\\
&\quad= \big\|x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n} g_1 x_{\alpha_n}^{-\varepsilon_n}\ldots x_{\alpha_1}^{-\varepsilon_1}\ldots x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n} g_k x_{\alpha_n}^{-\varepsilon_n}\ldots x_{\alpha_1}^{-\varepsilon_1}\big\|_Y\\
&\quad\leqslant \sum_{i\leqslant k}\big\|x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n} g_i x_{\alpha_n}^{-\varepsilon_n}\ldots x_{\alpha_1}^{-\varepsilon_1}\big\|_Y \leqslant \sum_{i\leqslant k}\|g_i\|_{\{\mathfrak{s}_{\alpha_1,\ldots,\alpha_n}\}} \leqslant \sum_{i\leqslant k}\overline{N}_{\alpha_1\ldots\alpha_n}\big([g_i]\big)\\
&\quad= \overline{N}_{\alpha_1\ldots\alpha_n}\big([g]\big) = \|g\|_{\{\mathfrak{s}_{\alpha_1,\ldots,\alpha_n}\}}.
\end{aligned}$$

(B) The word $[g]$ is nonfactorable, i.e., $[g] = [x^\varepsilon[\tilde{g}]y^{-\varepsilon}]$ (and $l(\tilde{g}) < l(g)$). We have

$$N_{\alpha_1\ldots\alpha_n}\big([g]\big) = \inf_k\big\{{}^kN_{\alpha_1\ldots\alpha_n}\big([g]\big)\big\}.$$

Let us show that for each $k \in \mathbb{N}$,

$${}^kN_{\alpha_1\ldots\alpha_n}\big([g]\big) \geqslant \big\|x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n} g x_{\alpha_n}^{-\varepsilon_n}\ldots x_{\alpha_1}^{-\varepsilon_1}\big\|_Y.$$

Clearly, this inequality holds when ${}^kN_{\alpha_1\ldots\alpha_n}([g]) \geqslant 1$. Now suppose that

$${}^kN_{\alpha_1\ldots\alpha_n}\big([g]\big) < 1$$

(this, in particular, implies that $k > 0$). We have

$$\begin{aligned}
&{}^kN_{\alpha_1\ldots\alpha_n}\big([g]\big)\\
&\quad= 2^k \cdot \sum_{\alpha\in A_{\alpha_1\ldots\alpha_n k}} \min\big\{f_\alpha(x), f_\alpha(y)\big\}\cdot \overline{N}_{\alpha_1\ldots\alpha_n\alpha}\big([\tilde{g}]\big) + \frac{1}{2^k} + 2^k\cdot d_{\alpha_1\ldots\alpha_n k}(x,y)\\
&\quad< 1;
\end{aligned}$$

therefore,

$$d_{\alpha_1\ldots\alpha_n k}(x,y) \leqslant 1/2^k \leqslant 1 \quad \text{and}$$

$$\begin{aligned}
\sum_{\alpha\in A_{\alpha_1\ldots\alpha_n k}} \min\big\{f_\alpha(x), f_\alpha(y)\big\} &\geqslant \sum_\alpha f_\alpha(x) - \sum_\alpha \big|f_\alpha(x) - f_\alpha(y)\big|\\
&\geqslant \sum_\alpha f_\alpha(x) - \frac{1}{2} \geqslant \frac{1}{2}
\end{aligned}$$

(by conditions 2°(a) and (b) from the definition of $\mathfrak{S}$). Let us denote the element of the finite set

$$\big\{\alpha \in A_{\alpha_1\ldots\alpha_n k}\colon \min\big\{f_\alpha(x), f_\alpha(y)\big\} \neq 0\big\}$$

that minimizes $\overline{N}_{\alpha_1\ldots\alpha_n\alpha}([\tilde{g}])$ as $\alpha_{\min}$. For $k > 0$, we have

$$
\begin{aligned}
&{}^k N_{\alpha_1\ldots\alpha_n}\big([g]\big)\\
&\quad\geqslant 2\cdot\sum_{\alpha\in A_{\alpha_1\ldots\alpha_n k}}\min\big\{f_\alpha(x),\,f_\alpha(y)\big\}\cdot\overline{N}_{\alpha_1\ldots\alpha_n\alpha_{\min}}\big([\tilde g]\big)+\frac{1}{2^k}\\
&\quad\geqslant \overline{N}_{\alpha_1\ldots\alpha_n\alpha_{\min}}\big([\tilde g]\big)+\frac{1}{2^k}.
\end{aligned}
$$

Since $g\in S(Y)$ and $g\equiv x^{\varepsilon}\tilde g y^{-\varepsilon}$, we have $x,y\in Y$ and $\tilde g\in F^*(Y)$. The relation $\min\{f_{\alpha_{\min}}(x),\,f_{\alpha_{\min}}(y)\}\neq 0$ and $0^{\circ\circ}(6)$ imply that $\operatorname{supp} f_{\alpha_{\min}}=U_{\alpha_{\min}}$ and $x,y\in U_{\alpha_{\min}}\in\gamma_{\alpha_1\ldots\alpha_n k}$. Therefore, $U_{\alpha_{\min}}$ intersects Y. It follows from condition $0^{\circ\circ}(5)$ that $x_{\alpha_{\min}}\in U_{\alpha_{\min}}\cap Y$. By the induction hypothesis,

$$
\overline{N}_{\alpha_1\ldots\alpha_n\alpha_{\min}}\big([\tilde g]\big)\geqslant\big\|x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n}x_{\alpha_{\min}}^{\varepsilon}\tilde g x_{\alpha_{\min}}^{-\varepsilon}x_{\alpha_n}^{-\varepsilon_n}\ldots x_{\alpha_1}^{-\varepsilon_1}\big\|_Y,
$$

and by conditions $2^{\circ\circ}(1)$ and (4), since $x, y, x_{\min}\in U_{\alpha_{\min}}\in\gamma_{\alpha_1\ldots\alpha_n k}$,

$$
\begin{aligned}
&2\cdot\big\|x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n}x^{\varepsilon}x_{\alpha_{\min}}^{-\varepsilon}x_{\alpha_n}^{-\varepsilon_n}\ldots x_{\alpha_1}^{-\varepsilon_1}\big\|_Y\leqslant 1/2^k,\\
&2\cdot\big\|x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n}x_{\alpha_{\min}}^{\varepsilon}y^{-\varepsilon}x_{\alpha_n}^{-\varepsilon_n}\ldots x_{\alpha_1}^{-\varepsilon_1}\big\|_Y\leqslant 1/2^k.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&{}^k N_{\alpha_1\ldots\alpha_n}\big([g]\big)\\
&\quad\geqslant\big\|x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n}x_{\alpha_{\min}}^{\varepsilon}\tilde g x_{\alpha_{\min}}^{-\varepsilon}x_{\alpha_n}^{-\varepsilon_n}\ldots x_{\alpha_1}^{-\varepsilon_1}\big\|_Y\\
&\qquad+\big\|x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n}x^{\varepsilon}x_{\alpha_{\min}}^{-\varepsilon}x_{\alpha_n}^{-\varepsilon_n}\ldots x_{\alpha_1}^{-\varepsilon_1}\big\|_Y\\
&\qquad+\big\|x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n}x_{\alpha_{\min}}^{\varepsilon}y^{-\varepsilon}x_{\alpha_n}^{-\varepsilon_n}\ldots x_{\alpha_1}^{-\varepsilon_1}\big\|_Y\\
&\quad\geqslant\big\|x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n}x^{\varepsilon}\tilde g y^{-\varepsilon}x_{\alpha_n}^{-\varepsilon_n}\ldots x_{\alpha_1}^{-\varepsilon_1}\big\|_Y,
\end{aligned}
$$

as required. □

Principal Statement 2 immediately follows from the lemma with $n=0$ (the words $x_{\alpha_1}^{\varepsilon_1}\ldots x_{\alpha_n}^{\varepsilon_n}$ and $x_{\alpha_n}^{-\varepsilon_n}\ldots x_{\alpha_1}^{-\varepsilon_1}$ are then empty, and $\mathfrak{s}_{\alpha_1\ldots\alpha_n}$ coincides with $\mathfrak{s}=\mathfrak{s}_\emptyset$) and the definition of $\|\cdot\|_{\{\mathfrak{s}\}}$: for $g\in F(Y)\setminus F^*(Y)$, $\|g\|_{\{\mathfrak{s}\}}$ is equal to the cardinality of $\{\mathfrak{s}\}$, i.e., 1, while $\|g\|_Y$ has an upper bound of $1/8$. □

Remark. If $Y=X$ and $\dim X=0$, then all pseudometrics from $\mathcal{D}$ and functions from $\mathcal{F}$ in the proof of Principal Statement 2 can be chosen rational-valued. Using Lemma 10, it is easy to verify by induction on word lengths that the function $\overline{N}$ is then also rational-valued. Therefore, the seminorm $\|\cdot\|_{\{\mathfrak{s}\}}$ is rational-valued, too. Thus, if $\dim X=0$, then the topology of $F_M(X)$ is generated by a family of rational-valued seminorms.

8. Main theorems

Theorem 1. *Let X be a completely regular T_1 space and Y be its subspace. Then the topological subgroup of $F_M(X)$ generated by Y is the free topological group $F_M(Y)$ if and only if each bounded continuous pseudometric on Y can be extended to a continuous pseudometric on X.*

Proof. Sufficiency was proved by Pestov [5]. To prove necessity, we need the following Markov theorem [3]:

Theorem. *Let G be a topological group and U be an open neighborhood of the identity element in G. Then there exists a continuous seminorm $\|\cdot\|$ on G such that the set $\{x \in X\colon \|x\| < 1\}$ is contained in U.*

Clearly, we can replace 1 by $1/8$ and assume that $\|\cdot\|$ has an upper bound of $1/8$ in Markov's theorem. Applying Principal Statements 1 and 2 completes the proof. □

Corollary 1 (see also papers [7] by this author). *If a completely regular T_1 space X is Dieudonné complete, then the group $F_M(X)$ is Weil complete.*

Proof. Since X is Dieudonné complete, it can be embedded into a product P of metric spaces as a closed subspace in such a way that every bounded continuous pseudometric on X can be extended over P; therefore, Theorem 1 can be applied. It says that $F_M(X)$ is a topological subgroup of $F_M(P)$; obviously, $F_M(X)$ is closed in $F_M(P)$. Uspenskiĭ [9] proved that the free topological group of a product of metric spaces is Weil complete. Therefore, $F_M(P)$ and its closed subgroup $F_M(X)$ are Weil complete. □

Pestov proved that the Dieudonné completeness of X is also necessary for the completeness of $F_M(X)$ (see the proof of Theorem 1 in [5]). This result and Corollary 1 imply the equivalence of the Dieudonné completeness of a completely regular T_1 space X and the Weil completeness of its free topological group.

Corollary 2. *Any T_0 topological group G is a quotient group of a Weil complete T_0 topological group.*

Proof. Any completely regular T_1 space is an image of a paracompact space under a quotient map. Let X be a paracompact space and f be a quotient map of X onto G. Consider an extension of f to a continuous homomorphism $\hat{f} : F_M(X) \to G$. This homomorphism is open, because f is quotient. Therefore, G is a quotient group of $F_M(X)$. The space X is Dieudonné complete as a paracompact space. According to Corollary 1, the group $F_M(X)$ is Weil complete. □

Theorem 2 (see also [7]). *If* $\dim X = 0$, *then* $\operatorname{ind} F_M(X) = 0$.

This immediately follows from the remark to Principal Statement 2.

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