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MORE GAME-THEORETIC PROPERTIES OF BOOLEAN ALGEBRAS

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The following infinite game $\mathcal{G}$ was investigated in [5]: Let B be a Boolean algebra. Two players, White and Black, take turns to choose successively a sequence

$$w_1 \geq b_1 \geq w_2 \geq b_2 \geq \cdots \geq w_n \geq b_n \geq \cdots \tag{1}$$

of nonzero elements of B. White wins the play (1) if and only if the sequence (1) converges to zero. This game is related to distributivity laws for Boolean algebras. It was proved in [5] that B is $(\aleph_0, \infty)$-distributive if and only if White does not have a winning strategy in $\mathcal{G}$.

In the present paper we introduce several other games and show how they are related to various other properties of Boolean algebras. We shall also illustrate our discussion with several examples.

The cut & choose games

Let B be a Boolean algebra. The game $\mathcal{G}_{c\&c}$ is played as follows: At the beginning, White chooses some nonzero $a \in B$. In the first move, White cuts a into two disjoint pieces a_1^0 and a_1^1; let $W_1 = \{a_1^0, a_1^1\}$ where $a_1^0 \neq 0 \neq a_1^1$, $a_1^0 + a_1^1 = a$, and $a_1^0 \cdot a_1^1 = 0$. Then Black chooses either a_1^0 or a_1^1. Then White cuts a again: $W_2 = \{a_2^0, a_2^1\}$ such that $a_2^0 \neq 0 \neq a_2^1$, $a_2^0 + a_2^1 = a$, and $a_2^0 \cdot a_2^1 = 0$, and Black chooses either a_2^0 or a_2^1. The game continues in this fashion ad infinitum: At move n, White cuts and Black chooses. Let $W_n = \{a_n^0, a_n^1\}$ where $a_n^0 \neq 0 \neq a_n^1$, $a_n^0 + a_n^1 = a$, and $a_n^0 \cdot a_n^1 = 0$; let $a_a^{f(n)}$ be the element of W_n chosen by Black on his nth move. White wins the play

$$W_1, a_1^{f(1)}, W_2, a_2^{f(2)}, \ldots, W_n, a_n^{f(n)}, \ldots \tag{2}$$

if and only if

$$\prod_{n=1}^{\infty} a_n^{f(n)} = 0.$$

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Notice that there is a somewhat simpler way of describing the game $\mathcal{G}_{c\&c}$: After starting the game with $a \neq 0$ and cutting it, White always cuts Black's previous choice (rather than a itself). Black chooses an element of White's partition. I am using the less intuitive description merely because it lends itself to generalizations: viz. the games $\mathcal{G}_1, \mathcal{G}_\omega$ and $\mathcal{G}_{\text{fin}}$.

Cut and choose games, played with sets (rather than on a Boolean algebra) have been investigated before; see Mycielski [10] or Ulam [11]. Such games have been studied extensively by Galvin and others.

We recall the distributivity laws for Boolean algebras [9]: Let κ and λ be cardinal numbers. A Boolean algebra B is (κ, λ)-*distributive* if it satisfies the distributive law

$$\prod_{\alpha<\kappa} \sum_{\beta<\lambda} a_{\alpha\beta} = \sum_{f:\kappa\to\lambda} \prod_{\alpha<\kappa} a_{\alpha,f(\alpha)}. \tag{3}$$

B is (κ, ∞)-*distributive* if it is (κ, λ)-distributive for all λ. An equivalent formulation of (κ, λ)-distributivity is:

(4) For every nonzero $a \in B$ and every collection $\{W_\alpha : \alpha < \kappa\}$ of partitions of a such that $|W_\alpha| \leq \lambda$ for each α, there exists a nonzero $b \leq a$ such that for each α there is $a_\alpha \in W_\alpha$ with $b \leq a_\alpha$.

The distributivity laws are of particular interest in the theory of forcing [6, p. 263]: A complete Boolean algebra B is (κ, λ)-distributive if and only if every generic extension by B has the property that every function $f : \kappa \to \lambda$ belongs to the ground model.

Theorem 1. *A Boolean algebra b is* $(\aleph_0, 2)$-*distributive if and only if White does not have a winning strategy in the game* $\mathcal{G}_{c\&c}$.

(Note that as a corollary, White has a winning strategy in $\mathcal{G}_{c\&c}$ if and only if White has a winning strategy in $\mathcal{G}$.)

Proof. First assume that B is not $(\aleph_0, 2)$-distributive. It follows that there exists a nonzero element $a \in B$ and a countable collection of partitions $\{W_n : n = 1, 2, \ldots\}$ of size 2 such that $\prod_{n=1}^{\infty} a_n = 0$ for every sequence $\{a_n : n = 1, 2, \ldots\}$ such that $a_n \in W_n$ for all n. There is an obvious winning strategy for White in the game $\mathcal{G}_{c\&c}$. As his nth move, White plays W_n. Thus if B is not $(\aleph_0, 2)$-distributive, White has a winning strategy in the game $\mathcal{G}_{c\&c}$.

Conversely, let us assume White has a winning strategy σ in the game $\mathcal{G}_{c\&c}$. Let $a \neq 0$ be the element of B chosen by White. We shall construct a countable collection of partitions $\{P_n : n = 1, 2, \ldots\}$ of a, of size 2, such that $\prod_{n=1}^{\infty} b_n = 0$ whenever $b_n \in P_n$ for all n. Let W be the first move of White by σ; let $W = \{a_0, a_1\}$. We let $b_1^0 = a_0$, $b_1^1 = a_1$ and $P_1 = \{b_1^0, b_1^1\}$. For $i = 0, 1$ let W_i be the second move of White by σ in response to Black playing a_i; let $W_i = \{a_{i0}, a_{i1}\}$. We let P_2 be the partition that results from partitioning a_0 by W_0 and a_1 by W_1: $P_2 = \{b_2^0, b_2^1\}$ where

$$b_2^0 = a_0 \cdot a_{00} + a_1 \cdot a_{10}, \qquad b_2^1 = a_0 \cdot a_{01} + a_1 \cdot a_{11}.$$

By induction on the length of s, we define a_s and W_s for all finite 0–1-sequences s: Let $s=\langle s(1),\ldots,s(n)\rangle$. We denote $W_s=\{a_{s^\frown 0},a_{s^\frown 1}\}$ the $(n+1)$st move of White by σ in the position

$$W,\ a_{s(1)},\ W_{s(1)},\ a_{\langle s(1),s(2)\rangle},\ W_{\langle s(1),s(2)\rangle},\ldots,\ a_s.$$

For each n, we let $P_{n+1}=\{b^0_{n+1},b^1_{n+1}\}$ where

$$b^0_{n+1}=\sum_s a_s\cdot a_{s^\frown 0},$$

$$b^1_{n+1}=\sum_s a_s\cdot a_{s^\frown 1},$$

the sums being taken over all 0–1-sequences of length n. Now if $\langle\varepsilon(1),\varepsilon(2),\ldots,\varepsilon(n),\ldots\rangle$ is any infinite sequence of 0's and 1's, we have

$$\prod_{n=1}^{\infty} b_n^{\varepsilon(n)}=\prod_{n=1}^{\infty} a_{\langle\varepsilon(1),\ldots,\varepsilon(n)\rangle}$$

and the right-hand side is zero because the $a_{\langle\varepsilon(0),\ldots,\varepsilon(n)\rangle}$'s are Black's moves in a play in which White plays by his winning strategy σ. Thus the partitions P_n of a witness that B is not $(\aleph_0,2)$-distributive. □

Before I proceed further, I shall present two examples. Note that if B has an $\aleph_0$-closed dense subset P, then Black has a winning strategy in $\mathscr{G}_{c\&c}$: As his nth move, when White plays W_n, Black picks $a_n\in W_n$ and $p_n\in P$ such that $p_n\leqslant a_n$ and $p_n\leqslant p_{n-1}$. Thus

$$\begin{gathered}\aleph_0\text{-closed dense subset}\Rightarrow\text{Black wins }\mathscr{G}_{c\&c}\Rightarrow\text{White does not win}\\ \mathscr{G}_{c\&c}\Leftrightarrow(\aleph_0,2)\text{-distributive.}\end{gathered}\tag{5}$$

The following examples show that the first two implications cannot be reversed.

Example 1. *Prikry forcing.* (Black wins $\mathscr{G}_{c\&c}$.)

Let κ be a measurable cardinal and let D be a normal measure on κ. P is the following notion of forcing (cf. [8]):

A forcing condition is a pair (s,A) where s is a finite increasing sequence of ordinals below κ and $A\in D$. A condition (t,B) is stronger than (s,A) just in case t extends s, $B\subseteq A$, and range$(t-s)\subseteq A$. Let B be the complete Boolean algebra corresponding to P.

As forcing P adjoins a new countable subset of κ, B is not $(\aleph_0,\kappa)$-distributive and so does not have a dense $\aleph_0$-closed subset. I claim that Black has a winning strategy in the game $\mathscr{G}_{c\&c}$.

We recall the following basic fact about Prikry forcing [8]: If (s,A) is a condition and σ is a sentence of the forcing language, then there is a set A' of measure 1, $A'\subseteq A$, such that the condition (s,A') decides σ. This suggests a winning strategy for Black: Given $a\in B$, let $p=(s,A)$ be such that $p\leqslant a$. As his

nth move, when White plays a partition W_n of a, let Black pick $a_n \in W_n$ and $A_n \in D$ such that $A_n \subseteq A_{n-1}$ and $(s, A_n) \leq a_n$. (If $n=1$, $a_0 = a$ and $A_0 = A$.) This is a winning strategy since

$$\prod_{n=1}^{\infty} a_n \geq \left(s, \bigcap_{n=1}^{\infty} A_n\right). \qquad \square$$

Note that in the game $\mathscr{G}$ played on Prikry forcing, White has a winning strategy (White can win by systematically increasing the size of the finite part s). Thus a winning strategy for Black in $\mathscr{G}_{c\&c}$ does not guarantee a winning strategy for Black in $\mathscr{G}$, although the opposite is true (see Theorem 3).

Example 2. *A Suslin algebra.* (Neither player wins $\mathscr{G}_{c\&c}$.)

A Suslin algebra is an atomless $\aleph_1$-saturated $(\aleph_0, 2)$-distributive Boolean algebra. If T is a Suslin tree, then the complete Boolean algebra corresponding to (the upside down) T is a Suslin algebra. If B is a Suslin algebra, then by Theorem 1, White does not have a winning strategy in $\mathscr{G}_{c\&c}$. I shall sketch a construction (using Jensen's principle $\diamondsuit$) of a Suslin algebra B such that Black does not have a winning strategy either.

Thus assume that $\diamondsuit$ holds (for its formulation, see e.g. [6]). Our construction follows closely Jensen's construction of a Suslin tree, as described in [6]. We assume that the reader is familiar with the construction. Let $\{S_\alpha : \alpha < \omega_1\}$ be a $\diamondsuit$-sequence; $S_\alpha \subseteq \alpha$ for all α. We construct a tree T by levels; at stage α, we have constructed T_α, the first α levels of T. If S_α is a maximal antichain in T_α, then we destroy it in the usual fashion. We amend the construction if the following happens (at a limit α): We say that a function f is an *α-good partial strategy for Black* if

(6) (a) The arguments of f are finite sequences $\langle W_1, \ldots, W_n \rangle$ such that each W_i is a partition of some level $\beta < \alpha$ of T_α into two disjoint sets, and $f(\langle W_1, \ldots, W_n \rangle) \in W_n$.

(b) If $\langle W_1, \ldots, W_n \rangle \in \mathrm{dom}(f)$, then $\langle W_1, \ldots, W_k \rangle \in \mathrm{dom}(f)$ for every $k < n$.

(c) There exists a sequence $\{W_n : n = 1, 2, \ldots\}$ such that for each n, $\langle W_1, \ldots, W_n \rangle \in \mathrm{dom}(f)$, and there is a unique α-branch b in T_α with the property that for each n, b goes through $f(\langle W_1, \ldots, W_n \rangle)$.

Now suppose that S_α codes an α-good partial strategy for Black. (The coding method is unspecified but fixed throughout.) Let $\{W_n : n = 1, 2, \ldots\}$ and b be as in (6c). We construct the αth level of T so that b does not have an extension in T.

The resulting tree T is a Suslin tree. Let B be the corresponding Suslin algebra. I claim that Black does not have a winning strategy in $\mathscr{G}_{c\&c}$. Let σ be a strategy for Black. Every partition $W = \{a_0, a_1\}$ can be represented as a partition of some level β of the tree T into two disjoint sets.

Let $S \subseteq \omega_1$ code σ. It can be verified that there is a closed unbounded set C of α's such that $S \cap \alpha$ codes an α-good partial strategy $f \subseteq \sigma$ for Black. By $\diamondsuit$, there is $\alpha \in C$ such that $S \cap \alpha = S_\alpha$. Then it follows that σ cannot be a winning strategy: Let $\{W_n : n = 1, 2, \ldots\}$ and b be the sequence and the α-branch we eliminated in the construction of level α. If White plays $W_1, W_2, \ldots, W_n, \ldots$ and if Black plays by σ, then Black loses; thus σ is not a winning strategy. $\square$

Cutting into more pieces

We now change the game by allowing White to cut into more than two pieces. Let κ be a cardinal number, and let B be a given Boolean algebra. The game $\mathscr{G}_1(\kappa)$ (the subscript signifies that Black chooses one piece) is played as follows: White chooses a nonzero element $a \in B$, and as his nth move, he plays a partition W_n of a such that $|W_n| \leqslant \kappa$. Black's nth move consists of choosing one $a_n \in W_n$. White wins the play

$$W_1, a_1, W_2, a_2, \ldots, W_n, a_n, \ldots \tag{7}$$

just in case

$$\prod_{n=1}^{\infty} a_n = 0. \tag{8}$$

Thus the game $\mathscr{G}_{c\&c}$ is a special case of this game, indeed $\mathscr{G}_{c\&c}$ is just $\mathscr{G}_1(2)$. We also consider the game $\mathscr{G}_1 = \mathscr{G}_1(\infty)$ in which White cuts some $a \in B$ by partitions W_n (without limitation on their size) and Black chooses $a_n \in W_n$. Again, White wins (7) if and only if $\prod_{n=1}^{\infty} a_n = 0$. Just as $\mathscr{G}_{c\&c}$ is related to distributive laws, so is $\mathscr{G}_1(\kappa)$:

Theorem 2. *A Boolean algebra B is $(\aleph_0, \kappa)$-distributive if and only if White does not have a winning strategy in the game $\mathscr{G}_1(\kappa)$..*

Proof. If B is not $(\aleph_0, \kappa)$-distributive, then White has a winning strategy by playing $a, \{W_n : n = 1, 2, \ldots\}$, a counterexample to $(\aleph_0, \kappa)$-distributivity (just as in the proof of Theorem 1).

Conversely, let White have a winning strategy σ in $\mathscr{G}_1(\kappa)$, for cutting $a \in B$. If κ is finite, then the proof is exactly as in Theorem 1: so let κ be infinite.

We shall construct partitions P_n, $n = 1, 2, \ldots$ for a, each P_n of size $\leqslant\kappa$ which will constitute a counterexample to $(\aleph_0, \kappa)$-distributivity. First let $P_1 = \sigma(\langle\ \rangle)$ be the first move of White. For each $x_1 \in P_1$, let

$$W_2(x_1) = \{x_1 \cdot z : z \in \sigma(\langle x_1 \rangle)\}$$

and let

$$P_2 = \bigcup \{W_2(x_1) : x_1 \in \sigma(\langle\ \rangle)\}.$$

Each $W_2(x_1)$ is a partition of x_1, and P_2 is a partition of a.

Clearly, $|P_2| \leqslant \kappa$. Then for each $x_1 \in \sigma(\langle\ \rangle)$ and each $x_2 \in \sigma(\langle x_1 \rangle)$, let

$$W_3(x_1, x_2) = \{x_1 \cdot x_2 \cdot z : z \in \sigma(\langle x_1, x_2 \rangle)\}$$

and

$$P_3 = \bigcup \{W_3(x_1, x_2) : x_1 \in \sigma(\langle\ \rangle), x_2 \in \sigma(\langle x_1 \rangle)\}.$$

Each $W_3(x_1, x_2)$ is a partition of $x_1 \cdot x_2$, and P_3 is a partition of a, $|P_3| \leqslant \kappa$. We continue in this manner and define $P_4, P_5, \ldots$. Now if $\{a_n : n = 1, 2, \ldots\}$ is a sequence such that $a_n \in P_n$ for each n, there exists a sequence $\{x_n : n = 1, 2, \ldots\}$ such that for each n, $x_n \in \sigma(\langle x_1, \ldots, x_{n-1} \rangle)$ and $a_n = x_1 \cdot \ldots \cdot x_n$. Since σ is a winning strategy for White, we have $\prod_{n=1}^{\infty} a_n = 0$ and it follows that B is not $(\aleph_0, \kappa)$-distributive. □

Corollary. *A Boolean algebra B is $(\aleph_0, \infty)$-distributive if and only if White does not have a winning strategy in the game $\mathscr{G}_1$.*

I should remark at this point that the existence of a winning strategy (for either player) in the game $\mathscr{G}_1$ (but not in $\mathscr{G}_1(\kappa)$ if $\kappa < \mathrm{sat}(B)$) is an invariant for the completion of B. In fact, if P is an arbitrary partially ordered set, we can define the game $\mathscr{G}_1 = \mathscr{G}_1(P)$ played on P in the obvious way. And we have:

Proposition. *Let P be a dense subset of a Boolean algebra B. Then White (Black) has a winning strategy in $\mathscr{G}_1(P)$ if and only if White (Black) has a winning strategy in $\mathscr{G}_1(B)$.*

I leave the easy proof to the reader.

Let us recall that $\mathscr{G}$ is the game from [5] described in the introductory paragraph. A relation between $\mathscr{G}$ and $\mathscr{G}_1$ is given by the following theorem:

Theorem 3. *For every Boolean algebra B, if Black has a winning strategy in $\mathscr{G}$, then Black has a winning strategy in $\mathscr{G}_1$.*

Proof. Let σ be a winning strategy for Black in $\mathscr{G}$. We shall describe a winning strategy τ for Black in $\mathscr{G}_1$. Let W_1, a partition of some $a_1 \neq 0$, be the first move of White in $\mathscr{G}_1$. Let $b_1 = \sigma(\langle a_1 \rangle)$. There exists $c_1 \in W_1$ such that $b_1 \cdot c_1 \neq 0$. We let $\tau(W_1) = c_1$, and let $a_2 = b_1 \cdot c_1$. Note that $a_1 \geqslant b_1 \geqslant a_2$ and $c_1 \geqslant a_2$. Now let W_2 be the second move of White. Let $b_2 = \sigma(\langle a_1, a_2 \rangle)$. There exists $c_2 \in W_2$ such that $b_2 \cdot c_2 \neq 0$. We let $\tau(W_1, W_2) = c_2$, and let $a_3 = b_2 \cdot c_2$. We have $a_2 \geqslant b_2 \geqslant a_3$ and $c_2 \geqslant a_3$. We continue this ad infinitum: When White plays W_n, we choose $c_n = \tau(W_1, \ldots, W_n)$ such that $a_{n+1} = b_n \cdot c_n \neq 0$ where $b_n = \sigma(\langle a_1, \ldots, a_n \rangle)$. For each n, we have

$$c_n \geqslant a_{n+1} \geqslant b_{n+1}. \tag{9}$$

It follows that τ is a winning strategy, since for each sequence $\{W_n : n = 1, 2, \ldots\}$

of White's moves, (29) implies that

$$\prod_{n=1}^{\infty} c_n \geq \prod_{n=1}^{\infty} b_n. \tag{10}$$

However, the right-hand side of (10) is nonzero since the b_n are Black's moves against the a_n in $\mathcal{G}$, by his winning strategy σ. Thus the left-hand side of (10) is nonzero and so τ is a winning strategy. □

I shall now interrupt to give an example of a Boolean algebra B for which neither White nor Black has a winning strategy in $\mathcal{G}_1$. I would like to know if there is a B for which Black has a winning strategy in $\mathcal{G}_1$ (or even in $\mathcal{G}$) but which does not have a dense $\aleph_0$-closed subset. We have

$$\begin{aligned}&\text{dense } \aleph_0\text{-closed subset} \Rightarrow \text{Black wins } \mathcal{G} \Rightarrow \text{Black wins } \mathcal{G}_1\\ &\quad \Rightarrow \text{White does not win } \mathcal{G}_1 \Leftrightarrow \text{White does not win } \mathcal{G}\\ &\quad \Leftrightarrow (\aleph_0, \infty)\text{-distributive.}\end{aligned} \tag{11}$$

The third implication cannot be reversed but I don't know about the first two[1].

Example 3. *Shooting a closed unbounded subset through a stationary set.* (Neither player wins $\mathcal{G}_1$.)

Let A be a stationary subset of $\aleph_1$ such that the complement of A is also stationary. Let P be the following notion of forcing (see [2]). A forcing condition p is a countable closed set of ordinals, $p \subseteq A$. A condition q is stronger than p if q is an end extension of p. It is proved in [2] that P is $(\aleph_0, \infty)$-distributive and so White does not have a winning strategy in $\mathcal{G}_1$ by Theorem 2. (In [5] it is proved that Black does not have a winning strategy in $\mathcal{G}$.) In Theorem 7 below, we show that if Black has a winning strategy in $\mathcal{G}_\omega$ which is an easier game for Black than $\mathcal{G}_1$, then stationary sets are preserved in the generic extension. However, forcing with P adjoins a closed unbounded subset of A, hence destroying the stationary set ω_1-A. Thus Black does not have a winning strategy in $\mathcal{G}_1$. □

Choosing finitely many pieces

The next change in the rules is that we allow Black to choose more than one piece (but only a finite number). Let κ be a cardinal number, and let B be a given Boolean algebra. The $\mathcal{G}_{\text{fin}}(\kappa)$ is played as follows: White chooses a nonzero element $a \in B$, and as his nth move, he plays a partition W_n of a such that $|W_n| \leq \kappa$. Black, as his nth move, chooses a finite subset F_n of W_n and lets $a_n = \sum F_n$. White wins the play

$$W_1, F_1, W_2, F_2, \ldots, W_n, F_n, \ldots \tag{12}$$

[1] See Addenda for a discussion about the first implication.

if and only if

$$\prod_{n=1}^{\infty} a_n = 0. \tag{13}$$

We also consider the same $\mathcal{G}_{\text{fin}} = \mathcal{G}_{\text{fin}}(\infty)$ in which White cuts some $a \in B$ by partitions W_n and Black chooses a finite subset $F_n \subseteq W_n$. Again, White wins (12) just in case $\prod_{n=1}^{\infty} (\sum F_n) = 0$.

Note that the game $\mathcal{G}_{\text{fin}}(\kappa)$ is easier for Black than $\mathcal{G}_1(\kappa)$ (and $\mathcal{G}_{\text{fin}}$ is easier for Black than $\mathcal{G}_1$). Also, as was the case with $\mathcal{G}_1$, existence of a winning strategy in $\mathcal{G}_{\text{fin}}$ is an invariant for dense subsets of B.

The game $\mathcal{G}_{\text{fin}}$ is related to weak distributivity of Boolean algebras. Let κ and λ be cardinal numbers and let B be a Boolean algebra. We recall (see [9]) that B is *weakly* (κ, λ)*-distributive* if it satisfies the weak distributive law

$$\prod_{\alpha<\kappa} \sum_{\beta<\lambda} a_{\alpha\beta} = \sum_{f:\kappa \to [\lambda]^{\text{fin}}} \prod_{\alpha<\kappa} \sum_{\beta \in f(\alpha)} a_{\alpha\beta} \tag{14}$$

(where $[\lambda]^{\text{fin}}$ is the set of all finite subsets of λ). B is *weakly* (κ, λ)*-distributive* if it is weakly (κ, λ)-distributive for all λ.

(In the language of the theory of forcing, B is weakly (κ, λ)-distributive iff for every function $f: \kappa \to \lambda$ in the generic extension there is a function $g: \kappa \to [\lambda]^{\text{fin}}$ in the ground model such that $f(\alpha) \in g(\alpha)$ for all $\alpha < \kappa$.)

Theorem 4. *Let B be a Boolean algebra. If White does not have a winning strategy in $\mathcal{G}_{\text{fin}}(\kappa)$, then B is weakly $(\aleph_0, \kappa)$-distributive.*

Corollary. *If White does not have a winning strategy in $\mathcal{G}_{\text{fin}}$,* then *B is weakly $(\aleph_0, \infty)$-distributive.*

Proof. If B is not weakly $(\aleph_0, \kappa)$-distributive, then there is a nonzero $a \in B$ and partitions W_n, $n = 1, 2, \ldots$ of a such that $|W_n| \leq \kappa$ for each n, and $\prod_{n=1}^{\infty} (\sum F_n) = 0$ whenever $\{F_n : n = 1, 2, \ldots\}$ is a sequence of finite sets, $F_n \subseteq W_n$. White has a winning strategy in $\mathcal{G}_{\text{fin}}(\kappa)$, namely he plays these W_n. $\square$

I don't know if the converse of Theorem 4 is true. Anyway, it is time for another example. Consider the following diagram.

$$\begin{tikzcd}
 & \text{Black wins } \mathcal{G}_{\text{fin}} \arrow[dr, Rightarrow] & \\
\text{Black wins } \mathcal{G}_1 \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \text{White does not win } \mathcal{G}_{\text{fin}} \arrow[d, Rightarrow] \\
 & \begin{array}{c}\text{White does not win } \mathcal{G}_1 \\ ((\aleph_0, \infty)\text{-distributive})\end{array} \arrow[ur, Rightarrow] & \text{weakly } (\aleph_0, \infty)\text{-distributive}
\end{tikzcd} \tag{15}$$

I don't know if the last implication can be reversed. No other implication can be reversed however. This is because (a) $(\aleph_0, \infty)$-distributivity does not imply that Black wins $\mathscr{G}_{\text{fin}}$ (P from Example 3 is a counterexample; Black does not win $\mathscr{G}_\omega$ which is easier for Black than $\mathscr{G}_{\text{fin}}$), and (b) the next example exhibits a Boolean algebra B which is not $(\aleph_0, \infty)$-distributive but is such that Black wins $\mathscr{G}_{\text{fin}}$.

Example 4. *A measure algebra.* (Black wins $\mathscr{G}_{\text{fin}}$.)

Let $\mathscr{B}$ be the σ-algebra of all Borel subsets of the interval $[0, 1]$ and let I be the ideal of sets of Lebesgue measure 0. Let $B = \mathscr{B}/I$. B is a complete Boolean algebra and it is well known that B is not $(\aleph_0, 2)$-distributive but is weakly $(\aleph_0, \infty)$-distributive [9]. In fact, Black has a winning strategy in $\mathscr{G}_{\text{fin}}$:

For each $a \in B$, let $m(a)$ be the Lebesgue measure of (any) $A \in B$ such that $a = A/I$; m is a σ-additive measure on B, and $m(a) > 0$ for all $a \neq 0$. Let $a \neq 0$ and let W_n, $n = 1, 2, \ldots$, be partitions of a. The strategy σ for Black is defined as follows: For each n, there is a finite set $F_n \subseteq W_n$ such that

$$m(a - \textstyle\sum F_n) \leqslant \frac{m(a)}{2} \cdot \frac{1}{2^n}. \tag{16}$$

We let $\sigma(\langle W_1, \ldots, W_n \rangle) = F_n$. To see that σ is a winning strategy, note that by (16),

$$m\Big(\prod_{n=1}^{\infty} \sum F_n\Big) \geqslant m(a) - \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{m(a)}{2} = \frac{m(a)}{2}. \qquad \square$$

Choosing countably many pieces

The next change makes it still easier for Black: we allow Black to choose countably many pieces. Let κ be a cardinal number (or $\kappa = \infty$) and let B be a given Boolean algebra. The game $\mathscr{G}_\omega(\kappa)$ is played as follows: White chooses a nonzero element $a \in B$, and as his nth move, he plays a partition W_n of a such that $|W_n| \leqslant \kappa$. Black, as his nth move, chooses a countable subset C_n of W_n. White wins the play

$$W_1, C_1, W_2, C_2, \ldots, W_n, C_n, \ldots \tag{17}$$

just in case

$$\prod_{n=1}^{\infty} \sum C_n = 0. \tag{18}$$

(Note that (18) makes sense even if the infinite sums $\sum C_n$ do not exist. In fact, (18) can be expressed for an arbitrary partially ordered set P in place of B: For every $p \leqslant a$ there is n and there is $q \leqslant p$ such that q is incompatible with every $x \in C_n$.)

Let $\mathcal{G}_\omega$ be the game $\mathcal{G}_\omega(\infty)$. As before, existence of winning strategies in $\mathcal{G}_\omega$ is an invariant for dense subsets of B.

For any set A, let $[A]^{\aleph_0}$ denote the set of all at most countable subsets of A. A Boolean algebra B is $(\kappa, \aleph_0, \lambda)$-*distributive* if it satisfies the following distributive law:

$$\prod_{\alpha<\kappa} \sum_{\beta<\lambda} a_{\alpha\beta} = \sum_{f:\kappa\to[\lambda]^{\aleph_0}} \prod_{\alpha<\kappa} \sum_{\beta\in f(\alpha)} a_{\alpha\beta}. \tag{19}$$

B is $(\kappa, \aleph_0, \infty)$-*distributive* if it is $(\kappa, \aleph_0, \lambda)$-distributive for all λ. The significance of this property is that, equivalently, if f is a function from κ to λ in the generic extension by B, then there is a function $g: \kappa \to [\lambda]^{\aleph_0}$ in the ground model such that $f(\alpha) \in g(\alpha)$ for all $\alpha < \kappa$. In particular, B is $(\aleph_0, \aleph_0, \infty)$-distributive if and only if every countable set of ordinals in the generic extension is included in a countable set from the ground model.

Theorem 5. *Let B be a Boolean algebra. If White does not have a winning strategy in $\mathcal{G}_\omega(\kappa)$, then B is $(\aleph_0, \aleph_0, \kappa)$-distributive.*

Corollary. *If White does not have a winning strategy in $\mathcal{G}_\omega$, then B is $(\aleph_0, \aleph_0, \infty)$-distributive.*

Proof. If B is not $(\aleph_0, \aleph_0, \kappa)$-distributive, then there is $a \neq 0$ and partitions W_n, $n = 1, 2, \ldots$ of a of size $\leqslant\kappa$ such that $\prod_{n=1}^{\infty} (\sum C_n) = 0$ whenever $\{C_n : n = 1, 2, \ldots\}$ is a sequence of countable sets, $C_n \subseteq W_n$. White wins the game $\mathcal{G}_\omega(\kappa)$ by playing these W_n. $\square$

The converse of Theorem 5 is false. I shall return to this in Example 6.

Let us consider now the following implications

$$\begin{array}{ccccc} & & \text{Black wins } \mathcal{G}_\omega & & \\ & \nearrow & & \searrow & \\ \text{Black wins } \mathcal{G}_{\text{fin}} & & & & \text{White does not win } \mathcal{G}_\omega \\ & \searrow & & \nearrow & \\ & & \text{White does not win } \mathcal{G}_{\text{fin}} & & \end{array} \tag{20}$$

None of the implications in (20) can be reversed:

(a) There is B for which White does not win $\mathcal{G}_{\text{fin}}$ and Black does not win $\mathcal{G}_\omega$. Such B is described in Example 3 (adding a closed unbounded subset of a stationary set). Since B is $(\aleph_0, \infty)$-distributive, White does not have a winning strategy in $\mathcal{G}_1$ and hence White does not have a winning strategy in $\mathcal{G}_{\text{fin}}$. Since B destroys a stationary set, it follows from Theorem 7 below that Black does not have a winning strategy in $\mathcal{G}_\omega$.

(b) There is B for which Black wins $\mathcal{G}_\omega$ and White wins $\mathcal{G}_{\text{fin}}$. See the next example.

Example 5. *A countable Boolean algebra.* (White wins $\mathscr{G}_{\text{fin}}$, Black wins $\mathscr{G}_\omega$.)

Let B be the (unique) countable atomless Boolean algebra. It is clear that Black has a winning strategy in $\mathscr{G}_\omega$. On the other hand, it is well known that B is not weakly $(\aleph_0, \aleph_0)$-distributive (see [9]), and so by Theorem 4, White has a winning strategy in $\mathscr{G}_{\text{fin}}$. □

We shall now investigate the relation of the game $\mathscr{G}_\omega$ to two other properties of Boolean algebras that have been recently considered in the theory of forcing.

Axiom A (Baumgartner [1]). A partial ordering $(P, \leqslant)$ satisfies Axiom A if there exist partial orderings $\leqslant_n$, $n \in \omega$, such that

(21) (i) $p \leqslant_0 q$ iff $p \leqslant q$.

(ii) If $p \leqslant_{n+1} q$, then $p \leqslant_n q$.

(iii) If $\{p_n : n \in \omega\}$ is such that $p_{n+1} \leqslant_n p_n$ for all n, then there is q such that $q \leqslant_n p_n$ for all n.

(iv) If W is a partition of p, then for every n there is $q \leqslant_n p$ such that q is compatible with at most countably many $x \in W$.

Every $\aleph_0$-closed notion of forcing P satisfies[2] Axiom A, as does every $\aleph_1$-saturated P, as well as several other familiar notions such as Sacks, Silver, Laver or Mathias forcing (see [1]).

Theorem 6. *Let P be a notion of forcing that satisfies Axiom* A. *Then Black has a winning strategy in the game $\mathscr{G}_\omega$ (for P).*

Proof. Assume that P satisfies Axiom A. We shall describe a winning strategy σ for Black in the game $\mathscr{G}_\omega$. Let W_1, a partition of p, be the first move of White. By (iv), there exist $p_1 \leqslant_0 p_0$ and a countable $C_1 \subseteq W_1$ such that $p_1 \leqslant \sum C_1$. Then if W_2 is another partition of p_0, there exist, again by (iv), $p_2 \leqslant_1 p_1$ and a countable $C_1 \subseteq W_2$ such that $p_2 \leqslant \sum C_2$. And so on: When White plays $W_1, W_2, \ldots, W_n, \ldots$, we find countable sets $C_1 \subseteq W_1$, $C_2 \subseteq W_2, \ldots, C_n \subseteq W_n, \ldots$ and conditions $p_1, p_2, \ldots, p_n, \ldots$ such that

$$p_0 \geqslant_0 p_1 \geqslant_1 p_2 \geqslant_2 \cdots \geqslant_n p_{n+1} \geqslant_{n+1} \cdots$$

and that $p_n \leqslant \sum C_n$ for every n. By (iii) there exists a condition q such that $q \leqslant P_n$ for all n. Hence

$$q \leqslant \prod_{n=1}^{\infty} \sum C_n$$

and so Black wins. Thus the strategy σ described above is a winning strategy for Black in $\mathscr{G}_\omega$. □

The converse of Theorem 6 is false. I shall return to this following Theorem 8 below.

[2] The referee has pointed out that if Black has a winning *positional* strategy in the game $\mathscr{G}$ on P, then P satisfies Axiom A.

Proper forcing (Shelah)

Let κ be an uncountable cardinal. A set $C \subseteq [\kappa]^{\aleph_0}$ is *closed unbounded* (see [4] or [7]) if

(22) (i) For every $P \in [\kappa]^{\aleph_0}$ there is $Q \in C$ such that $P \subseteq Q$.
(ii) If $P_1 \subseteq P_2 \subseteq \cdots \subseteq P_n \subseteq \cdots$ are elements of C, then $\bigcup_{n=1}^{\infty} P_n \in C$.

A set $S \subseteq [\kappa]^{\aleph_0}$ is *stationary* if $S \cap C \neq \emptyset$ for every closed unbounded set C.

A notion of forcing P is *proper* if for every uncountable cardinal κ and every stationary set $S \subseteq [\kappa]^{\aleph_0}$, S remains stationary in the generic extension by P.

If P satisfies Axiom A, then it is proper, but the converse is not true (see the discussion following Theorem 8). If P is proper, then P is $(\aleph_0, \aleph_0, \infty)$-distributive, but Example 3 shows that the converse is false. (To see that every proper P is $(\aleph_0, \aleph_0, \infty)$-distributive, notice that the $(\aleph_0, \aleph_0, \lambda)$-distributivity is equivalent to the statement that the set $[\lambda]^{\aleph_0}$ in the ground model remains unbounded in the generic extension.)

Theorem 7. *Let B be a Boolean algebra such that Black has a winning strategy in the game $\mathcal{G}_\omega$. Then B is proper.*

Proof. Let B be a complete Boolean algebra and assume that Black has a winning strategy σ in $\mathcal{G}_\omega$. Let κ be an infinite cardinal and let S be a stationary subset of $[\kappa]^{\aleph_0}$. We want to show that S is stationary in the generic extension.

We use this useful characterization of closed unbounded sets in $[\kappa]^{\aleph_0}$ (for proof, see [3]): If C is closed unbounded, then there is a function $f: [\kappa]^{<\omega} \to \kappa$ such that $C \supseteq C_f$ where $C_f = \{P \in [\kappa]^{\aleph_0} : P \text{ is closed under } f\}$. ($[\kappa]^{<\omega}$ is the set of all finite sequences in κ.)

In order to show that the set S is stationary in the generic extension, let f be a B-valued name for a function $f: [\kappa]^{<\omega} \to \kappa$, and let p be a condition. It suffices to find $P \in S$ and a condition $q \leq p$ such that

$$q \Vdash P \text{ is closed under } f. \tag{23}$$

Thus let f be such, and without loss of generality assume that $p = 1$.

For every $\vec{\alpha} \in [\kappa]^{<\omega}$ and every $\beta < \kappa$, let

$$a(\vec{\alpha}, \beta) = [\![f(\vec{\alpha}) = \beta]\!] \tag{24}$$

and let $W(\vec{\alpha})$ be the collection of all the nonzero $a(\vec{\alpha}, \beta)$. Each $W(\vec{\alpha})$ is a partition of B.

We shall show that the set

$$A = \{P \in [\kappa]^{\aleph_0} : [\![P \text{ is closed under } f]\!] \neq 0\} \tag{25}$$

contains a closed unbounded subset. Then $S \cap A \neq \emptyset$ and so (23) follows.

Let g be the following function, from $[[\kappa]^{<\omega}]^{<\omega}$ into $[\kappa]^{\aleph_0}$: if s is a finite sequence $s = \langle \vec{\alpha}_1, \ldots, \vec{\alpha}_n \rangle$, let

$$g(s) = \{\beta < \kappa : a(\vec{\alpha}_n, \beta) \in \sigma(\langle W(\vec{\alpha}_1), \ldots, W(\vec{\alpha}_n)\rangle)\} \tag{26}$$

(when White plays the partitions $W(\vec{\alpha}_1), \ldots, W(\vec{\alpha}_n)$, σ tells Black to choose a countable subset C_n of $W(\vec{\alpha}_n)$; $g(s)$ is the countable set of those β's that index the elements of C_n).

Let C be the set of all those $P \in [\kappa]^{\aleph_0}$ that are closed under g, in the following sense: if $s \in [[P]^{<\omega}]^{<\omega}$, then $g(s) \subseteq P$. Clearly, C is closed unbounded.

We shall complete the proof by showing that $C \subseteq A$. Let $P \in C$. Let $\{\vec{\alpha}_n : n = 1, 2, \ldots\}$ be an enumeration of $[P]^{<\omega}$. For each n let C_n be the countable set $C_n = \sigma(W(\vec{\alpha}_1), \ldots, W(\vec{\alpha}_n))$. Since $g(\langle \vec{\alpha}_1, \ldots, \vec{\alpha}_n \rangle) \subseteq P$, we have

$$\sum C_n \leq \sum_{\beta \in P} a(\vec{\alpha}_n, \beta) = [\![f(\vec{\alpha}_n) \in P]\!]. \tag{27}$$

Now σ is a winning strategy for Black and so

$$\prod_{n=1}^{\infty} \sum C_n \neq 0.$$

This and (27) gives

$$\|(\forall \vec{\alpha} \in [P]^{<\omega}) f(\vec{\alpha}) \in P\| = \prod_{n=1}^{\infty} \|f(\vec{\alpha}_n) \in P\| \neq 0$$

and so $P \in A$. $\square$

The converse of Theorem 7 is false; I shall shortly present an example. Let us consider the following diagram:

$$\begin{tikzcd}
 & \text{proper} & \\
\text{Black wins } \mathcal{G}_\omega \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & (\aleph_0, \aleph_0, \infty)\text{-distributive} \arrow[ul, Rightarrow] \\
 & \text{White does not win } \mathcal{G}_\omega \arrow[ur, Rightarrow] &
\end{tikzcd} \tag{28}$$

None of the implications in (28) can be reversed:

(a) Example 3 (Adding a closed unbounded subset of a stationary set) gives a Boolean algebra B, for which White does not win $\mathcal{G}_\omega$ (because White does not win $\mathcal{G}_1$ since B is $(\aleph_0, \infty)$-distributive), and which is not proper (B destroys a stationary subset of $\aleph_1$).

(b) The algebra B in the next example is proper but White has a winning strategy in $\mathcal{G}_\omega$.

Example 6. *Adding a closed unbounded set with finite conditions.* (White wins $\mathcal{G}_\omega$.)

Let P be the following notion of forcing (due to Baumgartner, see [1]): A condition p is a finite function with $\mathrm{dom}(p) \subseteq \omega_1$, $\mathrm{ran}(p) \subseteq \omega_1$, with the property that there exists a normal (i.e. increasing and continuous) function $f: \omega_1 \to \omega_1$ such that $p \subseteq f$. A condition q is stronger than p if $q \supseteq p$.

The notion of forcing P is proper. We shall show that White has a winning strategy in the game $\mathcal{G}_\omega$.

Let $W_1 = \{\{(0, \beta)\} : \beta < \omega_1\}$. W_1 is a partition. Let $C_1 \subseteq W_1$ be Black's first move. There is a countable ordinal α_1 such that $\omega^{\alpha_1} = \alpha_1$ and that $\beta < \alpha_1$ for every $\{(0, \beta)\} \in C_1$. Let $W_2 = \{\{(\alpha_1, \beta)\} : \{(\alpha_1, \beta)\}$ is a condition$\}$. W_2 is a partition. And so on. When Black plays $C_1 \subseteq W_1$, $C_2 \subseteq W_2, \ldots, C_n \subseteq W_n$, we find $\alpha_{n+1} = \omega^{\alpha_{n+1}} > \alpha_n$ such that $\beta < \alpha_{n+1}$ for all $\{(\alpha_n, \beta)\} \in C_n$ and let

$$W_{n+1} = \{\{(\alpha_{n+1}, \beta)\} : \{(\alpha_{n+1}, \beta)\} \text{ is a condition}\}.$$

I claim that White wins. Assume that White loses; thus there is $p \in P$ such that

$$p \leqslant \sum C_n \quad \text{for all } n. \tag{29}$$

Since $p \in P$, there is a normal function $f: \omega_1 \to \omega_1$ such that $p \subseteq f$. Let $\lambda = \lim_{n\to\infty} \alpha_n$ and let m be such that

$$\mathrm{dom}(p) \cap \lambda = \mathrm{dom}(p) \cap \alpha_m.$$

If there is $n \geqslant m$ such that $f(\alpha_n) \geqslant \alpha_{n+1}$, then we let

$$q = p \cup \{(\alpha_n, f(\alpha_n))\}.$$

Now q is a condition, $q \leqslant p$, and q is incompatible with every $\{(\alpha_n, \beta)\} \in C_n$; this contradicts (29).

Thus assume that $f(\alpha_n) < \alpha_{n+1}$, for all $n \geqslant m$. (Note that this implies that $f(\lambda) = \lambda$.) We define a new normal function g as follows:

(30) (i) $g(\alpha) = f(\alpha)$ when $\alpha < \alpha_m$,

(ii) $g(\alpha_m + 1) = \alpha_{m+2}$,

(iii) $g(\alpha + 1) = g(\alpha) + 1$ when $\alpha_m + 1 \leqslant \alpha < \lambda$,

(iv) $g(\alpha) = \sup_{\xi < \alpha} g(\xi)$ when $\alpha_m < \alpha < \lambda$ and α is a limit,

(v) $g(\alpha) = f(\alpha)$ when $\alpha \geqslant \lambda$.

Since $\omega^\lambda = \lambda$, we have $g(\alpha) < \lambda$ for all $\alpha < \lambda$ and so g is a normal function.

Now let

$$q = p \cup \{\alpha_{m+1}, g(\alpha_{m+1})\}.$$

Since $q \subseteq g$, q is a condition. But

$$q(\alpha_{m+1}) = g(\alpha_{m+1}) > g(\alpha_m + 1) = \alpha_{m+2}$$

and so q is incompatible with every $\{(\alpha_{m+1}, \beta)\} \in C_{m+1}$. Again, this contradicts (29). $\square$

The last theorem is devoted to the implication

$$\text{Axiom A} \Rightarrow \text{Black wins } \mathcal{G}_\omega.$$

from Theorem 6. It turns out that this implication cannot be reversed.

Shelah calls a notion of forcing P *ω-proper* if for a sufficiently large cardinal κ the following holds:

If $N_1 < N_2 < \cdots < N_n < \cdots$ is a sequence of countable elementary submodels of $(V_\kappa, \in)$ such that $P \in N_1$ and $N_n \in N_{n+1}$ for all n, then for every condition $p \in N_1$ there is a stronger condition q such that for all n,

$$q \leq \sum (W \cap N_n) \tag{31}$$

for every $W \in N_n$ which is a partition of p.

Theorem 8. *Let p be an ω-proper notion of forcing. The Black has a winning strategy in the game $\mathcal{G}_\omega$ for P.*

Proof. Assume that P is ω-proper. We shall describe a winning strategy σ for Black in $\mathcal{G}_\omega$. Let κ be a sufficiently large cardinal. Let $p \in P$ and let W_1 be a partition of p. Let N_1 be countable elementary submodel of V_κ such that $P \in N_1$, $p \in N_1$ and $W_1 \in N_1$. Let $\sigma(W_1) = W_1 \cap N_1$. Let W_2 be another partition of p. Let N_2 be a countable elementary submodel of V_κ such that $N_1 \subset N_2$, $N_1 \in N_2$ and $W_2 \in N_2$. Let $\sigma(W_1, W_2) = W_2 \cap N_2$. And so on. I claim that if Black follows σ, he wins. Let $W_1, W_2, \ldots, W_n, \ldots$ be a sequence of partitions of p. There exists a sequence $N_1 < N_2 < \cdots < N_n < \cdots$ of elementary submodels of V_κ such that $W_n \in N_n$ and $N_n \in N_{n+1}$ for every n, and $\sigma(W_1, \ldots, W_n) = W_n \cap N_n$. Since P ω-proper, there exists $q \leq p$ such that (31) holds for all n and all $W \in N_n$. In particular, we have $q \leq \sigma(W_1, \ldots, W_n)$ for every n and so Black wins. $\square$

If P satisfies Axiom A, then P is ω-proper. However, Shelah and Baumgartner have counterexamples showing that the converse is not true. It follows that the implication

$$\text{Axiom A} \Rightarrow \text{Black wins } \mathcal{G}_\omega$$

cannot be reversed. (I have been told by Shelah that the implication

$$\omega\text{-proper} \Rightarrow \text{Black wins } \mathcal{G}_\omega$$

cannot be reversed either.)

Open problems

First consider the implications

$$\aleph_0\text{-closed dense subset} \Rightarrow \text{Black wins } \mathcal{G}_{c\&c} \Rightarrow (\aleph_0, 2)\text{-distributive}.$$

Examples 1 and 2 show that these implications cannot be reversed. However, both examples use additional set-theoretic axioms: the existence of a measurable cardinal and $\diamondsuit$. Thus:

Problem 1. Prove in ZFC that "Black wins $\mathscr{G}_{c\&c}$" does not imply "B has an $\aleph_0$-closed dense subset". (Or show that a large cardinal assumption is necessary.)

Problem 2. Prove in ZFC that "$(\aleph_0, 2)$-distributive" does not imply "Black wins $\mathscr{G}_{c\&c}$".

The results of this paper can be presented in the form of a diagram:

$$\begin{tikzcd}
 & & & \boxed{\text{weakly } (\aleph_0,\infty)\text{-distr.}} \arrow[ddr] & \\
 & \boxed{(\aleph_0,\infty)\text{-distr.}} \arrow[r] & \boxed{\text{White does not win } \mathscr{G}_{\mathrm{fin}}} \arrow[ur, "?"] \arrow[dr] & & \\
\boxed{\aleph_0\text{-closed dense subset}} \arrow[ur] \arrow[r] \arrow[dr] & \boxed{\text{Black wins } \mathscr{G}_{\mathrm{fin}}} \arrow[ur] \arrow[dr] & & \boxed{\text{White does not win } \mathscr{G}_\omega} \arrow[r] & \boxed{(\aleph_0,\aleph_0,\infty)\text{-distr.}} \\
 & \boxed{\text{Axiom A}} \arrow[r] & \boxed{\text{Black wins } \mathscr{G}_\omega} \arrow[ur] \arrow[r] & \boxed{\text{proper}} \arrow[ur] &
\end{tikzcd} \tag{32}$$

None of the implications in diagram (32) can be reversed, except possibly the one with a question mark. (The reader can verify this with the help of Examples 3, 4, 5 and 6.)

Problem 3. Show that "Weakly $(\aleph_0, \infty)$-distributive" does not imply "White does not win $\mathscr{G}_\omega$".

Or at least:

Problem 4. Show that "weakly $(\aleph_0, \infty)$-distributive" does not imply "White does not win $\mathscr{G}_{\mathrm{fin}}$".

The following diagram is a detail of diagram (32):

$$\begin{tikzcd}
 & & & (\aleph_0,\infty)\text{-distr.} & \\
 & & \text{Black wins } \mathscr{G}_1 \arrow[ur] \arrow[dr] & & \\
 & \text{Black wins } \mathscr{G} \arrow[ur, "?"] & & \text{Black wins } \mathscr{G}_{\mathrm{fin}} \arrow[dr] & \\
\aleph_0\text{-closed dense subset} \arrow[ur, "?"] \arrow[dr] & & & & \text{Black wins } \mathscr{G}_\omega \\
 & \text{Axiom A} \arrow[rr] & & \omega\text{-proper} \arrow[ur] &
\end{tikzcd} \tag{33}$$

The meaning of arrows and question marks is the same as in diagram (32).

Problem 5. Show that "Black wins $\mathscr{G}_1$" does not imply "Black wins $\mathscr{G}$".

Problem 6. Show that "Black wins $\mathscr{G}$" does not imply "B has an $\aleph_0$-closed dense subset".

Problem 7. Show that "Black wins $\mathscr{G}_1$" does not imply Axiom A.

Addenda (June 1983)

1. Some of the games described in this work were independently formulated and investigated by Charlie Gray in his Berkeley thesis [12]. In particular, he considered the games $\mathscr{G}_\omega$ and $\mathscr{G}_{\text{fin}}$ (in my notation). One of the main themes of his work is that the property "Black has a winning strategy" is preserved (for several games) under iteration of forcing with countable support.

2. One of Gray's results gives a game-theoretic characterization of properness (cf. [12] or [13]). Consider the following modification of the game $\mathscr{G}_\omega$: the new game is played as $\mathscr{G}_\omega$, *except* that Black is allowed at stage n to change all his previous moves (i.e. to increase C_k for each $k<n$). The payoff is defined as in $\mathscr{G}_\omega$, but each C_n is the union of all Black's attempts for C_n.

Gray's theorem states that Black has a winning strategy in this game if and only if the forcing notion is proper.

3. In connection with our Problem 4, Gray formulates an interesting conjecture. To solve Problem 4, one needs to find a weakly $(\aleph_0, \infty)$-distributive complete Boolean algebra for which White wins the game $\mathscr{G}_{\text{fin}}$. This algebra has to be therefore quite different from the measure algebra (used in Example 4).

Now, it had been conjectured by von Neumann (in the Scottish book, around 1930) that every ccc, countably generated, weakly $(\aleph_0, \infty)$-distributive cBa is isomorphic to the measure algebra. A result of Jensen from early seventies gives a counterexample (using $\diamondsuit$).

Gray's Conjecture. If B is a ccc, countably generated complete Boolean algebra such that Black has a winning strategy in $\mathscr{G}_{\text{fin}}$, then B is isomorphic to the algebra Borel sets mod measure zero.

4. Matt Foreman made considerable progress on our Problem 6. His main result [14] is:

Theorem (Foreman). *If Black has a winning strategy in the game $\mathscr{G}$ on B and if $|B| \leq \aleph_1$, then B has a dense subset that is ω-closed.*

Foreman's theorem has some interesting consequences. First, the property 'Black has a winning strategy $\mathscr{G}$'' is *hereditary*; i.e. if Black has a winning strategy in $\mathscr{G}(B)$ and if A is a complete subalgebra of B, then Black has a winning strategy in $\mathscr{G}(A)$. Thus Foreman's theorem has the following

Corollary. *If B has a dense ω-closed subset and if A is a complete subalgebra of B of size at most $\aleph_1$, then A has a dense ω-closed subset.*

Problem 6a. Is the property "B has a dense ω-closed subset" hereditary?

A counterexample to heredity would of course be a solution of Problem 6. We show below that conversely, a solution of Problem 6 would give a counterexample to heredity, and therefore these two problems are equivalent.

5. Let P and Q be partially ordered sets and assume that their product $P\times Q$ has a dense ω-closed subset. Then Black has a winning strategy in the game $\mathscr{G}(P)$: he chooses his moves $p_1, p_3, \ldots, p_{2n+1}, \ldots$ along with 'witnesses' $q_{2n+1}\in Q$ so that $(p_1, q_1)\geqslant(p_3, q_3)\geqslant\cdots$ is a descending sequence in the ω-closed dense subset.

Conversely, let P be a partial ordering with the property that Black has a winning strategy in the game $\mathscr{G}(P)$; I claim that there is a partial ordering Q such that $P\times Q$ has a dense ω-closed subset. Namely, let Q be the collapse of the cardinal $|P|$ with countable conditions. In the generic extension by Q, P has size at most $\aleph_1$, and Black still has (the same) winning strategy in $\mathscr{G}(P)$. By Foreman's theorem, P has (in V^Q) an ω-closed dense subset E.

Let $D=\{(p, q)\in P\times Q : q\Vdash p\in\mathbf{E}\}$. D is a dense subset of $P\times Q$ and I claim that D is ω-closed. Let $\{(p_n, q_n)\}_{n=0}^{\infty}$ be a descending sequence in D. The condition $q_\infty=\bigcup_{n=0}^{\infty} q_n$ forces that $\{p_n\}_n$ is a descending sequence in E and therefore there is $q\leqslant q_\infty$ and $p\in P$ such that $q\Vdash p\in\mathbf{E}$ and $p\leqslant p_n$ for all n.

Thus we have

Theorem. *Black has a winning strategy in $\mathscr{G}(P)$ if and only if there exists a Q such that $P\times Q$ has a dense ω-closed subset.*

Problem 6b. Find P and Q such that $P\times Q$ has a dense ω-closed subset but P does not.

6. If P has a dense ω-closed subset and if $\Vdash$ ($\mathbf{Q}$ has a dense ω-closed subset), then $P*\mathbf{Q}$ has a dense ω-closed subset. If $P*\mathbf{Q}$ has a dense ω-closed subset, and if $|P|\leqslant\aleph_1$, then by Foreman's theorem, P has a dense ω-closed subset. How about $\mathbf{Q}$?

If $P*\mathbf{Q}$ is (literally) ω-closed, then it is easy to see that $\Vdash$($\mathbf{Q}$ is ω-closed); but this is misleading:

Example. P and $\mathbf{Q}$ *such that* P *is* ω*-closed and* $P * \mathbf{Q}$ *has a dense* ω*-closed subset, but* $\Vdash \mathbf{Q}$ *does not.*

Let P be the forcing that adjoins a Cohen subset of ω_1 (with countable conditions). In V^P, let $\mathbf{Q}$ be the forcing that shoots a closed unbounded set through the P-generic set $\mathbf{G} \subset \omega_1$ (with conditions that are countable closed subsets of $\mathbf{G}$ ordered by end-extension). P is ω-closed, and $\mathbf{Q}$ is not even proper (it destroys the stationary set $\omega_1 - \mathbf{G}$). The forcing $P * \mathbf{Q}$ has the following dense ω-closed subset D: $(p, q) \in D$ iff for some $\alpha < \omega_1$, $p \in \{0, 1\}^{\alpha+1}$ and $p(\alpha) = 1$, and q is a closed subset of $\{\xi : p(\xi) = 1\}$ and $\alpha \in q$.

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