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Theoretical Computer Science 303 (2003) 187–213

Theoretical Computer Science

www.elsevier.com/locate/tcs

A termination proof for epsilon substitution using partial derivations

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Abstract

Epsilon substitution method introduced by Hilbert is a successive approximation process providing numerical realizations from proofs of existential formulas. Most convergence (termination) proofs for it use assignments of decreasing ordinals to stages of the process and work only for predicative systems. We describe a new ordinal assignment for the case of first-order arithmetic admitting extension to impredicative systems. It is based on an interpretation of individual epsilon substitutions forming the substitution process as incomplete finite proofs, each encoding a complete but infinite proof.

MSC: 03F05; 03F07; 03F15; 03F30

Keywords: Epsilon substitution method; Infinite proofs; Incomplete proofs; Extraction of programs

1. Introduction

The theory of deductive program synthesis and verification relies on complete proofs of specifications. Such proofs are assumed to be found by an automated deduction program or constructed manually using a proof-checking system. This contradicts practice: even in mathematics most proofs are very far from being complete, and verification of programs usually checks only "principal" parts. However this practice can be supported by some existing and new theory.

When specifications do not require inductive proofs, the main program synthesis tool is Herbrand's theorem. For existential formulas $\exists x R(x)$ with quantifier-free $R(x)$ there is a transformation of any first-order proof $\pi : \exists x R(x)$ into a set of witnesses $t_1, \ldots, t_n$ such that $R(t_1) \vee \cdots \vee R(t_n)$. The whole proof π is needed in the standard formulation, while in fact only quantifier inferences are used, and the whole propositional part is

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PII: S0304-3975(02)00451-6

redundant. Predicate inferences contain mathematically and algorithmically interesting part of the proof; propositional part is usually the most labor-consuming and often non-interesting part. An exact formulation of the observation above uses ε-calculus (see below) that works for classical logic.

The main pragmatic reason for having constructive or intuitionistic proofs is a possibility to extract programs from proofs $\pi : \exists x A(x)$ without any restriction for $A(x)$. In this new logic one cannot completely ignore the propositional part of π: implications contribute significantly into the complexity of the eventual program. Most program extraction methods here are based on functional interpretations that are based on Brouwer–Heyting–Kolmogorov interpretation of constructive logical connectives. These interpretations differ in the amount of information they need. For example modified realizability *mr*, a functional interpretation introduced by G. Kreisel, ignores negative premises of implications:

$$x\ mr(\neg A \rightarrow B) \equiv \neg A \rightarrow x\ mr\ B.$$

Another manifestation of the same phenomenon is Harrop's theorem.

Theorem 1.1. *For arbitrary* A, B, *if* $\neg A \rightarrow \exists x B(x)$ *is derivable* (*in intuitionistic first- or higher-order logic*, *intuitionistic first- or higher-order arithmetic*, *etc.*), *then* $\neg A \rightarrow B(t)$ *for some* t *is derivable in the same theory.*

In fact $\neg A$ can be replaced by any $\vee, \exists$-free formula C. Proofs of such lemmas C, even of number-theoretic identities, to say nothing about Riemann hypothesis or Fermat's last theorem, can be very complicated, but they can be skipped if we are interested only in the program.

1.1. Finite and infinite proofs of existential sentences

This paper extends to ε-calculus the approach of [8], where a set of reductions (cut-elimination transformations) for ordinary finite derivations in first-order arithmetic with induction schema was derived from similar reductions for infinitary derivations. Using the apparatus and results from [4,5] one can describe a motivation for these reductions as follows. Let

$$h \rightarrow h^{\infty}$$

be a standard translation of finite arithmetic derivations into infinitary derivations that essentially replaces induction axiom $A0 \wedge \forall x(Ax \rightarrow A(sx)) \Rightarrow An$ by its derivation consisting of series of cuts over $A0, A1, \ldots, A(n-1)$. Let $\mathscr{E}$ be a canonical operator reducing cut-degree (the maximal complexity of cuts) by one. Then the standard definition of a finite derivation can be modified so that it is possible to define a "pre-image" of $\mathscr{E}$ on finite derivations: there is a primitive recursive operation E such that

$$\mathscr{E}(h^{\infty}) \equiv (\mathsf{E}h)^{\infty}. \tag{1.1}$$

More specifically, there are primitive recursive operations $tp(h), o(h), h[i]$ providing for every finite derivation h the last rule and the ordinal measure $tp(h), o(h)$ of h^{∞} as well as the notation $h[i]$ for the infinitary derivation of the ith premise of the last rule. More precisely, $o(h)$ is an ordinal less than ε_0 providing an approximation of the ordinal height $\|h^\infty\|$, and $h[i]$ is a derivation in H such that $(h[i])^\infty = h^\infty(i)$:

$$\frac{\ldots(h[i])^\infty:\ \Gamma(h_i)\ldots}{h^\infty : \Gamma(h)}\ tp(h).$$

Let $h \in H$ be a derivation of an existential sentence $\exists y P y$. Let $h' := \mathsf{E}\ldots\mathsf{E}h$, where E is applied $cutdegree(h)$ times. Then $(h')^\infty$ is cutfree, and consists essentially of a finite number of $\exists$-rules

$$\begin{array}{l} Axiom:\ Pn_1,\ldots,Pn_k,\exists yPy \quad o_k \\ \qquad\qquad\vdots \\ \dfrac{Pn_1,\exists yPy \quad o_1}{\exists yPy \quad o(h')}, \end{array}$$

where Pn_k is a true sentence. In other words, n_k is the required value for y and a sequence $Pn_1,\ldots,Pn_k$ provides a computation of this value, while ordinals $o(h') > o_1 > \ldots$, etc. assigned to subderivations provide estimates of convergence.

1.2. ε-Substitution and incomplete proofs

In this paper we apply this schema to epsilon substitution method introduced by Hilbert (cf. [6]). It is a successive approximation process providing numerical realizations from proofs of existential formulas. The language uses epsilon terms $\varepsilon x F[x]$, read as the least x satisfying $F[x]$. The main axioms of the corresponding formalism are *critical formulas*

$$F[t] \to F[\varepsilon x F[x]]. \tag{1.2}$$

The H-process for a given finite system Cr of critical formulas generates finite substitutions of numerals for closed canonical epsilon-terms (having no closed ε-subterms). All canonical ε-terms not mentioned in the substitution have default value 0. The initial substitution S_0 is identically 0. If substitutions

$$S_0,\ldots,S_i \tag{1.3}$$

are already generated, and S_i is not yet a solving substitution (satisfying all critical formulas in Cr), take the first formula in Cr which is false under S_i, i.e. for which $S_i(F[t]) = \mathbf{true}$, $S_i(F[\varepsilon x F]) = \mathbf{false}$. Set

$$S_{i+1}(\varepsilon x F) = (the\ least\ n \leqslant t)\ (S_i(F[n]) = \mathbf{true})$$

drop the values of higher rank (Section 2.3) and preserve remaining values.

Ackermann [2] proved that the sequence (1.3) terminates for every system Cr of critical formulas in first-order arithmetic after a finite number of steps in a *solution* S_k satisfying all critical formulas in Cr. He assigned ordinals less than ε_0 to sections of the H-process consisting of consecutive substitutions of restricted rank, and proved that ordinals of the sections strictly decrease.

In the present paper we assign ordinals $o(S_0) > o(S_1) > \cdots$ to substitutions in the H-process (1.3) by embedding the H-process into normalization (cut elimination) process for certain infinite derivation, called original derivation in [11], Section 6.3. In more detail, a formal system $PA\varepsilon*$ where e-substitutions are derived is introduced in Section 5 below. Derivations in $PA\varepsilon*$ are finite, but treated as notations for infinite derivations in the system εPA (cf. [11] and Section 4 below) via translation h^∞ (Section 6 below). Some of the rules of $PA\varepsilon*$ are transformed into the same rules of the infinitary system εPA, but rules $\mathsf{R}_e, \mathsf{E}_r, \mathsf{D}_r, \mathsf{W}_\Sigma$ are "invisible": they are modeled in εPA by transformations $\mathscr{R}_e, \mathscr{E}, \mathscr{D}, \mathscr{W}$ of infinite derivations defined in Section 4.

An ordinal $o(h)$ is assigned to every $PA\varepsilon*$-derivation h by a simple primitive recursive definition (Definition 5.5). This recursion models a standard definition of the ordinal height $\|h^\infty\|$ of the infinite translation h^∞. In fact operations $^\infty$ and o use additional argument π (see below).

To assign an ordinal to an ε-substitution S_i generated by an H-process, the sequence (1.3) is enriched by steps of adding default zero values needed for all computations, so that (1.3) becomes a finite sequence π_i of inferences by the rules Fr, H of Section 3. The pair (S_i, π_i) is interpreted as a $PA\varepsilon*$-derivation h_i, but this interpretation uses in an intermediate step of (primitive) recursion more complicated $PA\varepsilon*$-derivations (Section 8). The main goal of introduction of the system $PA\varepsilon*$ was to explicate these intermediate steps. They appear here naturally as notations for stages of cut elimination applied to the infinite original derivation. This connection easily proves

$$o(S_i, \pi_i) > o(S_{i+1}, \pi_{i+1})$$

(cf. (7.5)) and hence termination of the ε-substitution process.

Derivable objects (sequents) of $PA\varepsilon*$ contain components $(e, ?), (e, ?^0)$ indicating that e has the default value 0, as well as components $(e, +)$. New (compared to previous literature on epsilon substitution method, for example [10,11]) component $(e, +)$ indicates that the value of the ε-term e is defined, but unknown yet. Presence of such components makes our derivations only partial. Operations on partial derivations defined here are successful because the values of undefined ε-terms turn out to be computed by the time these values are actually needed for further computations.

Most definitions related to ε-substitution are taken from [11].

The definitions of a computable expression, correct substitution, H-step, axioms of PA-systems are slightly changed in the same direction as Definitions 5.2, 5.3 in [3]. The change makes the definitions closer to [6]. It remains to be seen whether present approach can help in correcting a (defective) termination proof from [1].

2. ε-Substitution process

2.1. The language of ε-substitutions

Definition 2.1. *Variables* $x, y, z, \ldots$ are for natural numbers. *Numerical terms* are variables, $0, St$ and εxF for all formulas F. There are many (as much as needed) primitive recursive *predicates* including $=$. *Formulas* are constructed from atomic formulas by propositional connectives: $\wedge FG, \rightarrow FG, \neg F, \ldots$ written as $(F \wedge G), (F \rightarrow G), \ldots$.

Quantifiers can be defined from ε in a standard way.
Critical formulas:

$$(pred)\ s \neq 0 \rightarrow s = S\varepsilon x(s = Sx),$$
$$(\varepsilon) \qquad F[t] \rightarrow F[\varepsilon vF].$$

Definition 2.2. An ε-term is *canonical* if it is closed and contains no proper closed ε-subterms. An expression e is *simple* if it is closed and contains no ε. TRUE (FALSE) denotes the set of all true (false) simple formulas. [A simple formula contains no variables and is constructed from computable atomic formulas by Boolean connectives. Every simple term is a numeral].

$\mathbb{N}$ is the set of natural numbers.

Definition 2.3. A *sequent* is a finite function from canonical ε-terms into the set $\{?, ?^0, +\} \cup \mathbb{N}$.

A sequent will be written as a finite list consisting of components of the form

$$(e, ?), (e, ?^0), (e, +), (e, n).$$

An ε-*substitution* is a sequent without components of the form $(e, ?^0)$.

Definition 2.4. Two sequents Σ, Θ are *multiplicable* if $\Theta \cup \Sigma$ is a function after $?^0$ is identified with ? and $(e, +)$ with (e, n), if both are present. In this case we write $\Theta * \Sigma$ for $\Theta \cup \Sigma$, and say that $\Theta * \Sigma$ is defined.

The set $FV(e)$ of free variables of an expression e is defined in the standard way: εx binds x. An expression e is *closed* iff $FV(e) = \emptyset$.

We identify expressions which are equivalent modulo renaming of bound variables; $e[x/u]$ denotes the result of substituting u for each free occurrence of x in e, where bound variables in e are renamed if necessary. If x is known from the context we write $e[u]$ for $e[x/u]$.

We assume as always a fixed system

$$Cr = \{Cr_0, \ldots, Cr_N\}$$

of closed critical formulas.

2.2. Computations with the ε-substitutions

Definition 2.5. An ε-substitution S is *total* if $dom(S)$ is the set of all canonical ε-terms.

$\bar{S} := S \cup \{(e, ?) : e$ is a canonical ε-term $\notin dom(S)\}$ is called the *standard extension of* S.

(1) If $(e, u) \in S$ and $u \neq ?$, then $e \hookrightarrow^1_S u$.
(2) If $(e, ?) \in S$, then $e \hookrightarrow^1_S 0$.

(3) If $0 \leqslant i \leqslant n$, $e_i \hookrightarrow^1_S e'_i$ then $e_0 e_1 \dots e_n \hookrightarrow^1_S e_0 \dots e_{i-1} e'_i e_{i+1} \dots e_n$.
(4) If $F \hookrightarrow^1_S F'$ then $\varepsilon x F \hookrightarrow^1_S \varepsilon x F'$.

Definition 2.6. e is *S-reducible* if there exists an e' with $e \hookrightarrow^1_S e'$. Otherwise e is *S-irreducible* or in *S-normal form*. $\hookrightarrow_S$ denotes the transitive and reflexive closure of $\hookrightarrow^1_S$.

The unique S-irreducible expression e^* with $e \hookrightarrow_S e^*$ is called the *S-normal-form of* e and denoted by $|e|_S$.

Definition 2.7. Let S be an ε-substitution.

An expression e is *S-computable* if $|e|_S$ does not contain closed ε-terms.

S *computes* a set Φ of closed formulas iff all formulas in Φ are S-computable.

For a pair $(\varepsilon x F[x], n) \in S$ define

$$Cr(e, S) := F[\![n]\!] := F[x/n] \wedge \neg F[x/0] \wedge \cdots \wedge \neg F[x/(n-1)].$$

If t is not a numeral, then $F[\![t]\!] := F[t]$. Let

$$\mathscr{F}(S) := \{Cr(e, S)\colon\ (e, n) \in S \text{ for } n \in \mathbb{N}\}.$$

S is *computationally inconsistent* (ci) if $A \hookrightarrow_S \mathsf{FALSE}$ for some $A \in \mathscr{F}(S)$. Otherwise S is *computationally consistent* (cc). [For example S does not compute some $A \in \mathscr{F}(S)$].

S is *correct* if $\wedge \mathscr{F}(S) \hookrightarrow_S \top$.

Let

$$\mathscr{CR}(S) := \{F[\![|t|_S]\!]\colon \text{critical formula } F[t] \to F[\varepsilon x F[x]] \text{ is in } Cr\}$$

S is *solving* iff S is *cc* and $Cr \hookrightarrow_S \top$.

S is *+-free* if it has no components $(e, +)$.

S is *computing* iff all formulas $A \in \mathscr{F}(S)$ are S-computable.

S is *deciding* iff S is computing and the critical formulas $Cr_0, \dots, Cr_N$ are S-computable.

Note: +-Components of a substitution S are never used in a computation $\hookrightarrow_S$. A sequent $(e, +), \Theta$ is correct (cc,ci) iff Θ is correct (cc,ci).

If S is total then every expression is S-computable.

Definition 2.8. The H-rule applies to an ε-substitution S if S is *cc*, non-solving and computes $Cr \cup \mathscr{CR}(S)$.

Definition 2.9. For any finite set Φ of expressions let N_Φ be the number of distinct closed ε-terms occurring in expressions in Φ.

2.3. The rank function

The rank is a measure of nesting of bound variables. For closed expressions it will be the same as in [6,11]. Note that an arbitrary closed ε-term $\varepsilon x F$ can be written as

$$\varepsilon x F \equiv \varepsilon x F'[x_1/t_1, \dots, x_n/t_n], \quad n \geqslant 0, \tag{2.1}$$

where $\varepsilon x F'$ is canonical, and $t_1, \dots, t_n$ are closed ε-terms.

Definition 2.10. If e does not contain ε, then $rk(e):=0$.
If εxF is canonical, then

$$rk(\varepsilon xF) := \max\{rk(f)\text{: } f \text{ is a closed } \varepsilon\text{-subterm of } F[x/0]\} + 1.$$

In particular, if F does not contain ε, then $rk(\varepsilon xF)=1$.
If (2.1) holds with a canonical $\varepsilon xF'$, then

$$rk(\varepsilon xF) := \max\{rk(\varepsilon xF'[x_1/0,\dots,x_n/0], rk(t_1),\dots,rk(t_n)\}.$$

For an arbitrary closed expression e,

$$rk(e) := \max\{rk(t)\text{: } t \text{ is a closed } \varepsilon\text{-subterm of } e\}.$$

Definition 2.11 (Truncation to a given rank). For each ε-substitution S and $r<\omega$ we set

$$S_{\leqslant r} := \{(e,u) \in S\text{: } rk(e) \leqslant r\}.$$

Analogously we define $S_{\geqslant r}$, $S_{<r}$, $S_{>r}$.

Lemma 2.1. *If S,S' are ε-substitutions with $S_{\leqslant r}=S'_{\leqslant r}$ then $|e|_S=|e|_{S'}$ holds for all closed expressions e of rank $\leqslant r$.*

2.4. H-process

Let us recall some definitions from [11].

Definition 2.12. Let S be an ε-substitution such that $\bar{S}$ is non-solving. (Then $|Cr_I|_{\bar{S}} \in$ FALSE for some $I \leqslant N$.)
Set $r_I := rk(\varepsilon x|F|_{\bar{S}})$, where $Cr_I = F_0 \to F[\varepsilon xF]$.
$Cr(S) := Cr_I$, where $I \leqslant N$ is such that
$|Cr_I|_{\bar{S}} \in \mathsf{FALSE}\,\&\,\forall J \leqslant N[|Cr_J|_{\bar{S}} \in \mathsf{FALSE} \Rightarrow r_I < r_J \vee (r_I = r_J \wedge I \leqslant J)]$.
Let $Cr(S) = F_0 \to F[\varepsilon xF]$:
$\varepsilon x|F|_{\bar{S}}$ is called the *H-term of S*.
The H-value v of S is defined as follows:
(a) if $F_0 = (s \neq 0)$, and $F = (s = \mathrm{S}x)$ then $v := |s|_{\bar{S}} - 1$,
(c) if $F_0 = F[t]$ then $v :=$ the unique $n \in \mathbb{N}$ with $|F|_{\bar{S}}[\![n]\!] \hookrightarrow_{\bar{S}}$ TRUE.

Definition 2.13 (The step of the ε-substitution process). If $\bar{S}$ is non-solving then

$$H(S) := (S \setminus \{(e,?)\})_{\leqslant rk(e)} \cup \{(e,v)\},$$

where e is the H-term and v the H-value of S.

The following properties of $H(S)$ are well known (cf. [6,11]).

Lemma 2.2 (Properties of $\mathsf{H}(S)$). *Let S be an ε-substitution such that $\bar{S}$ is correct and non-solving, and let e be the H-term, v the H-value of S. Then the following holds*:
(a) $(e,?)\in\bar{S}$,
(b) $|e|_{H(S)}=v\neq 0$,
(c) $\overline{H(S)}$ *is correct.*

Definition 2.14. The *H-process* for the system Cr of critical formulas $Cr_0,\dots,Cr_N$ *with an initial substitution* S_0 is defined as follows:

$$S_{n+1} := \begin{cases} H(S_n) & \text{if } \overline{S_n} \text{ is non-solving,} \\ \emptyset & \text{otherwise.} \end{cases}$$

The H-process *terminates* iff there exists an $n\in\mathbb{N}$ such that $\overline{S_n}$ is solving.

If the initial substitution is not mentioned (as will be mostly the case), it is assumed that

$$S_0 \equiv \emptyset.$$

3. The system εPA

The system εPA is the arithmetical part of the infinitary system εEA from [11] with the changes in the definitions of computations, H-rule, etc. made in Section 2.4.

Sequents are $+$-free.

Axioms:

$\mathsf{AxF}(\Theta)$ $\quad\Theta$ is ci,

$\mathsf{AxS}(\Theta)$ $\quad\Theta$ is solving,

$\mathsf{AxH}_{e,v}(\Theta)$ e is the H-term, v is the H-value of Θ.

Rules of inference:

$$\frac{(e,?^0),\Theta\dots\quad (e,n),\Theta\dots\quad (n\in\mathbb{N})}{\Theta}\ \mathsf{Cut}_e,$$

$$\frac{(e,?),\Theta\dots\quad (e,n),\Theta\dots\quad (n\in\mathbb{N})}{\Theta}\ \mathsf{CutFr}_e,$$

$$\frac{(e,?),\Theta}{\Theta}\ \mathsf{Fr}_e \qquad \frac{(e,v),\Theta_{\leqslant rk(e)}}{(e,?),\Theta}\ \mathsf{H}_{e,v}$$

if the H-rule applies to $(e,?),\Theta$, and e is the H-term, v the H-value of $(e,?),\Theta$.

Note: The definition of AxS here is changed compared to [11]: it is not required that Θ be deciding, i.e. compute all formulas in $Cr\cup\mathscr{CR},\mathscr{F}(S)$. As a consequence, such axioms can become c.i., that is AxF as a result of further “computations”, for example if formulas Σ are added by weakening $\mathscr{W}_\Sigma$ (Section 4.2.2). However, branches containing such axioms are cut off anyway during cut elimination.

In the above rules e denotes a canonical ε-term not in $dom(\Theta)$.

Definition 3.1. We call e the *main term* of the respective inference.

Definition 3.2. A *derivation* $d \in \varepsilon PA$ is defined in a standard way (cf. [5]) using inference symbols $\mathsf{AxX}, \mathsf{X} \in \{\mathrm{F}, \mathrm{S}, \mathrm{H}\}$, $\mathsf{Cut}, \mathsf{CutFr}, \mathsf{Fr}, \mathsf{H}$.

For each inference symbol $\mathscr{T}$ in this list $|\mathscr{T}|$ denotes the set of indices for the premises of $\mathscr{T}$:

$$|\mathsf{Cut}_e| := \{?^0, 0, 1, \dots,\}, \ |\mathsf{CutFr}_e| := \{?, 0, 1, \dots,\}, \ |\mathsf{Fr}_e| \equiv |\mathsf{H}_{e,v}| := \{0\}.$$

The result of a $\mathscr{T}$-inference with premises derived by d_i is written $\mathscr{T}\{d_i\}_{i \in \|\mathscr{T}\|}$.

The last sequent of d is denoted $\Gamma(d)$.

For example the following derivation:

$$\dfrac{\dfrac{\dfrac{\overset{\mathsf{AxH}}{(g,?),(f,?),(e,?)}}{(f,?),(e,?)}\,\mathsf{Fr}_g}{(e,?)}\,\mathsf{Fr}_f}{\emptyset}\,\mathsf{Fr}_e$$

is $\mathsf{Fr}_e\mathsf{Fr}_f\mathsf{Fr}_g\mathsf{AxH}((g,?),(f,?),(e,?))$ in our notation. The proof-figure

$$\dfrac{\dfrac{\dfrac{\overset{\mathsf{AxH}_{e,v}}{(e,?^0),(f,?),(g,?)}}{(e,?^0),(f,?)}\,\mathsf{Fr}_g \qquad \overset{\mathsf{AxS}}{\dots(e,?^0),(f,m)\dots}}{(e,?^0)}\,\mathsf{CutFr}_f \qquad \overset{\mathsf{AxS}}{\dots(e,n)\dots}}{\emptyset}\,\mathsf{Cut}_e$$

becomes

$$\mathsf{Cut}_e\mathsf{CutFr}_f\mathsf{Fr}_g\mathsf{AxH}_{e,v}(\Theta)\{\mathsf{AxS}((e,?^0),(f,m))\}_{m \in \mathbb{N}}\{\mathsf{AxS}((e,n))\}_{n \in \mathbb{N}},$$

where $\Theta \equiv (e,?^0),(f,?),(g,?)$.

Definition 3.3. The ordinal height $\|h\|$ is determined in a standard way beginning with 1 for the axioms:

$$\|\mathsf{AxX}(\Theta)\| = 1; \quad \|\mathscr{T}\{d_i\}_{i \in |\mathscr{T}|}\| := \sup(\|d_i\| + 1)_{i \in |\mathscr{T}|}.$$

Definition 3.4. Let d be a deduction (from some assumptions) in εPA.

d is an *r-deduction* iff $\mathsf{Cut}(d) < r \,\&\, \mathsf{CutFr}(d) < 0 \,\&\, \mathsf{Fr}(d) \geqslant r \,\&\, H(d) > r$.

d is an *r^+-deduction* iff $\mathsf{Cut}(d) < r \,\&\, \mathsf{CutFr}(d) = r \,\&\, \mathsf{Fr}(d) > r \,\&\, H(d) \geqslant r$.

Let us recall from [11] the definition of a path (from an empty sequent $\emptyset$ to a sequent Θ in a derivation of Θ).

We indicate a path here not by sequents constituting it as in [11], Definition 30, but by a sequence of branches of inferences beginning with lowermost one and leading from $\emptyset$ to Θ. For example, if the sequence is

$$\dfrac{\dfrac{\dfrac{(g,?),(f,3),(e,?^0)}{(f,3),(e,?^0)}\,\mathsf{Fr}_g}{(e,?^0)}\,\mathsf{CutFr}_f}{\emptyset}\,\mathsf{Cut}_e$$

then $\pi \equiv \mathsf{Cut}_e ?^0 \mathsf{CutFr}_f 3 \mathsf{Fr}_g$.

The *length* $lth(\pi)$ is the number of components in π.

For example, $lth(\mathsf{Cut}_e ?^0 \mathsf{CutFr}_f 3 \mathsf{Fr}_g) = 3$.

A ρ-deduction for $\rho \in \{r, r^+\}$ is defined below exactly as in [11].

Definition 3.5. If $\mathscr{T}$ is an inference then $rk(\mathscr{T})$ denotes the rank of its main term.

If d is a deduction, X is one of the symbols Cut, CutFr, Fr, H, and $\bowtie$ is one of the symbols $<, \leqslant, >, \geqslant, =$ then

$\mathsf{X}(d) \bowtie r :\Leftrightarrow rk(\mathscr{T}) \bowtie r$ for every X-inference $\mathscr{T}$ in d.

Hence "$\mathsf{Cut}(d) < r$" means that all cuts in d have rank $<r$, and "$\mathsf{X}(d) < 0$" means that there are no X-inferences in d.

The same notation is used for path in a deduction, so that "$\mathsf{Cut}(\pi) < r$" means that all cuts in π have rank $<r$, and "$\mathsf{X}(\pi) < 1$" means that there are no X-inferences in π.

d is an *r-deduction* iff $\mathsf{Cut}(d) < r \;\&\; \mathsf{CutFr}(d) < 0 \;\&\; \mathsf{Fr}(d) \geqslant r \;\&\; \mathsf{H}(d) \geqslant r$.

d is an *r^+-deduction* iff $\mathsf{Cut}(d) < r \;\&\; \mathsf{CutFr}(d) = r \;\&\; \mathsf{Fr}(d) > r \;\&\; \mathsf{H}(d) \geqslant r$.

An *r-path* π is defined like *r-deduction*:

$\mathsf{Cut}(\pi) < r \;\&\; \mathsf{CutFr}(\pi) < 0 \;\&\; \mathsf{Fr}(\pi) \geqslant r \;\&\; \mathsf{H}(\pi) \geqslant r$.

A proper *r^+-path* π is defined like an r^+-deduction:

$\mathsf{Cut}(\pi) < r \;\&\; \mathsf{CutFr}(\pi) = r \;\&\; \mathsf{Fr}(\pi) > r \;\&\; \mathsf{H}(\pi) \geqslant r$.

Lemma 3.1. *For a given e there can be at most one occurrence of $\mathscr{T}_e$ for $\mathscr{T} \in \{\mathsf{Cut}, \mathsf{CutFr}\}$ in a given path.*

Proof. Induction using the condition $e \notin dom(\Theta)$. □

4. Operations on infinite derivations

4.1. Original derivation

The next definition describes construction of the original derivation, $\mathsf{orig}(\Theta)$ for a $+$-free sequent Θ, cf. Section 6.3 of [11].

Definition 4.1. (1) Θ is AxX, $\mathsf{X} \in \{\mathsf{F}, \mathsf{S}, \mathsf{H}\}$. $\mathsf{orig}(\Theta) := \mathsf{AxX}(\Theta)$.

(2) Otherwise. Put

$$\mathsf{orig}(\Theta) := \mathsf{Cut}_e\{\mathsf{orig}((e,u),\Theta)\}_{u \in \{?^0\} \cup \mathbb{N}}, \tag{4.1}$$

where term e is chosen below and AxX is AxA unless $(e,u),\Theta$ is an axiom of other kind.

(a) Θ computes Cr. Then e is the first (in some fixed ordering) canonical ε-term in $|\mathscr{CR}(\Theta)|_\Theta$.

(b) Θ does not compute Cr. Then e is the first (in some fixed ordering) canonical ε-term in $|Cr|_\Theta$.

Lemma 4.1. *Let Θ be a +-free sequent, not* AxX *for $X \neq A$.*

(1) $\mathsf{orig}(\Theta)$ *is a derivation in εPA consisting of axioms and* Cut-*inferences of rank* $\leqslant r_0$ (*Definition* 5.3.2)

(2) *If Θ computes Cr, then* $\|\mathsf{orig}(\Theta)\| \leqslant N_{|\mathscr{CR}(\Theta)|_\Theta} + 1$

(3) *In general,* $\|\mathsf{orig}(\Theta)\| \leqslant \omega + N_{|Cr|_\Theta}$.

Proof. Induction on $|Cr|_\Theta$ with induction on $|\mathscr{CR}(\Theta)|_\Theta$ in the basis. □

4.2. Cut-reduction operator $\mathscr{R}_e$

We formalize here the most essential part of the Cut-elimination proof from Section 6 of [11]: cut-reduction of one Cut:

$$\frac{(e,?^0),\Theta \quad \ldots(e,n),\Theta\ldots}{\Theta}\ \mathsf{Cut}_e. \tag{4.2}$$

Corresponding operator $\mathscr{R}_e$ eventually produces a derivation of Θ from derivations $d^*:(e,?^0),\Theta$ and $\tilde{d}:\{d_n:(e,n),\Theta\}_{n\in\mathbb{N}}$ plus the place of the Cut_e in a bigger derivation of the sequent $\emptyset$. This place is given by an r-path π for Θ where $r = rk(e)$.

$\mathscr{R}_e d^* \tilde{d}\pi$ is defined by recursion on d^*, so that d^* in general is a subderivation of the left premise of Cut_e (4.2) (cf. Definition 2.4 of multiplication *):

$$\frac{(e,?^0),\ \Upsilon \quad \ldots d_n:\ (e,n)\Theta\ldots}{(e,?),\ \Upsilon * \Theta}\ \mathscr{R}_e d^* \tilde{d}\pi.$$

Below we present schematically original derivation with Cut which is reduced, and the result of reduction by the operator $\mathscr{R}_e$:

$$\begin{array}{l}
\mathsf{AxH}_{e,v} \\
(e,?),\ \Upsilon \\
\quad\vdots\, d^0 \qquad\qquad \vdots\, d_v \\
\dfrac{(e,?^0),\Theta \quad \ldots d_v:(e,v)\Theta\ldots}{d:\ \Theta}\ \mathsf{Cut}_e \\
\qquad\qquad \vdots\,\pi \\
\qquad\qquad \emptyset
\end{array}$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots\ \mathscr{W}_\Sigma d_v}{(e,v)\Upsilon_{\leqslant r} * \Theta}\ \vdots\ \mathscr{F}\mathscr{R}\mathscr{H}\pi}{(e,v)\Upsilon_{\leqslant r} * \Theta_{\leqslant r}}\ H_{e,v}}{(e,?)\Upsilon * \Theta}\ \vdots d^0 \quad}{(e,?),\Theta} \qquad \dfrac{\vdots\, d_v}{\ldots d_v : (e,v)\Theta \ldots}
$$

$$
\dfrac{(e,?),\Theta \qquad \ldots d_v : (e,v)\Theta \ldots}{\Theta}\ \mathsf{CutFr}_e
$$

$$
\begin{array}{c} \Theta \\ \vdots \pi \\ \emptyset \end{array}
$$

The definition uses auxiliary operators $\mathscr{F}\mathscr{R}\mathscr{H}\pi, \mathscr{W}_\Sigma$ to be described in the next subsections.

Definition 4.2. Let $d^* : (e,?^0), \Upsilon$ and $d_n : (e,n)\Theta$ for all $n \in \mathbb{N}$ be r^+-derivations, sequents $(e,?^0), \Upsilon$ and Θ be multiplicable. Let π be an $r+1$-path for Θ,

$$\tilde{d} := \{d_n:\ (e,n),\Theta\}_{n\in\mathbb{N}}. \tag{4.3}$$

Let $d^* \equiv \mathscr{T}\{d_i^*\}_{i\in|\mathscr{T}|}$.

(1) $d^* \equiv \mathsf{AxX}((e,?^0),\Delta)$ and $\mathsf{AxX} \neq \mathsf{AxH}_{e,v}$ for any v:

$$\mathscr{R}_e d^* \tilde{d}\pi := \mathsf{AxX}((e,?),\Delta);$$

(2) $\mathscr{T} \neq \mathsf{Ax}$:

$$\mathscr{R}_e d^* \tilde{d}\pi := \mathscr{T}\{\mathscr{R}_e d_i^* \tilde{d}\pi\}_{i\in|\mathscr{T}|};$$

(3) $d^* \equiv \mathsf{AxH}_{e,v}((e,?^0),\Upsilon)$:

$$\mathscr{R}_e d^* \tilde{d}\pi := \mathsf{H}_{e,v}(\mathscr{F}\mathscr{R}\mathscr{H}\pi)\mathscr{W}_\Sigma\, d_v$$

where d_v is taken from (4.3) and

$$\Sigma := ((e,v), \Upsilon_{\leqslant r}). \tag{4.4}$$

Lemma 4.2. *Let the conditions of Definition* 4.2 *be satisfied, that is* $d^* : (e,?^0), \Upsilon$ *and* $d_n : (e,n), \Theta$ *for* $n \in \mathbb{N}$ *are* r^+*-derivations, sequents* $(e,?), \Upsilon$ *and* Θ *are multiplicable,* π *is an* $r+1$*-path for* Θ.

Then $\mathscr{R}_e d^* \tilde{d}\pi$ *is an* r^+*-derivation and if* $\|d_n\| \leqslant \alpha$ *for all* $n \in \mathbb{N}$, *then*

$$\|\mathscr{R}_e d^* \tilde{d}\pi\| \leqslant \alpha + lth(\pi) + 1 + \|d^*\|$$

Proof. Induction on d^* with the same cases as in Definition 4.2.

(1) $\|\mathscr{R}_e d^* \tilde{d}\pi\| = 1 \leqslant \alpha + lth(\pi) + 1 + \|d^*\|$.

(2) $\|\mathscr{R}_e d^* \tilde{d}\pi\| = \|\mathscr{T}\{\mathscr{R}_e d_i^* \tilde{d}\pi\}_{i\in|\mathscr{T}|} =$

$$\sup(\|\mathscr{R}_e d_i^* \tilde{d}\pi\}_{i\in|\mathscr{T}|}\| + 1) \leqslant (IH) \sup(\alpha + lth(\pi) + 1 + \|d_i^*\| + 1) =$$

$$\alpha + lth(\pi) + 1 + sup(\|d_i^*\| + 1) \text{ (continuity of } +) = \alpha + lth(\pi) + 1 + \|d^*\|$$

(3) $\|\mathscr{R}_e d^* \tilde{d}\pi\| = \|\mathsf{H}_{e,v}(\mathscr{F}\mathscr{R}\mathscr{H}\pi)\mathscr{W}_\Sigma\, d_v\| = \|\mathscr{F}\mathscr{R}\mathscr{H}\pi\mathscr{W}_\Sigma\, d_v\| + 1$

$$\leqslant \text{(Lemma 4.3) } \|\mathscr{W}_\Sigma\, d_v\| + lth(\pi) + 1 \leqslant \text{(Lemma 4.4)}$$

$$\|d_v\| + lth(\pi) + 1 \leqslant (\text{since} \|d_v\| \leqslant \alpha)\alpha + lth(\pi) + 1 \qquad \square$$

4.2.1. Repetition of Fr, H*-inferences:* $\mathscr{F}\mathscr{R}\mathscr{H}\pi$*.*

Definition 4.3. If π is an $(r+1)$-path, then $\mathscr{F}\mathscr{R}\mathscr{H}\pi$ is the result of deleting from π all inferences except Fr, H.

Lemma 4.3. *Let π be an $(r+1)$-path for Θ and $\Sigma \leqslant r$ be a correct sequent such that $\Theta_{\leqslant r} \subseteq \Sigma$. Then $\mathscr{F}\mathscr{R}\mathscr{H}\pi$ is a deduction of Σ from $\Theta * \Sigma$ and*

$$rk(e) \geqslant r+1; \quad lth(\mathscr{F}\mathscr{R}\mathscr{H}\pi) \leqslant lth(\pi).$$

Proof (Cf. Lemma 6.4 in Mints et al. [11]). Use induction on π with subcases corresponding to the uppermost inference $\mathscr{T}$. The case $\mathscr{T} \equiv \mathsf{CutFr}_e$ is impossible, since π is an $r+1$-path. $\mathscr{T} \equiv \mathsf{Cut}_e$ is pruned since its main formula is already in Σ. Only $\mathsf{Fr}_e, \mathsf{H}_{e,v}$-inferences are retained since $rk(e) > r$. $\square$

Example. $\pi \equiv \mathsf{Fr}_e\mathsf{Cut}_f u\mathsf{H}_{g,v}\mathsf{Fr}_h$; $\Sigma \equiv f^u, q^0$; $\Theta \equiv e^? f^u g^v h^?$. Then $\mathscr{F}\mathscr{R}\mathscr{H}\pi \equiv \mathsf{Fr}_e\mathsf{H}_{g,v}\mathsf{Fr}_h$.

$$\begin{array}{c}\Theta\\ \pi\\ \emptyset\end{array} \quad \dfrac{\dfrac{\dfrac{\dfrac{e^? f^u g^v h^?}{e^? f^u g^v}\,\mathsf{Fr}_h}{e^?, f^u}\,\mathsf{H}_{g,v}}{e^?}\,\mathsf{Cut}_f u}{\emptyset}\,\mathsf{Fr}_e \qquad \begin{array}{c}\Theta * \Sigma\\ \mathscr{F}\mathscr{R}\mathscr{H}\pi\\ \Sigma\end{array} \quad \dfrac{\dfrac{\dfrac{e^?, g^v, h^?, \Sigma}{e^?, g^v, \Sigma}\,\mathsf{Fr}_h}{e^?, \Sigma}\,\mathsf{H}_{g,v}}{\Sigma}\,\mathsf{Fr}_e$$

4.2.2. Weakening: $\mathscr{W}_\Sigma d$ or $d * \Sigma$.

The next definition corresponds to Lemma 6.3 of [11].

Definition 4.4. Let d be an r^+-derivation of a sequent Θ. Let $\Sigma \leqslant r$ be a correct sequent such that $\Theta * \Sigma$ is defined and $(\Sigma f)_{\geqslant r} \subseteq \Theta$, $\Sigma t \geqslant r$. We define by recursion on d a derivation $\mathscr{W}_\Sigma d$ (denoted in [11] and below by $d * \Sigma$) of $\Theta * \Sigma$ obtained from d by deleting inferences $\mathsf{Cut}_e, \mathsf{CutFr}_e$ with $e \in dom(\Sigma)$.

(1) $d \equiv \mathsf{Cut}_e\{d_u\}_{u\in\{?^0\}\cup\mathbb{N}}$. Here $rk(e) < r$.

  (a) $e \notin dom(\Sigma)$: $d * \Sigma := \mathsf{Cut}_e\{d_u * \Sigma\}_{u\in\{?^0\}\cup\mathbb{N}}$.

(b) $(e,?)\in\Sigma$. Impossible, since $\Sigma t\geqslant r$, $rk(e)<r$.
(c) $(e,u)\in\Sigma$, $u\in\{?^0\}\cup\mathbb{N}: d*\Sigma:=d_u*\Sigma$.

(2) $d\equiv\mathsf{CutFr}_e\{d_u\}_{u\in\{?\}\cup\mathbb{N}}$. Here $rk(e)=r$.
(a) $e\notin dom(\Sigma): d*\Sigma:=\mathsf{CutFr}_e\{d_u*\Sigma\}_{u\in\{?\}\cup\mathbb{N}}$.
(b) $(e,?^0)\in\Sigma$. Impossible, since $rk(e)=r$, hence $(e,?^0)\in(\Sigma f)_{\geqslant r}\subseteq\Theta$, the conclusion of CutFr_e.
(c) $(e,u)\in\Sigma$, $u\in\{?\}\cup\mathbb{N}: d*\Sigma:=d_u*\Sigma$.

(3) $d\equiv\mathscr{T}d_0$, $\mathscr{T}\in\{\mathsf{Fr}_e,\mathsf{H}_{e,v}\}: d*\Sigma:=\mathscr{T}d_0*\Sigma$.

(4) $d\equiv\mathsf{AxX}(\Theta): d*\Sigma:=\mathsf{AxX}'(\Theta*\Sigma)$ for a suitable X'.

This definition can be abbreviated as follows. Let $d\equiv\mathscr{T}\{d_i\}_i$.
If $\mathscr{T}\neq\mathsf{Cut},\mathsf{CutFr}$ or the main formula of $\mathscr{T}$ is not in Σ, then

$$d*\Sigma:=\mathscr{T}\{d_i*\Sigma\}_i.$$

Otherwise d has a branch ending in its endsequent and consisting of inferences to be skipped over:

$$\begin{array}{c} d' \\ / \\ \vdots \\ / \\ | \\ d \end{array}$$

where d' is the derivation of the uppermost sequent of this branch, i.e. d' does not end in a redundant inference, but the inference following d' is redundant. Then $d*\Sigma:=d'*\Sigma$.

Note: The number of consecutive inferences to be skipped over is bounded by the number of components in Σ.

Lemma 4.4. $\|d*\Sigma\|\leqslant\|d\|$.

Proof. No new inferences are added to d. □

4.2.3. Elimination of degree-r cuts: $\mathscr{E}_r\,\mathrm{d}\pi$.

To define $\mathscr{E}$, operation $\mathscr{R}_e$ is used, so argument π is needed.

Definition 4.5. Let $d:\Theta$ be an $(r+1)$-derivation, and π be an $(r+1)$-path for Θ.

(1) $d\equiv\mathsf{AxX}(\Theta):\mathscr{E}_r d\pi:=d$,

(2) $d\equiv\mathsf{Cut}_e\{d_u\}_{u\in\{?^0\}\cup\mathbb{N}}$, $rk(e)=r$.

$\mathscr{E}_r d\pi := \mathsf{CutFr}_e(\mathscr{R}_e(\mathscr{E}_r d_{?^0}(\pi\mathsf{Cut}_e ?^0))\tilde{d}\pi)\tilde{d}$, where $\tilde{d} \equiv \{\mathscr{E}_r d_n(\pi\mathsf{Cut}_e n)\}_{n\in\mathbb{N}}$.

$$\dfrac{\begin{matrix}\vdots & & \vdots \\ \mathscr{R}_e(\mathscr{E}_r d_{?^0}(\pi\mathsf{Cut}_e ?^0))\tilde{d}\pi & & \mathscr{E}_r d_n(\pi\mathsf{Cut}_e n) \\ (e, ?^0), \Theta & & \tilde{d}:\ \ldots (e,n), \Theta \ldots\end{matrix}}{\begin{matrix}\mathscr{E}_r d\pi :\ \Theta \\ \vdots\pi \\ \emptyset\end{matrix}}\ \mathsf{CutFr}_e.$$

(3) Otherwise. $d \equiv \mathscr{T}\{d_i\}_{i\in|\mathscr{T}|} : \mathscr{E}_r d\pi := \mathscr{T}\{\mathscr{E}_r d_i(\pi\mathscr{T} i)\}_{i\in|\mathscr{T}|}$.

Lemma 4.5. *Let $d:\Theta$ be an $(r+1)$-derivation, π be an $(r+1)$-path for Θ. Then*

$$\mathscr{E}_r d\pi \text{ is an } r^+\text{-derivation}; \quad \|\mathscr{E}_r d\pi\| \leqslant \omega^{\|d\|}.$$

Proof. Cf. Lemma 6.6 in [11]. The only non-trivial case is 2.

Let $\alpha := \sup\{\|d_n\| : n\in\mathbb{N}\}$. We have by IH and Lemma 4.2:

$$\|\mathscr{E}_r d_n(\pi\mathsf{Cut}_e n)\}\| \leqslant \omega^\alpha$$

$$\|\mathscr{R}_e(\mathscr{E}_r d_{?^0}(\pi\mathsf{Cut}_e ?^0))\tilde{d}\pi\| \leqslant \omega^\alpha + lth(\pi) + 1 + \omega^{\|d_{?^0}\|} = \omega^\alpha + \omega^{\|d_{?^0}\|}$$

since $\|d_{?^0}\| > 0$ by definition, hence $\omega^{\|d_{?^0}\|}$ is a principal number for addition. Now use $\|d_{?^0}\|, \alpha < \|d\|$:

$$\begin{aligned}\|\mathscr{E}_r d\pi\| &= \|\mathsf{CutFr}_e(\mathscr{R}_e(\mathscr{E}_r d_{?^0}(\pi\mathsf{Cut}_e ?^0))\tilde{d}\pi)\tilde{d} \\ &= \sup(\|\mathscr{R}_e(\mathscr{E}_r d_{?^0}(\pi\mathsf{Cut}_e ?^0))\tilde{d}\pi\| + 1, \{\|\mathscr{E}_r d_n(\pi\mathsf{Cut}_e n)\}\| + 1\}_{n\in\mathbb{N}}) \\ &\leqslant \omega^\alpha + \omega^{\|d_{?^0}\|} + 1 < \omega^{\|d\|} \qquad \square\end{aligned}$$

4.2.4. Operation $\mathscr{D}$: replacing CutFr *by* Fr.

Definition 4.6. (1) $\mathscr{D}\mathsf{AxX}(\Theta) := \mathsf{AxX}(\Theta)$.

(2) $\mathscr{D}\mathscr{T}\{d_i\}_i := \mathscr{T}\{\mathscr{D}d_i\}_i$, if $\mathscr{T} \neq \mathsf{CutFr}$.

(3) $\mathscr{D}\mathsf{CutFr}_e\{d_i\}_i := \mathsf{Fr}_e\mathscr{D}d_?$.

5. System *PAε∗*

Derivable sequents have at most one +-component.

Deductions will have a linear form

$$\dfrac{\dfrac{\begin{matrix}\dfrac{\dfrac{\Gamma_{n+1} \quad \Gamma'_{n+1}}{\Gamma_n}}{} \\ \vdots \\ \dfrac{\Gamma_{i+1}}{\Gamma_i} \\ \vdots \\ \Gamma_1\end{matrix}}{} \qquad \begin{matrix}\vdots \\ \Gamma'_1\end{matrix}}{\Gamma_0}$$

with the leftmost main branch consisting of +-free sequents $\Gamma_0, \Gamma_1, \ldots$ and the right branches containing sequents $\Gamma'_1, \ldots, \Gamma'_{n+1}$ that have exactly one +-component.

Axioms

$\mathsf{AxF}(\Theta)$	Θ is ci
$\mathsf{AxS}(\Theta)$	Θ is solving
$\mathsf{AxH}_{e,v}(\Theta)$	e is the H-term, v is the H-value of Θ
$\mathsf{AxA}(\Theta)$	Θ is not $\mathsf{AxF}, \mathsf{AxS}, \mathsf{AxH}$

Rules of inference

$$\frac{(e,?^0),\Theta \quad (e,+),\Theta}{\Theta}\ \mathsf{Cut}_e \quad \frac{(e,?),\Theta \quad (e,+),\Theta}{\Theta}\ \mathsf{CutFr}_e$$

$$\frac{(e,?^0),\ \Upsilon \quad (e,+),\Theta}{(e,?),\ \Upsilon * \Theta}\ \mathsf{R}_e \quad \frac{\Theta}{\Theta}\ \mathsf{E}_r \quad \frac{\Theta}{\Theta}\ \mathsf{D}_r$$

$$\frac{(e,?),\Theta}{\Theta}\ \mathsf{Fr}_e. \quad \frac{(e,v),\Theta_{\leqslant rk(e)}}{(e,?),\Theta}\ \mathsf{H}_{e,v} \quad \frac{\Theta}{\Sigma * \Theta}\ \mathsf{W}_\Sigma$$

In these rules e is a canonical term such that premises and conclusion are legal sequents. In the rules $\mathsf{Cut}_e, \mathsf{CutFr}_e, \mathsf{Fr}_e, \mathsf{H}_{e,v}$ the conclusion Θ should be +-free.

Definition 5.1. A *quasi-derivation* in $PA\varepsilon*$ is a finite tree proceeding from axioms (in its leaves) by inference rules.

Quasi-derivations in $PA\varepsilon*$ are written as linear sequences using inference symbols in the same way as derivations in εPA (Section 3).

The definition of a ρ-path is extended to sequents $\Theta \equiv (e,+),\Theta'$ with +-free Θ': the path contains a unique occurrence of Cut_e+ or CutFr_e+ corresponding to $(e,+)$.

Definition 5.2. Let $\pi'\mathscr{T}0\pi''$ be a ρ-path ($\rho \in \{r, r^+\}$) in ePA for a +-free sequent $(e,0),\Theta'$ with $\mathscr{T} \in \{\mathsf{Cut}_e, \mathsf{CutFr}_e\}$ and the component $\mathscr{T}0$ corresponding to $(e,0)$. Then

$\pi'\mathscr{T}+\pi''$ is a ρ-path in $PA\varepsilon*$ for $(e,+),\Theta'$ and the component $\mathscr{T}+$ corresponds to $(e,+)$.

In this case, $\pi[e/n] := \pi'\mathscr{T}n\pi''$

Otherwise, $\pi[e/n] := \pi$.

The next definition singles out derivations $d \in PA\varepsilon*$ among quasi-derivations. In fact additional arguments are needed. We define a relation $(h,\pi) \in \rho$ (read "h over a path π is a ρ-derivation") by primitive recursion on h for $\rho \in \{r, r^+\}$, $r > 0$, a quasi-derivation h and a path π for $\Gamma(h)$. In most cases the component π is just extended in a natural way. Expression r^+ is treated as $r + \frac{1}{2}$ in inequalities like $r^+ > s$.

Definition 5.3. (1) $h \equiv \mathsf{AxX}(\Theta)$, $X \neq A : (h, \pi) \in \rho$ for all ρ.

(2) $h \equiv \mathsf{AxA}(\Theta)$. Let

$$f_\Theta := \begin{cases} 0 & \Theta \text{ is } + \text{-free} \\ f & \text{if } \Theta \equiv (f, +), \Theta' \end{cases}$$

$r_0 := \max\{rk(f) : f \text{ is a closed } \varepsilon\text{-term in } |Cr \cup \mathscr{CR}(\Theta)|_\Theta,\ f \neq f_\Theta\}$.

Then $(h, \pi) \in \rho$ iff $r_0 < r$.

(3) $h \equiv \mathsf{Cut}_e h_0 h_1$. This case is similar to the AxA-clause: $(h, \pi) \in \rho$ iff

$$(h_0, \pi\mathsf{Cut}_e ?^0) \in \rho, \quad (h_1, \pi\mathsf{Cut}_e +) \in \rho \text{ and } rk(e) < r. \tag{5.1}$$

(4) $h \equiv \mathsf{CutFr}_e h_0 h_1 : (h, \pi) \in r^+$ iff

$$(h_0, \pi\mathsf{CutFr}_e ?) \in r^+, \quad (h_1, \pi\mathsf{CutFr}_e +) \in r^+ \text{ and } rk(e) = r. \tag{5.2}$$

(5) $h \equiv \mathsf{R}_e h_0 h_1 : (h, \pi) \in r^+$ iff
$rk(e) = r$; $\Gamma(h_0) \equiv (e, ?^0), \Upsilon$; $\Gamma(h_0) \equiv (e, +), \Theta$,
$\pi \equiv \pi' \mathsf{CutFr}_e ? \kappa$, where $\tilde{\pi} := \pi' \mathsf{Cut}_e ?^0 \kappa$ is a path for $(e, ?^0), \Upsilon$
with $(\mathsf{Cut}_e ?^0)$ corresponding to $(e, ?^0)$,
π' is an $r+1$-path for Θ, κ is an r^+-path,
$(h_0, \tilde{\pi}) \in r^+$, $(h_1, \pi', \mathsf{Cut}_e +) \in r^+$.

$$\begin{array}{c} \vdots h_0 \\ (e, ?^0), \Upsilon \end{array} \qquad \dfrac{\begin{array}{c}\vdots h_0 \\ (e, ?^0), \Upsilon\end{array} \quad \begin{array}{c}\vdots h_1 \\ (e, +), \Theta\end{array}}{(e, ?), \Upsilon * \Theta}\ \mathsf{R}_e h_0 h_1$$

$$\dfrac{\begin{array}{c}\vdots \kappa \quad r^+ \\ (e, ?^0), \Theta\end{array} \quad \begin{array}{c}\vdots h_1 \\ (e, +), \Theta\end{array}}{\begin{array}{c}\Theta \\ \vdots \pi' \quad r+1 \\ \emptyset\end{array}}\ \mathsf{Cut}_e \qquad \dfrac{\begin{array}{c}\vdots \kappa \\ (e, ?), \Theta\end{array} \qquad \begin{array}{c}\vdots h_1 \\ (e, +), \Theta\end{array}}{\begin{array}{c}\Theta \\ \vdots \pi' \\ \emptyset\end{array}}\ \mathsf{CutFr}_e$$

(6) $h \equiv \mathsf{E}_r h_0 : (h, \pi) \in r^+$ iff $(h_0, \pi) \in r+1$ and π is an $r+1$-path.

(7) $h \equiv \mathsf{H}_{e,v} h_0 : (h, \pi) \in \rho$ iff $(h_0, \pi\mathsf{H}_{e,v}) \in \rho$ and $rk(e) \geqslant r$.

(8) $h \equiv \mathsf{Fr}_e h_0 : (h, \pi) \in \rho$ iff $(h_0, \pi\mathsf{Fr}_e) \in \rho$ and $rk(e) \geqslant \rho$ [$rk(e) \geqslant r^+$ means $rk(e) > r$].

(9) $h \equiv \mathsf{W}_\Sigma h_0 : (h, \pi) \in r^+$ iff
$(h_0, \pi') \in r^+$ for some π', $\Sigma t \geqslant r$, $\Sigma \leqslant r$ is a correct $+$-free sequent, $\Sigma * \Gamma(h_0)$ is defined, $(\Sigma f)_{\geqslant r} \subseteq \Gamma(h_0)$.

$$\begin{array}{cc} \vdots h_0 & \vdots \mathsf{W}_\Sigma h_0 \\ \Theta & \Sigma * \Theta \\ \vdots \pi' & \vdots \pi \end{array}$$

(10) $h \equiv \mathsf{D}_r h_0$: $(h, \pi) \in r$ iff $(h_0, \pi) \in r^+$.

Let us summarize Definition 5.3.

Lemma 5.1. (1) $(\mathscr{T}\{h_i\}_{i\in|\mathscr{T}|},\pi)\in\rho$ *implies* $(h_i,\pi_i)\in\rho'$ *for* $\rho\in\{r,r^+\}$,
with $\pi_i\equiv\pi\mathscr{T}_i$, $\rho'\equiv\rho$ *except*
$\mathscr{T}\equiv\mathsf{R}_e$, *where* $\pi_0\equiv\pi'\mathsf{CutFr}_e?\kappa$, $\pi_1\equiv\pi\mathsf{Cut}_e+$,
$\mathscr{T}\equiv\mathsf{E}_r$, *where* $\pi_0\equiv\pi$, $\rho'\equiv r+1$,
$\mathscr{T}\equiv\mathsf{D}_r$, *where* $\pi_0\equiv\pi$, $\rho'\equiv r^+$,
(2) $(h_i,\pi_i)\in\rho\in\{r,r^+\}$, *for all* $i\in|\mathscr{T}|$ *and suitable paths* π_i *implies* $(\mathscr{T}\{h_i\}_{i\in|\mathscr{T}|},\pi)\in\rho$ *for a suitable* π *except*
$\mathscr{T}\equiv\mathsf{Cut}_e$ *with* $rk(e)\geqslant r$,
$\mathscr{T}\equiv\mathsf{CutFr}_e$ *with* $rk(e)\neq r$ or $\rho\neq r^+$,
$\mathscr{T}\equiv\mathsf{H}_{e,v}$ *with* $rk(e)<r$,
$\mathscr{T}\equiv\mathsf{Fr}_e$ *with* $rk(e)<\rho$,
or $\mathscr{T}\in\{\mathsf{R}_e,\mathsf{E}_r,\mathsf{W}_\Sigma,\mathsf{D}_r\}$ *with similar exceptions.*

Proof. By inspection. □

Let us define an ordinal $o(h)$ of a $PA\varepsilon*$-derivation h intended to be a good approximation to the ordinal height of h^∞. When the last sequent $\Gamma(h)$ contains a +-component

$$\Gamma(h)\equiv(e,+),\Gamma'$$

we will have

$$o(h[e/n])\leqslant o(h)\quad\text{for all } n\in\mathbb{N}. \tag{5.3}$$

Definition 5.4.

$$\delta_\Theta:=\begin{cases}0 & \text{if } \Theta \text{ is } +\text{-free}\\ 1 & \text{if } \Theta\equiv(e,+),\Theta'\\ & \text{and either } \Theta \text{ computes } Cr \text{ and } e \text{ occurs in } |\mathscr{CR}(\Theta)|_\Theta\\ & \text{or } \Theta \text{ does not compute } Cr \text{ and } e \text{ occurs in } |Cr|_\Theta.\end{cases}$$

Definition 5.5. Let $h\in PA\varepsilon*$, π be a path for $\Gamma(h)$.

$$o(h,\pi):=\begin{cases}1 & \text{if } h\equiv\mathsf{AxX}(\Theta),\ X\neq A\\ N_{|\mathscr{CR}(\Theta)|_\Theta}+1-\delta_\Theta & \text{if } h\equiv\mathsf{AxA}(\Theta),\\ & \Theta \text{ computes } Cr\\ \omega+N_{|Cr|_\Theta}-\delta_\Theta & \text{if } h\equiv\mathsf{AxA}(\Theta)\\ & \text{otherwise}\\ o(h_1,\pi'\mathsf{Cut}_e+)+lth(\pi)+1+o(h_0,\pi) & \text{if } h=\mathsf{R}_e h_0 h_1 \text{ and}\\ & \pi\equiv\pi'\mathsf{Cut}_e?^0\kappa\\ \omega^{o(h_0,\pi)} & \text{if } h=\mathsf{E}_r h_0\\ o(h_0,\pi\mathscr{T})+1 & \text{if } h=\mathscr{T}h_0,\\ & \mathscr{T}\in\{\mathsf{Fr}_e,\mathsf{H}_{e,v},\mathsf{D}_r,\mathsf{W}_\Sigma\}\\ \max(o(h_0,\pi\mathscr{T}?'),o(h_1,\pi\mathscr{T}+))+1 & \text{if } h\equiv\mathscr{T}h_0h_1,\\ & \mathscr{T}\in\{\mathsf{Cut}_e,\mathsf{CutFr}_e\}\end{cases}$$

where $?'$ is $?^0$ for Cut and is $?$ for CutFr.

We define substitution $h[e/n]$, $h\in PA\varepsilon*$, $n\in\mathbb{N}$.

Definition 5.6. If $\Gamma(h)$ does not contain $(e,+)$, then $h[e/n] := h$.
In the following:

$$\Gamma(h) \equiv (e,+), \Gamma', \ \Gamma(h[e/n]) := (e,n), \Gamma' \tag{5.4}$$

(1) $h \equiv \mathsf{AxX}(\Gamma(h)),\ X \neq A : h[e/n] := \mathsf{AxX}'((e,n),\Gamma')$ for a suitable X'.
(2) $h \equiv \mathsf{AxA} : h[e/n] := \mathsf{AxX}((e,n),\Gamma')$ for a suitable X.
(3) $h \equiv \mathscr{T} h_0 : h[e/n] := \mathscr{T} h_0[e/n]$.
(4) $h \equiv \mathsf{R}_f h_0 h_1 : h[e/n] := \mathsf{R}_f h_0[e/n] h_1$.

Comments: Condition (5.4) excludes $\mathsf{Cut}, \mathsf{CutFr}, \mathsf{Fr}, \mathsf{H}$ as a last rule in h.
To 4: If $h \equiv \mathsf{R}_f h_0 h_1$, then $(e,+)$ occurs only in $\Gamma(h_0)$, not in $\Gamma(h_1)$.

Lemma 5.2. (1) *If* π *is a* ρ-path *for* $\Gamma(h)$, *then* $\pi[e/n]$ *is a* ρ*-path for* $\Gamma(h[e/n])$.
(2) *If* $(h,\pi) \in \rho$, *then* $(h[e/n], \pi[e/n]) \in \rho,\ o(h[e/n], \pi[e/n]) \leqslant o(h)$.

Proof. 1. If $\pi \equiv \pi' \mathscr{T} + \pi'',\ \Gamma(h) \equiv (e,+), \Theta'$ (Definition 5.2), where $\pi' \mathscr{T} 0 \pi''$ is a ρ-path for $(e,0), \Theta'$, then $\pi[e/n] \equiv \pi' \mathscr{T} n \pi''$ is a ρ-path for $(e,n)\Theta'$.
2. Consider cases in Definition 5.6 assuming
$\Gamma(h) \equiv (e,+), \Gamma',\ \pi := \pi'(\mathscr{T}'+)\pi'',\ \mathscr{T}' \in \{\mathsf{Cut}_e, \mathsf{CutFr}_e\}$.
Cases 1,2: $\pi[e/n]$ is again a ρ-path, $h[e/n]$ is again an axiom.
Case 3: $h \equiv \mathscr{T} h_0$.
Case 3.1: $\mathscr{T} \equiv \mathsf{E}_s$. Then $\Gamma(h) \equiv \Gamma(h_0),\ \rho \equiv r^+,\ s \equiv r,\ (h_0,\pi) \in r$. By IH and clause 1 of present lemma, $(h_0[e/n], \pi[e/n]) \in r$, hence $(\mathsf{E}_r h_0[e/n], \pi[e/n]) \in r^+$.
Case 3.2: $\mathscr{T} \equiv \mathsf{W}_\Sigma$. Again, $\pi[e/n]$ is a ρ-path for $\Gamma(h[e/n])$ and
$(h_0[e/n], \pi'[e/n]) \in r^+$, hence $(\mathsf{W}_\Sigma h_0[e/n], \pi[e/n]) \in r^+$.
Case 3.3: $\mathscr{T} \equiv \mathsf{D}_s h_0$. Like Case 3.2.
Case 4: $\mathscr{T} \equiv \mathsf{R}_f h_0 h_1$.

$$\frac{(f,?^0), \Upsilon \quad (f,+), \Theta}{(f,?), \Upsilon * \Theta}\ \mathsf{R}_e.$$

We have $h[e/n] \equiv \mathsf{R}_f h_0[e/n] h_1$. Assume $\pi \equiv \pi_1, \mathsf{Cut}_f ?^0, \kappa$ is a ρ-path for $(f,?^0), \Upsilon$, π_1 is a ρ-path for Θ and a component $(e,+)$ occurs in $\Gamma(h_0)$. Since this component is not in Θ, it occurs in $\Gamma(h_0)$ and $f \neq e$. Consider the occurrence of $\mathscr{T}_e'+$ in π corresponding to $(e,+)$. It is not in $\pi_1 \mathsf{Cut}_f ?^0$, hence

$$\kappa \equiv \kappa', \mathscr{T}'_e+, \kappa'',\ \ \pi[e/n] \equiv \pi_1, \mathsf{Cut}_f ?^0, \kappa', \mathscr{T}'_e n, \kappa''$$

and $\pi[e/n]$ is a ρ-path for $\Gamma(h_0)[e/n]$ by the part 1 of this lemma. Hence $(h[e/n], \pi[e/n]) \in \rho$. $\square$

6. Infinite translation

We define an infinite translation for $h \in PA\varepsilon*$ with an r-path π for $\Gamma(h)$ as an additional argument.

Definition 6.1. Let $h \in PA\varepsilon*$. Assume $\Gamma(h)$ to be $+$-free.
(1) $h \equiv \mathsf{AxX}(\Theta),\ X \neq A : (h,\pi)^\infty := h$.
(2) $h \equiv \mathsf{AxA}(\Theta) : (h,\pi)^\infty := \mathsf{orig}(\Theta)$, cf. Definition 4.1.
(3) $h \equiv \mathsf{Cut}_e h_0 h_1$: (Cf. Definition 5.6)

$$(h,\pi)^\infty := \mathsf{Cut}_e(h_0, \pi\mathsf{Cut}_e ?^0)^\infty \{(h_1[e/n], \pi\mathsf{Cut}_e n)^\infty\}_{n\in\mathbb{N}}.$$

(4) $h \equiv \mathsf{CutFr}_e h_0 h_1$: similarly to the case of Cut:

$$(h,\pi)^\infty := \mathsf{CutFr}_e(h_0, \pi\mathsf{CutFr}_e ?)^\infty \{(h_1[e/n], \pi\mathsf{CutFr}_e n)^\infty\}_{n\in\mathbb{N}}.$$

(5) $h \equiv \mathsf{R}_e h_0 h_1$:

$$(h,\pi)^\infty := \mathscr{R}_e(h_0, \tilde{\pi})^\infty \{(h_1[e/n], \pi'\mathsf{Cut}_e n)^\infty\}_{n\in\mathbb{N}}\ \pi',$$

where $\pi \equiv \pi'\mathsf{CutFr}_e ?^0\kappa,\ \tilde{\pi} \equiv \pi'\mathsf{Cut}_e ?^0\kappa$, cf. Definitions 5.3.5, 4.2.
(6) $h \equiv \mathsf{E}_r h_0 \pi : (h,\pi)^\infty := \mathscr{E}_r(h_0,\pi)^\infty \pi$, cf. Definition 4.5.
(7) $h \equiv \mathsf{D}_r h_0 : (h,\pi)^\infty := \mathscr{D}(h_0,\pi)^\infty$, cf. Definition 4.6.
(8) $h \equiv \mathscr{T} h_0,\ \mathscr{T} \equiv \mathsf{Fr}_e, \mathsf{H}_e : (h,\pi)^\infty := \mathscr{T}(h_0,\pi)^\infty$.
(9) $h \equiv \mathsf{W}_\Sigma h_0 : (h,\pi)^\infty := \mathscr{W}_\Sigma(h_0,\pi)^\infty$, cf. Definition 4.4.

If $\Gamma(h) \equiv (e,+), \Gamma'$, then $(h,\pi)^\infty := \{(h[e/n], \pi[e/n])^\infty\}_{n\in\mathbb{N}}$.

Then for $h \equiv \mathscr{T} h_0 h_1, \mathscr{T} \in \{\mathsf{Cut}_e, \mathsf{CutFr}_e\}$ one can write

$$(h,\pi)^\infty \equiv \mathscr{T}(h_0, \pi\mathscr{T}?')^\infty (h_1, \pi\mathscr{T}+)^\infty.$$

6.1. Correctness of the infinite translation

Theorem 6.1. *Let $h \in PA\varepsilon*$, $(h,\pi) \in \rho \in \{r, r^+\}$, where $r > 0$. Then*
(1) *If $\Gamma(h)$ is $+$-free, then $(h,\pi)^\infty \in \varepsilon PA$ is a ρ-derivation.*
(2) *If $\Gamma(h) = (e,+)\Gamma'$, then for every $n \in \mathbb{N}$, $(h[e/n], \pi[e/n])^\infty \in \varepsilon PA$ is a ρ-derivation.*

Proof. Induction on h.
(1) $h \equiv \mathsf{AxX}(\Theta),\ X \neq A$.
 (a) If $\Gamma(h)$ (that is Θ) is $+$-free, then $(h,\pi)^\infty \equiv h$, it does not contain any inference rules and is a ρ-derivation in εPA.
 (b) If $\Gamma(h) \equiv (e,+)\Theta'$, then $\pi[e/n]$ is a ρ-path for $(e,n), \Theta'$ in εPA (Lemma 5.2), and the previous argument applies.
(2) $h \equiv \mathsf{AxA}(\Theta)$. Let f_Θ, r_0 be as in the clause 2 of Definition 5.3.
 (a) Θ is $+$-free. Then $(h,\pi) \in \rho$ iff $r_0 < r$. By Lemma 4.1 $\mathsf{orig}(\Theta)$ is a derivation consisting only of axioms and cuts of rank $\leqslant r_0$. Hence $\mathsf{orig}(\Theta)$ is a ρ-derivation in εPA for $\rho \in \{r, r^+\}$.
 (b) $\Theta \equiv (e,+), \Theta'$. Then $h[e/n]$ is AxX or AxA, and one of cases 1a, 2a applies.

(3) $h \equiv \mathsf{Cut}_e h_0 h_1$. Here Θ is $+$-free and $(h,\pi) \in \rho$ iff (5.1) is true. By IH, $(h_0, \pi\mathsf{Cut}_e ?^0)^\infty$, $(h_1, \pi\mathsf{Cut}_e +)[e/n]^\infty$ are ρ-derivations in εPA for any n. Since only Cut_e is added to these derivations to form $(h,\pi)^\infty$, the latter is a ρ-derivation in εPA.

(4) $h \equiv \mathsf{CutFr}_e h_0 h_1$. Here $\rho = r^+$, sequent Θ is $+$-free and $(h,\pi) \in r^+$ iff (5.2) is true.
By IH, $(h_0, \pi\mathsf{CutFr}_e ?)^\infty, (h_1[e/n], \pi\mathsf{CutFr}_e n)^\infty$ are r^+-derivations in εPA for any n. Since only CutFr_e is added to these derivations to form $(h,\pi)^\infty$, the latter is an r^+-derivation in εPA.

(5) $h \equiv \mathsf{R}_e h_0 h_1$. We have $(h,\pi) \in r^+$ for $r = rk(e)$, and $\pi \equiv \pi'\mathsf{CutFr}_e ?\kappa$, where π' is an $r+1$-path for Θ,
$(h_0, \tilde{\pi}) \in r^+$, $(h_1, \pi'\mathsf{Cut}_e +) \in r^+$ for $\tilde{\pi} \equiv \pi'\mathsf{Cut}_e ?^0 \kappa$.
If $\Gamma(h)$ is $+$-free,

$$(h,\pi)^\infty := \mathscr{R}_e (h_0, \tilde{\pi})^\infty \{(h_1[e/n], \pi'\mathsf{Cut}_e n)^\infty\}_{n \in \mathbb{N}}\, \pi'$$

Apply IH, Lemma 5.2, Lemma 4.2.
If $\Gamma(h)$ contains $+$, then corresponding component $(f,+)$ belongs to $\Gamma(h_0)$, since $\Gamma(h_1)$ already contains $(e,+)$. Substitute $(h_0[f/m], \tilde{\pi}[f/m])^\infty$ for $(h_0, \tilde{\pi})^\infty$ in the argument for $+$-free case.

(6) $h \equiv \mathsf{E}_r h_0$. If $\Gamma(h)$ is $+$-free, we have $(h_0,\pi)^\infty \in r+1$ by IH and $(h,\pi)^\infty \in r^+$ by Lemma 4.5. $+$-case is treated as before.

(7) $h \equiv \mathscr{T} h_0$, $\mathscr{T} \equiv \mathsf{Fr}_e, \mathsf{H}_{e,v}$.
Apply IH and definition of ρ-derivation: one can add $\mathsf{Fr}_e, \mathsf{H}_{e,v}$ with $rk(e) \geqslant r$ to $(h_0,\pi)^\infty$ except the case $\mathscr{T} \equiv \mathsf{Fr}_e, \rho \equiv r^+$, when $rk(e) > r$ is needed.

(8) $h \equiv \mathsf{W}_\Sigma h_0 : (h,\pi) \in r^+$ iff $(h_0,\pi) \in r^+, \Sigma t \geqslant r$, $\Sigma \leqslant r$ is a correct $+$-free sequent, $\Gamma(h_0) * \Sigma$ is defined, $(\Sigma f)_{\geqslant r} \subseteq \Gamma(h_0)$. As before.

(9) $h \equiv \mathsf{D}_r h_0 : (h,\pi) \in r$ if $(h_0,\pi) \in r^+$. As before. □

7. Reduction of finite derivations: operations $tp_\pi(h), h_\pi[i]$

Let us define (following ideas in [5] and definition in [7]) for every derivation $h \in PA\varepsilon*$ with a $+$-free $\Gamma(h)$ the last rule $tp_\pi(h)$ of $(h,\pi)^\infty$ and the notations $h_\pi[i] \in PA\varepsilon*$ for premises of that last rule. A path π is used as an additional argument.

More precisely, if $tp_\pi(h) := \mathscr{T} \notin \{\mathsf{Cut}, \mathsf{CutFr}\}$, then $h_\pi[0]$ is a derivation in $PA\varepsilon*$ such that $(h,\pi)^\infty \equiv \mathscr{T}(h_\pi[0], \pi\mathscr{T})^\infty$:

$$\frac{(h_\pi[0], \pi\mathscr{T})^\infty : \Gamma(h_0)}{(h,\pi)^\infty : \Gamma(h)}\ \mathscr{T}. \tag{7.1}$$

If $tp_\pi(h) := \mathscr{T} \in \{\mathsf{Cut}_e, \mathsf{CutFr}_e\}$, then $h_\pi[0] : (e,?'), \Gamma(h)$ and $h_\pi[1] : (e,+), \Gamma(h)$ are derivations in $PA\varepsilon*$ such that

$$\frac{(h_\pi[0], \pi\mathscr{T}?')^\infty : (e,?'), \Gamma(h) \quad \dots (h_\pi[1][e/n], \pi\mathscr{T} n)^\infty : (e,n), \Gamma(h) \dots}{(h,\pi)^\infty : \Gamma(h)}\ \mathscr{T} \tag{7.2}$$

with $?' \equiv ?^0$ for $\mathscr{T} \equiv \mathsf{Cut}_e$ and $?' \equiv ?$ for $\mathscr{T} \equiv \mathsf{CutFr}_e$.

Leaving out π-subscripts and using notation

$$h' \equiv \mathscr{T} h[0]h[1] \text{ for } tp(h) \equiv \mathscr{T} \in \{\mathsf{Cut}_e, \mathsf{CutFr}_e\};\ h' \equiv \mathscr{T} h[0] \text{ otherwise} \quad (7.3)$$

we will have

$$h[0], h[1],\ h' \in PA\varepsilon*, \quad h \in \rho \ \Leftrightarrow\ h' \in \rho\ (\rho \in \{r, r^+\}). \quad (7.4)$$

Definition 7.1. Let $h \in PA\varepsilon*$, $\Gamma(h)$+-free.

(1) $h \equiv \mathsf{AxX}(\Theta)$.
  (a) $X \neq A : tp_\pi(h) := \mathsf{AxX}$.
  (b) $X = A$. Let e be the ε-term such that (4.1) holds. Put
    $tp_\pi(h) := \mathsf{Cut}_e$, $h_\pi[0] := \mathsf{AxX}((e, ?^0), \Theta)$, $h_\pi[1] := \mathsf{AxA}((e, +), \Theta)$ for a suitable X.

(2) $h \equiv \mathscr{T} h_0 h_1$ or $h \equiv \mathscr{T} h_0$ with $\mathscr{T} \in \{\mathsf{Cut}_e, \mathsf{CutFr}_e, \mathsf{Fr}_e, H_{e,v}\}$:
  $tp_\pi(h) := \mathscr{T}$; $h_\pi[i] := h_i$, $i \in |\mathscr{T}|$.

(3) $h \equiv \mathsf{E}_r h_0$:
  (a) $tp_\pi(h_0) \equiv \mathsf{Cut}_e$, $rk(e) = r$:

$$tp_\pi(h) := \mathsf{CutFr}_e;\ h_\pi[1] := \mathsf{E}_r h_{0\pi}[1],\ h_\pi[0] := \mathsf{R}_e(\mathsf{E}_r h_{0\pi}[0])h_\pi[1];$$

  Here is an original derivation h (with h_0 replaced by h'_0) and the new derivation $h' \equiv \mathsf{CutFr}_e h_\pi[0]h_\pi[1]$:

$$\dfrac{\dfrac{h_{0\pi}[0] : (e, ?^0), \Gamma \quad h_{0\pi}[1] : (e, +), \Gamma}{h'_0 : \Gamma}\ \mathsf{Cut}_e}{h : \Gamma}\ \mathsf{E}_r$$

$$\dfrac{\dfrac{\dfrac{h_{0\pi}[0] : (e, ?^0), \Gamma}{(e, ?^0), \Gamma}\ \mathsf{E}_r \quad \dfrac{h_{0\pi}[1] : (e, +), \Gamma}{h_\pi[1] : (e, +), \Gamma}\ \mathsf{E}_r}{h_\pi[0] : (e, ?), \Gamma}\ \mathsf{R}_e \quad \dfrac{h_{0\pi}[1] : (e, +), \Gamma}{h_\pi[1] : (e, +), \Gamma}\ \mathsf{E}_r}{h' : \Gamma}\ \mathsf{CutFr}_e$$

  (b) Otherwise. $tp_\pi(h) := tp_\pi(h_0)$; $h_\pi[i] := \mathsf{E}_r h_{0\pi}[i]$.
    Note that $tp_\pi(h_0) \equiv \mathsf{CutFr}_g$. is impossible. If $rk(g) = r' \geqslant r$, then $\mathsf{D}_{r'}$ must occur between CutFr_g and E_r. Otherwise E_r cannot occur below CutFr_g.

(4) $h \equiv \mathsf{R}_e h_0 h_1$. Let $\mathscr{T} := tp_\pi(h_0)$.
  (a) $\mathscr{T} \equiv \mathsf{AxX}, \mathsf{X} \neq \mathsf{H}_{e,v}$ for any $v : tp_\pi(h) := \mathsf{AxX}$
  (b) $\mathscr{T} \neq \mathsf{Ax} : tp_\pi(h) := \mathscr{T}$; $h_\pi[i] := \mathsf{R}_e h_{0\pi}[i]h_1$;
  (c) $\mathscr{T} \equiv \mathsf{AxH}_{e,v}((e, ?^0, \Upsilon))$:

$$tp_\pi(h) := \mathsf{H}_{e,v}; \quad h_\pi[0] := (\mathscr{F}\mathscr{R}\mathscr{H}\pi)\mathsf{W}_\Sigma h_1[e/v];$$

  where like in (4.4), $\Sigma := (e, v, \Upsilon_{\leqslant r})$ (cf. Definition 4.2).

(5) $h \equiv \mathsf{W}_\Sigma h_0$. Let $\mathscr{T} := tp_\pi(h_0)$.
  (a) $\mathscr{T} \notin \{\mathsf{Cut}_e, \mathsf{CutFr}_e\}$ or $e \notin dom(\Sigma)$:

$$tp_\pi(h) := tp_\pi(h_0); \quad h_\pi[i] := \mathsf{W}_\Sigma(h_{0\pi}[i])$$

(b) $\mathscr{T} \in \{\mathsf{Cut}_e, \mathsf{CutFr}_e\}$ and $(e,u) \in \Sigma$, $u \in \{?, ?^0\} \cup \mathbb{N}$:

$$tp_\pi(h) := tp(h^-);\ \ h_\pi[i] := h^-_\pi[i],$$

where

$$h^- := \begin{cases} \mathsf{W}_\Sigma h_{0\pi}[0] & \text{if } u = ?, ?^0 \\ \mathsf{W}_\Sigma h_{0\pi}[1][e/u] & \text{if } u \in \mathbb{N} \end{cases}$$

For example, if $\Sigma \equiv (e,n), \Sigma'$, then

$$\dfrac{\dfrac{h_{0\pi}[0] : (e, ?^0), \Gamma \quad h_{0\pi}[1] : (e, +), \Gamma}{h_0 : \Gamma}\ \mathsf{Cut}_e}{\Sigma * \Gamma}\ \mathsf{W}_\Sigma$$

$$\dfrac{h_{0\pi}[1][e/n] : (e,n), \Gamma}{h' : (e,n), \Sigma' * \Gamma}\ \mathsf{W}_\Sigma.$$

Comment: The number of recursion steps in 5b is bounded by $lth(\Sigma)$: it is the length of the branch leading to the derivation d' described at the end of Definition 4.4.

Lemma 7.1. *If* Θ *is* $\mathsf{AxA}, e \notin dom(\Theta)$, *then* $(e,+), \Theta$ is AxA.

Proof. The new sequent cannot become another kind of an axiom, since $(e,+)$ cannot be used in computations and does not contribute to $\mathscr{F}(\Theta)$. □

7.1. Correctness of $tp_\pi(h), o(h)$

Theorem 7.2. *Let* $h \in PA\varepsilon*$, $(h,\pi) \in \rho \in \{r, r^+\}$, *where* $r > 0$, *sequent* $\Gamma(h)$ *is* +*-free and* $h \notin \{\mathsf{AxF}, \mathsf{AxS}, \mathsf{AxH}\}$. *Then* (7.1,7.2), (7.4) *are satisfied and*

$$o(h_\pi[i], \pi tp_\pi(h) i) < o(h, \pi) \quad for\ i \in |tp_\pi(h)| \tag{7.5}$$

Proof. Induction on h. Below we often apply Lemma 5.1 when $\mathscr{T}$ is one of $tp_\pi(h)$, $tp_{\pi_i}(h[i])$ and get around most of the restrictions in that lemma, since $\mathscr{T}$ is an inference symbol of εPA, hence $\mathscr{T} \notin \{\mathsf{R}_e, \mathsf{E}_r, \mathsf{W}_\Sigma, \mathsf{D}_r\}$. In computations we often leave out paths and corresponding subscripts.

(1) $h \equiv \mathsf{AxX}(\Theta)$. Consider $\mathsf{X} \equiv \mathsf{A}$ when the axiom is expanded into a Cut_e. Since $(e, ?^0), \Theta$ and $(e,+), \Theta$ extend Θ, we have

$$\mathsf{AxX}((e, ?^0), \Theta) \in \rho,\ \ \mathsf{AxA}((e,+), \Theta) \in \rho.$$

Since the term e in (4.1) belongs to the set used for computation of r_0, $rk(e) \leqslant r_0 < r$. Hence by Definition 5.3

$$h' \equiv \mathsf{Cut}_e \mathsf{AxX}((e, ?^0), \Theta) \mathsf{AxA}((e,+), \Theta) \in \rho.$$

(2) $h \equiv \mathscr{T} h_0 h_1$ or $h \equiv \mathscr{T} h_0$ with $\mathscr{T} \in \{\mathsf{Cut}_e, \mathsf{CutFr}_e, \mathsf{Fr}_e, \mathsf{H}_{e,v}\}$: Here $h' \equiv h$.

(3) $h \equiv \mathsf{E}_r h_0 \pi$. Then $h \in r^+$, $h_0 \in r+1$.

(a) $tp_\pi(h_0) \equiv \mathsf{Cut}_e$ with $rk(e) = r$. We have

$$tp_\pi(h) := \mathsf{CutFr}_e;\ h_\pi[1] := \mathsf{E}_r h_{0\pi}[1],\ h_\pi[0] := \mathsf{R}_e(\mathsf{E}_r h_{0\pi}[0]) h_\pi[1];$$

Further, $h'_0 = \mathsf{Cut}_e h_0[0] h_0[1] \in r+1$ (IH,(7.4)), which implies $h_0[0]$, $h_0[1] \in r+1$ (Lemma 5.1). This implies in turn by Definition 5.3:

$$\mathsf{E}_r h_0[0], \mathsf{E}_r h_0[1] \in r^+;\ h[1] \in r^+;\ h[0] \in r^+;\ h' = \mathsf{CutFr}_e h[0] h[1] \in r^+$$

$$o(h) = \omega^{o(h_0)} \quad o(h_\pi[1]) = \omega^{o(h_{0\pi}[1])} < \omega^{o(h_0)} \text{ by IH}$$

$$o(\mathsf{E}_r h_{0\pi}[0]) = \omega^{o(h_{0\pi}[0])} < \omega^{o(h_0)} \text{ by IH}$$

$$o(h_\pi[0]) = o(h_\pi[1]) + lth(\pi) + 1 + o(\mathsf{E}_r h_{0\pi}[0]) < \omega^{o(h_0)}$$

since all terms are less than $\omega^{o(h_0)}$ which is a principal number for addition.

(b) $tp(h_0) \neq \mathsf{Cut}_e$ with $rk(e) = r$. With $\mathscr{T} := tp(h_0)$ we have $h'_0 \equiv \mathscr{T}\{h_0[i]\}_i \in r+1$ (IH, (7.4)), which implies $h_0[i] \in r+1$ (Lemma 5.1.1), $h[i] \equiv \mathsf{E}_r h_0[i] \in r^+$ (Definition 5.3) and $h' \equiv \mathscr{T}\{\mathsf{E}_r h_0[i]\} \in r^+$ by Lemma 5.1.2. Exceptions of Lemma 5.1.2 do not apply since $\mathscr{T} \not\equiv \mathsf{CutFr}_f$ (otherwise $h'_0 \in s^+ \neq r+1$) and if $\mathscr{T} \equiv \mathsf{Cut}_f$, then $rk(f) < r$ by the condition defining the present subcase.

$$o(h[i]) = \omega^{o(h_0[i])} < \omega^{o(h_0)} \text{ (IH)} = o(h)$$

(4) $h \equiv \mathsf{R}_e h_0 h_1$, $\Gamma(h)$ is +-free.

$$\frac{h_0 :\ (e, ?^0), \Upsilon \quad h_1 : (e, +)\Theta}{\mathsf{R}_e h_0 h_1 :\ (e, ?), \Upsilon * \Theta},$$

where $\Upsilon * \Theta$ is +-free, $h, h_0, h_1 \in r^+$ for $r = rk(e)$. Consider the same subcases as in the Definition 7.1 clause 4. Let $\mathscr{T} := tp_\pi(h_0)$. We have

$$o(h) \equiv o(h_1) + lth(\pi) + 1 + o(h_0).$$

(i) $\mathscr{T} \equiv \mathsf{AxX}$, $\mathsf{X} \neq \mathsf{H}_{e,v}$ for any v. Then $h' \equiv \mathsf{AxX}(\Upsilon * \Theta) \in \rho$ for any ρ.

(ii) $\mathscr{T} \neq \mathsf{Ax} : tp_\pi(h) \equiv \mathscr{T}$; $h_\pi[i] \equiv \mathsf{R}_e h_{0\pi}[i] h_1$;
$h'_0 \equiv \mathscr{T}\{h_0[i]\}_i \in r^+$(IH,(7.4)) implies $h_0[i] \in r^+$ (Lemma 5.1.1), $\mathsf{R}_e h_0[i] h_1 \in r^+$ (Definition 5.3), $h' \equiv \mathscr{T}\{\mathsf{R}_e h_0[i] h_1\}_i \in r^+$ (Lemma 5.1.2).

$$o(h) \equiv o(h_1) + lth(\pi) + 1 + o(h_0) > \text{ by IH},$$

$$o(h_1) + lth(\pi) + 1 + o(h_{0\pi}[i]) = o(h_\pi[i]).$$

(iii) $\mathscr{T} \equiv \mathsf{AxH}_{e,v}((e, ?^0, \Upsilon)) : tp_\pi(h) \equiv \mathsf{H}_{e,v}$;

$$h_\pi[0] \equiv (\mathscr{F}\mathscr{R}\mathscr{H}\pi) \mathsf{W}_\Sigma h_1[e/v]$$

$$o(h) \equiv o(h_1) + lth(\pi) + 1 + o(h_0) \geqslant \text{ by IH}$$

$$o(h_1[e/v]) + 1 + lth(\mathscr{F}\mathscr{R}\mathscr{H}\pi) + 1 = o(h_\pi[0]) + 1,$$

where like in (4.4), $\Sigma \equiv (e, v, \Upsilon_{\leqslant r})$ and $\mathscr{F}\mathscr{R}\mathscr{H}\pi$ is a sequence of Fr, H-inferences of rank $> r$ (Lemma 4.3).

$$
\begin{array}{ccc}
\dfrac{\begin{array}{cc}\begin{array}{c}\mathsf{AxH}\\ (e,?),\Upsilon \\ \vdots \kappa \\ (e,?^0),\Theta\end{array} & \begin{array}{c}\vdots\\ \ldots d_v : (e,v)\Theta \ldots\end{array}\end{array}}{d:\ \Theta} & \quad & \dfrac{\begin{array}{c}\vdots\ \mathscr{W}_\Sigma d_v \\ (e,v)\Upsilon_{\leqslant r} * \Theta \\ \vdots\ \mathscr{F}\mathscr{R}\mathscr{H}\pi \\ (e,v)\Upsilon_{\leqslant r} * \Theta_{\leqslant r}\end{array}}{(e,?)\Upsilon * \Theta}\, H_{e,v} \\
\vdots \pi & & \vdots \pi\kappa \\
\emptyset & & \emptyset
\end{array}
$$

$h_1 \in r^+$, $h_1[e/v] \in r^+$, $\mathsf{W}_\Sigma h_1[e/v] \in r^+$, (Definition 5.1), $h[0] \in r^+$ (Definition 5.1), since $\mathscr{F}\mathscr{R}\mathscr{H}\pi$ is a sequence of Fr, H-inferences of rank $> r$), $h' \equiv H_{e,v}h[0] \in r^+$ (Definition 5.3).

$h \equiv \mathsf{W}_\Sigma h_0$. Let $\mathscr{T} := tp_\pi(h_0) \neq \mathsf{Ax}$.

$(h_0, \pi') \in r^+$ for some π', $\Sigma t \geqslant r$, $\Sigma \leqslant r$ is a correct +-free sequent, $\Sigma * \Gamma(h_0)$ is defined, $(\Sigma f)_{\geqslant r} \subseteq \Gamma(h_0)$.

(5) (a) $\mathscr{T} \not\equiv \mathsf{Cut}_e, \mathsf{CutFr}_e$ or $e \notin dom(\Sigma): tp_\pi(h) = \mathscr{T}$; $h_\pi[i] = \mathsf{W}_\Sigma(h_0[i])$. We have $\rho = r^+, h_0 \in r^+$ (Definition 5.3), $h_0' \equiv \mathscr{T}\{h_0[i]\} \in r^+$ (IH), $h_0[i] \in r^+$ (Lemma 5.1.1), $h[i] \equiv \mathsf{W}_\Sigma(h_0[i]) \in r^+$ (Definition 5.3), $h' \equiv \mathscr{T}\{h[i]\} \in r^+$ (Lemma 5.1.2).

$$o(h) = o(h_0) + 1 > o(h_{0\pi}[i]) + 1 = o(\mathsf{W}_\Sigma(h_{0\pi}[i]) = o(h_\pi[i]).$$

(b) $\mathscr{T} \equiv \mathsf{Cut}_e, \mathsf{CutFr}_e$, $(e, u) \in \Sigma$, $u \in \{?, ?^0\} \cup \mathbb{N}$. Here
If $u \in \{?^0, ?\}$, then $u \equiv ?^0$ for $\mathscr{T} \equiv \mathsf{Cut}_e$ and $u \equiv ?$ for $\mathscr{T} \equiv \mathsf{CutFr}_e$ (cf. Definition 4.4).
$tp_\pi(h) := tp(h^-)$; $h_\pi[i] := h^-[i]$ where

$$h^- := \begin{cases} \mathsf{W}_\Sigma h_{0\pi}[0] & \text{if } u = ?, ?^0, \\ \mathsf{W}_\Sigma h_{0\pi}[1][e/u] & \text{if } u \in \mathbb{N}, \end{cases}$$

so $h' \equiv (h^-)'$, and $(h^-)' \in r^+$ by the same argument as in the previous subcase.

$$\begin{aligned} o(h_\pi[i]) &= o(h^-[i]) = o(\mathsf{W}_\Sigma h_{0\pi}[i][e/u]) = o(h_{0\pi}[i][e/u]) + 1 \\ &\leqslant o(h_{0\pi}[i] + 1 < o(h_0) + 1 = o(h) \qquad \square \end{aligned}$$

8. Ackermann-style termination proof

Theorem 8.1. *The* H*-process terminates*

Proof. Let

$$h := \mathsf{D}_1\mathsf{E}_1 \ldots \mathsf{D}_n\mathsf{E}_n\mathsf{AxA}(\emptyset),$$

where n is the maximal rank of ε-terms in Cr. Denoting

$$h^r := \mathsf{E}_r\mathsf{D}_{r+1}\mathsf{E}_{r+1}\dots\mathsf{D}_n\mathsf{E}_n\mathsf{AxA}(\emptyset)$$

we have: $(\mathsf{AxA}(\emptyset),\emptyset)\in n+1$; $(h^r,\emptyset)\in r^+$; $(\mathsf{D}_r h^r,\emptyset)\in r$. Hence $h\equiv \mathsf{D}_1 h^1$ describes a cut-free form $|d|$ of the original derivation d of the empty sequent: since $(h_0,\emptyset)\in 1$, $|d|\equiv(h_0,\emptyset)^\infty$ is a 1-derivation in $PA\varepsilon$, that is a cut-free derivation of $\emptyset$ by Theorem 6.1. Put

$$\pi_0 := \emptyset;\ \ h_{n+1} := h_{\pi_n}[0];\ \ \pi_{n+1} := (\pi_n\, tp(h_n,\pi_n 0))$$

or, dropping some arguments, $h_n\equiv h_0[0]\dots[0]$, n times. According to relations (7.1), (7.2) (cf. Theorem 6.1 and Lemma 6.9 of [11]), a cut-free derivation $|d|$ of the empty sequent is a protocol of a terminating H-process. More precisely, the top sequent of $|d|$ is an axiom AxS, and all other inferences are of the kind Fr or H.

By (7.5), for every n, if h_n is not yet an axiom, then

$$o(h_n) > o(h_{n+1}). \tag{8.1}$$

Hence (by induction on ε_0) the sequence h_n terminates in an axiom AxS, that is in a solution. □

Note: Our formulation of εPA and $PA\varepsilon *$ allow "irrelevant" applications of Cut, CutFr, Fr that do not "compute" any subterm in $Cr\cup\mathscr{CR}$. (It is possible they can accelerate an H-process). However our cut elimination transformations introduce such redundancies only to preserve periodicity of the H-process. This provides a bound for the number of Fr between any two applications H,H' of H-rules in a normal derivation: this is the number of Fr needed to make H' applicable after H plus the total number of Fr in the whole H-process before H. In view of this bound on the number of consecutive Fr-inferences, there is a primitive recursive function providing the numbers k_i, $i=1,2,\dots$ of the premises of H-inferences in this sequence, and it is possible to get rid of Fr. This is not necessary, since ordinals strictly decrease at all rules including Fr.

9. Reductions

Let us list reductions $h\mapsto h'$ with paths dropped.

(1) $\mathsf{AxA}(\Theta)\mapsto \mathsf{Cut}_e\mathsf{AxX}((e,?^0),\Theta)\mathsf{AxA}((e,+),\Theta)$.

(2) (a) $\mathsf{E}_r\mathsf{Cut}_e h_0 h_1\mapsto \mathsf{CutFr}_e(\mathsf{R}_e(\mathsf{E}_r h_0)(\mathsf{E}_r h_1))\mathsf{E}_r h_1$,

(b) $\mathsf{E}_r\mathscr{T}\{h_i\}\mapsto\mathscr{T}\{\mathsf{E}_r h_i\}$.

(3) (a) $\mathsf{R}_e\mathsf{AxX}((e,?^0),\Upsilon)h_1\mapsto \mathsf{AxX}((e,?),\Upsilon*\Gamma(h_1))$, if $X\neq \mathsf{H}_{e,v}$,

(b) $\mathsf{R}_e\mathscr{T}\{h_i\}h'\mapsto\mathscr{T}\{\mathsf{R}_e h_i h'\}$, $Ax\neq\mathscr{T}\in\varepsilon PA$,

(c) $\mathsf{R}_e\mathsf{AxH}_{e,v}((e,?^0),\Upsilon)h_1\mapsto \mathsf{H}_{e,v}\mathscr{FRH}\pi\mathsf{W}_\Sigma h_1[e/v]$,
where $\Sigma := (e,v)$, $\Upsilon_{\leqslant r}$.

(4) (a) If $\mathscr{T} \notin \{\mathsf{Cut}_e, \mathsf{CutFr}_e\}$ or $e \notin dom(\Sigma)$, then $\mathsf{W}_\Sigma \mathscr{T}\{h_i\} \mapsto \mathscr{T}\{\mathsf{W}_\Sigma h_i\}$.

(b) If $\mathscr{T} \in \{\mathsf{Cut}_e, \mathsf{CutFr}_e\}$, $e \in dom(\Sigma)$, then
$\mathsf{W}_\Sigma \mathscr{T} h_0 h_1 \mapsto \mathsf{W}_\Sigma h_0$, if $(e, ?') \in \Sigma$,
$\mathsf{W}_\Sigma \mathscr{T} h_0 h_1 \mapsto \mathsf{W}_\Sigma h_1[e/n]$, if $(e, n) \in \Sigma$.

It is in 3c that $(e, +)$ in $\Gamma(h_1)$ is replaced by (e, v), and $h_1[e/v]$ is transferred to the main branch.

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