p-adic Siegel–Eisenstein series of degree two

Sho Takemori

Department of Mathematics, Kyoto University, Kitashirakawa-Oiwake-Cho, Sakyo-Ku, Kyoto 606-8502, Japan

ABSTRACT

We prove an explicit formula for Fourier coefficients of Siegel–Eisenstein series of degree two with a primitive character of any conductor. Moreover, we prove that there exists the p-adic analytic family which consists of Siegel–Eisenstein series of degree two and a certain p-adic limit of Siegel–Eisenstein series of degree two is actually a Siegel–Eisenstein series of degree two.

1. Introduction

1.1. For a fixed prime $p$, a formal Fourier expansion $E = \sum_{h} a(h, E)q^{h}$ is called a p-adic Eisenstein series if there exists a sequence $\{E_{k_n}\}$ consisting of Eisenstein series $E_{k_n}$ of weight $k_n$ and $a(h, E_{k_n})$ is convergent to $a(h, E)$, p-adically and uniformly on $h$. Here $a(h, E_{k_n})$ is $h$-th Fourier coefficients of $E_{k_n}$. If the sequence $\{k_n\}$ converges p-adically in $X$, then we call the limit p-adic weight, where $X$ is as in (1.4). In elliptic modular case, it is easy to calculate the Fourier coefficients of p-adic Eisenstein series and to verify that a p-adic Eisenstein series of p-adic weight $(k, a)$ such that $k \in \mathbb{Z}$ and $k \geq 2$, is a modular form. In Siegel modular case, there exist some examples that show a p-adic Eisenstein series is a modular form. In [4], Katsurada and Nagaoka proved that the p-adic Siegel–Eisenstein series of p-adic weight $(k, k + (p - 1)/2)$ and degree two can be expressed as a linear combination of two genus theta series and twisted Eisenstein series with character $\chi_{p^r}$. Here $p$ is an odd prime, $k \geq 2$ and $\chi_{p^r}$ is the quadratic character mod $p$. In particular, their p-adic Siegel–Eisenstein series is a Siegel modular form of level $p$, character $\chi_{p^r}$ and degree two. In addition, Mizuno and Nagaoka proved that the p-adic Siegel–Eisenstein series of p-adic weight $(k, k)$ and degree two is a Siegel modular form of level $p$ and the trivial character [8]. They constructed a Jacobi form of level $p$ with the trivial character by a Hecke operator of level $p$ and Jacobi–Eisenstein series of level 1. And they...
showed that this Jacobi form corresponds to the $p$-adic Siegel–Eisenstein series of degree two by the Maass lift. In this paper, we extend their results to the case where the level and the character are general. The weight, however, is greater than three. This is because we only treat Eisenstein series that is absolutely convergent.

First, we prove an explicit formula for the Fourier coefficients of Siegel–Eisenstein series of degree two for every level. The explicit formula for the Fourier coefficients of Siegel–Eisenstein series was obtained by Kaufhold [5] for degree two and level one, by Katsurada [3] for general degree and level one, and by Mizuno [7] for degree two and a square free odd level. Also, in [2], Gunji calculated the $p$-Euler factor of the Fourier coefficients of Siegel–Eisenstein series of degree two and odd prime level $p$. The Euler factor is called the Siegel series. In Section 3, following the method of [2], we calculate the Siegel series explicitly. By this explicit calculation, we obtain the explicit formula for the Fourier coefficients of Siegel–Eisenstein series of degree two with a primitive character. By this explicit formula, we prove that a $p$-adic Siegel–Eisenstein series is a Siegel modular form and there exists the $p$-adic analytic family which consists of Siegel–Eisenstein series of degree two.

1.2. We state the main results of this paper precisely. For a field $K$ and positive integer $g$, we put

$$Sp_g(K) = \{ \alpha \in M_{2g}(K) \mid t^t \eta \alpha = \eta \},$$

$$P(K) = \left\{ \begin{pmatrix} a & b \\
                        c & d \end{pmatrix} \in Sp_g(K) \mid a, b, c, d \in M_g(K), c = 0 \right\},$$

where $\eta = \begin{pmatrix} 0_g & -1_g \\
                        1_g & 0_g \end{pmatrix}$. We denote by $\mathfrak{H}_g$ the Siegel upper half space of degree $g$. Let $N$ be a positive integer. We define $Sp_g(\mathbb{Z})$ and $\Gamma_0(N)$ by

$$Sp_g(\mathbb{Z}) = Sp_g(\mathbb{Q}) \cap GL_{2g}(\mathbb{Z}),$$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\
                        c & d \end{pmatrix} \in Sp_g(\mathbb{Z}) \mid a, b, c, d \in M_g(\mathbb{Z}), c \equiv 0 \mod N \right\}.$$

Let $\psi$ be a Dirichlet character mod $N$ and $k$ be an integer such that $\psi(-1) = (-1)^k$. We define Siegel–Eisenstein series $E_{k,\psi}^{(g)}(z)$ of degree $g$, weight $k$, character $\psi$ and level $N$ by

$$E_{k,\psi}^{(g)}(z) = \sum_{(c d) \in P(\mathbb{Q}) \cap \Gamma_0(N) \setminus \Gamma_0(N)} \overline{\psi}(\det(d)) \det(cz + d)^{-k}, \quad z \in \mathfrak{H}_g.$$ 

The right-hand side is absolutely convergent when $k > g + 1$. Let

$$E_{k,\psi}^{(g)}(z) = \sum_{0 \leq h \in Sym^*_g(\mathbb{Z})} a(h, E_{k,\psi}^{(g)}) \exp(2\pi i Tr(hz))$$

be the Fourier expansion of $E_{k,\psi}^{(g)}(z)$. Here $Sym^*_g(\mathbb{Z})$ is the set of half integral symmetric matrices of size $g$ and $h \geq 0$ means $h$ is positive semi-definite.

The first main result of this paper is the theorem for the explicit formula for the Fourier coefficients of the Siegel–Eisenstein series of degree two.
Theorem 4.1. Let $\psi$ be a primitive Dirichlet character mod $N$ and $h$ be a half integral positive-definite symmetric matrix. We denote the $h$-th Fourier coefficient of Siegel–Eisenstein series of degree two by $a(h, E^{(2)}_{k, \psi})$. Suppose $k > 3$. Then we have

$$a(h, E^{(2)}_{k, \psi}) = 2 \frac{L^{(N)}(2 - k, \chi_h \psi)}{L(1 - k, \psi)L^{(N)}(3 - 2k, \psi^2)} \times \prod_{q \text{ prime } q \mid N} F^{(2)}_q(h; \psi(q)q^{k-3}) \prod_{q \text{ prime } q \mid N} c_q(h, \psi; q^{k-3}).$$

The notation is as follows. For a Dirichlet $L$-function $L(s, \chi)$ and a positive integer $M$, we put $L^{(M)}(s, \chi) = \prod_{q \mid M} (1 - \chi(q)q^{-s}) L(s, \chi).$ The conductor of a Dirichlet character $\chi$ is denoted by $\nu(\chi).$ $F^{(2)}_q(h; \psi)$ is a polynomial of degree two associated with the $h$-th Fourier coefficient of Siegel–Eisenstein series of degree two. Let $\psi(q)q^{k-3}$ be the $q$-th Fourier coefficient of $E^{(2)}_{k, \psi}$.

If $(q, \psi, h)$ satisfies the condition (i) or (ii) below, then we define $c_q(h, \psi; T) = 0.$

1. If $(q, \psi, h)$ satisfies the condition (i) nor (ii), and $\nu_q \neq 1,$ then we define $c_q(h, \psi; T) = 1.$
2. If $(q, \psi, h)$ satisfies the condition neither (i) nor (ii), and $\nu_q = 1,$ then we define $c_q(h, \psi; T) = 1.$

We have

$$c_q(h, \psi; T) = 1 + q^{-1}(1 - q) \frac{1 - \chi_h \psi(q)q^{-2T-1}}{(1 - \psi(q)q^{-4T-2})(1 - \chi_h \psi(q)qT)} h^\beta_q - n_q + 1.$$

Here $\psi_q$ is the Dirichlet character mod $q$ such that $\psi = \prod_{q\mid N} \psi_q.$ $n_q$ and $\beta_q$ are given by

$$n_q = \text{ord}_q(f(\psi)),
\beta_q = 2\beta_q(h) = \text{ord}_q \left( \frac{f(\psi)f(\psi^2)}{f(\chi_h)} \right) + \text{ord}_q(\det 2h).$$

The conditions (i) and (ii) are as follows:

(i) $q = 2,$ $f(\psi_q) \geq 4$ and $f(\psi_q) \neq 8,$ and $h \in \text{Sym}_2^2(\mathbb{Z}) \setminus \text{Sym}_3^2(\mathbb{Z}).$
(ii) $q = 2,$ $f(\psi_q) = 8$ and $h$ is $\text{GL}_2(\mathbb{Z}_2)$-equivalent to a matrix of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad 2^m \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \quad \text{or} \quad 2^m \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix},$$

with $\alpha, \beta \in \mathbb{Z}_2^\times$ and $m \in \{0, 1\}.$

From now on, we fix a prime $p \geq 2.$ For the $p$-adic interpolation of Eisenstein series, we need to remove the $p$-Euler factor of Fourier coefficients. We construct an Eisenstein series $G^{(2)}_{k, \psi}$ which $p$-Euler factor of Fourier coefficients of is removed by using Hecke operators.

Let $M^{(g)}_k(\Gamma_0(N), \psi)$ be the space of Siegel modular forms of degree $g,$ weight $k,$ level $N$ and character $\psi.$ Suppose $f \in M^{(g)}_k(\Gamma_0(N), \psi).$ Then $f$ has the following Fourier expansion

$$f(z) = \sum_{0 \leq h \in \text{Sym}_g^\times(\mathbb{Z})} a(h, f) e(hz).$$
We define a Hecke operator \( U(p) \) as follows
\[
(f \mid U(p))(z) = \sum_{0 \leq h \in \text{Sym}_k^2(\mathbb{Z})} a(ph, f)e(hz).
\]

By the definition of \( U(p) \), we have
\[
f | U(p) \in \begin{cases} 
\mathcal{M}_k^\mathcal{S}(\Gamma_0(pN), \psi) & \text{if } p \nmid N, \\
\mathcal{M}_k^\mathcal{S}(\Gamma_0(N), \psi) & \text{if } p \mid N.
\end{cases}
\]

We define Hecke operators \( V_\psi(q) \) and \( W_\psi(p) \) as follows
\[
V_\psi(q) = \begin{cases} 
\frac{1 - \overline{\psi}(q)^2 q^{3-2k} U(q)}{1 - \overline{\psi}(q)^2 q^{3-2k}} & \text{if } q \neq 2, \\
\frac{U(q)^2 - \overline{\psi}(q)^2 q^{3-2k} U(q)^3}{1 - \overline{\psi}(q)^2 q^{3-2k}} & \text{if } q = 2,
\end{cases}
\]
\[
W_\psi(p) = \frac{(U(p) - \psi(p) p^{k-1})(U(p) - \psi^2(p) p^{2k-3})}{(1 - \psi(p) p^{k-1})(1 - \psi^2(p) p^{2k-3})}.
\]

By Theorem 4.1, we obtain the following proposition.

**Proposition 5.2.** Let \( N \) be a positive integer and \( \psi \) be a Dirichlet character mod \( N \). Put \( N = N_0 p^r \) with \( p \nmid N_0 \) and \( r \geq 0 \). Suppose that \( \psi_q \) is primitive for all \( q \mid N_0 \) and \( \psi_p \) is primitive if \( r > 1 \). We put
\[
E_{k, \psi} = \left| \prod_{q \mid N} V_\psi(q) \right|_{\psi_q = 1}.
\]

We define \( G_{k, \psi}^{(2)} \) as follows.

(i) If \( \psi_p \) is primitive, then we define
\[
G_{k, \psi}^{(2)} = \frac{1}{2} L(1 - k, \psi^2) L(N)(3 - 2k, \psi^2) E_{k, \psi}.
\]

(ii) If \( \psi_p \) is the trivial character mod \( p \), then we define
\[
G_{k, \psi}^{(2)} = \frac{1}{2} L(1 - k, \psi^2) L(N)(3 - 2k, \psi^2) E_{k, \psi} | W_{\xi}(p).
\]

Here \( \xi = \prod_{q \mid N_0} \psi_q \).

Let \( 0 \leq h \in \text{Sym}_k^2(\mathbb{Z}) \) be a half integral positive semi-definite symmetric matrix and suppose that \( k > 3 \). We denote \( h \)-th Fourier coefficient of \( G_{k, \psi}^{(2)} \) by \( a(h, G_{k, \psi}^{(2)}) \). Then the following assertions hold.

(1) If \( \text{rank } h = 0 \),
\[
a(h, G_{k, \psi}^{(2)}) = \frac{1}{2} L(1 - k, \psi^2) L(N)(3 - 2k, \psi^2).
\]

(2) If \( \text{rank } h = 1 \),
\[
a(h, G_{k, \psi}^{(2)}) = L(N)(3 - 2k, \psi^2) \prod_{q \text{ prime}} F_q^{(1)}(\epsilon(h); \psi(q) q^{k-2}).
\]
Here $F_q^{(1)}(m; T)$ is $1 + qT + \cdots + (qT)^{ord_q(m)}$ and $\varepsilon(h)$ is defined as follows
$$\varepsilon(h) = \max\{m \in \mathbb{Z}_{\geq 0} \mid m^{-1}h \in \text{Sym}^+_2(\mathbb{Z})\}.$$  

(3) If $\text{rank } h = 2$,
$$a(h, G^{(2)}_{k, \psi}) = L^{(N)}(2 - k, \chi_h \psi) \prod_{q \text{ prime}} F_q^{(2)}(h; \psi(q)q^{k-3}).$$

**Remark 1.1.** If $N = p$ is an odd prime and $\psi$ is trivial character mod $p$ then Mizuno [6] and Nagaoka [8] constructed $G^{(2)}_{k, \psi}$ by using Maass lift and a Hecke operator of level $p$ of Jacobi forms.

Next, we state the theorem for the $p$-adic analytic family which interpolates $G^{(2)}_{k, \psi}$. We fix embeddings $\mathbb{Q} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Let $N$ be a positive integer and $\chi$ be a Dirichlet character mod $N$. Put $N = N_0 p^r$ with $r \geq 0$ and $(N_0, p) = 1$. Then $\chi$ can be regarded as a character of $\lim_n (\mathbb{Z}/N_0 p^n \mathbb{Z})^\times = (\mathbb{Z}/p N_0 \mathbb{Z})^\times \times (1 + p \mathbb{Z}_p)$.

Here $p$ is given by
$$p = \begin{cases} p & \text{if } p \neq 2, \\ 4 & \text{if } p = 2. \end{cases} \quad (1.1)$$

We decompose
$$\chi = \chi_1 \chi_2, \quad (1.2)$$
where $\chi_1$ is the character of $(\mathbb{Z}/p N_0 \mathbb{Z})^\times$ and $\chi_2$ is the character of $(1 + p \mathbb{Z}_p)$. We put
$$P(\chi; T) = \begin{cases} 1 & \text{if } \chi_1 \neq 1, \\ 1 - \chi_2(1 + T)^{-1} & \text{if } \chi_1 = 1. \end{cases} \quad (1.3)$$

Then $P(\chi; T)$ is a “denominator” of the $p$-adic Dirichlet $L$-function.

**Theorem 5.2.** Let $N$ be a positive integer divisible by $p$ and $\psi$ be a Dirichlet character mod $N$. Put $N_0 = N_0 p^r$ with $p \nmid N_0$ and $r \geq 1$. Suppose that $\psi_q$ is primitive for all $q \mid N_0$ and $\psi_p$ is primitive if $r > 1$. We fix a topological generator $u$ of $1 + p \mathbb{Z}_p$. We denote by $\omega$ the Teichmüller character. For a half integral positive semi-definite symmetric matrix $0 \leq h \in \text{Sym}^+_2(\mathbb{Z})$, there exists $a(h, \psi; T) \in \text{Frac}(\mathbb{Z}_p[\psi][[T]])$ which satisfies the following interpolation property
$$a(h, \psi; \varepsilon(u)u^k - 1) = a(h, G^{(2)}_{k, \psi \omega \omega^{-k}}),$$
for any character of finite order $\varepsilon$ of $1 + p \mathbb{Z}_p$ and integer $k$ such that $k > 3$.

Put
$$Q(\psi; T) = P(\psi; T)P(\psi \omega^{-2}; u^{-2}(1 + T)^2 - 1)P'(\psi; u^{-1}(1 + T) - 1),$$
where $P(\psi; T)$ is as in (1.3) and $P'(\psi; T)$ is given by
$$P'(\psi; T) = \begin{cases} 1 & \text{if } \psi_1^2 \neq \omega^2, \\ 1 - \psi_2(u)(1 + T)^{-1} & \text{if } \psi_1^2 = \omega^2 \text{ and } p \neq 2, \\ (1 - \psi_2(u)(1 + T)^{-1})(1 + \psi_2(u)(1 + T)^{-1}) & \text{if } \psi_1^2 = \omega^2 \text{ and } p = 2. \end{cases}$$

Here $\psi_1$ and $\psi_2$ are as in (1.2). Then we have $Q(\psi; T)a(h, \psi; T) \in \mathbb{Z}_p[\psi][[T]].$
Finally, we state the theorem for a $p$-adic Siegel–Eisenstein series of degree two. We define $X$ and $X_\psi$ by

$$
X = \mathbb{Z}_p \times \mathbb{Z}/\phi(p)\mathbb{Z} \cong \text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times),
$$

$$
X_\psi = \{(s, a) \in X \mid (-1)^a = \psi(-1)\}. \tag{1.4}
$$

Here $\phi$ is Euler’s phi function, $\text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ is the set of continuous group homomorphisms from $\mathbb{Z}_p^\times$ to $\mathbb{Z}_p^\times$ and $p$ is as in (1.1). $X$ is equipped with the $p$-adic topology. We embed $\mathbb{Z}$ in $X$ by $\mathbb{Z} \ni m \mapsto (m \bmod \phi(p), m) \in X$. Let

$$
\mathbb{C}_p[[q]] = \left\{ f = \sum_{0 \leq h \in \text{Sym}_2^\times(\mathbb{Z})} a(h, f)e(hz) \mid a(h) \in \mathbb{C}_p \right\},
$$

be the space of formal Fourier expansions, where $\mathbb{C}_p$ is the completion of $\overline{\mathbb{Q}}_p$. We put $|f|_p = \sup_{0 \leq h \in \text{Sym}_2^\times(\mathbb{Z})} |a(h, f)|_p$.

**Theorem 5.3.** Let $N$ be a positive integer such that $p \mid N$ and $\psi$ be a primitive Dirichlet character mod $N$. Suppose $(k, a) \in X_\psi$ and let $k$ be an integer such that $k > 0$. For any sequence $(l_m)_m \subset X_\psi$ such that $l_m > 0$, $\lim_{m \to \infty} l_m = +\infty$ in usual topology of $\mathbb{R}$ and $\lim_{m \to \infty} l_m = (a, k) \in X_\psi$, we have

$$
\lim_{m \to \infty} |G_{l_m, \psi}^{(2)} - G_{k, \psi, \omega^{a-k}}^{(2)}|_p = 0.
$$

Here $\psi \cdot \omega^{a-k}$ is the Dirichlet character mod $Np$ induced by $n \mapsto \psi(n)\omega^{a-k}(n)$. Thus, the $p$-adic Siegel–Eisenstein series of level $N$, weight $k$, character $\psi$ and degree two is a Siegel–Eisenstein series of level $Np$, weight $k$, character $\psi \cdot \omega^{a-k}$ and degree two.

1.3. This paper is organized as follows. In Section 2, we calculate the Fourier coefficient of Siegel–Eisenstein series of degree $g$ by known techniques. We show that the Fourier coefficient has the Euler product expression. The Euler factor was examined by Feit, Katsurada, Kaufhold, Kitaoka and Shimura. In the last subsection of Section 2, we review some of their results. In Section 3, we explicitly calculate Siegel series at a finite place dividing the level in the case where $g = 2$, rank $h = 2$ and $\psi$ is a primitive character. In Section 4, using results of preceding two sections, we show the explicit formula for the Fourier coefficients of Siegel–Eisenstein series of degree two. In Section 5, we introduce a Siegel–Eisenstein series $G_{k, \psi}^{(2)}$ and prove Theorem 5.2 and Theorem 5.3.

**Notations.** For a commutative ring $R$, we denote by $M_{n,m}(R)$ the set of $n$-by-$m$ matrices with entries in $R$. We denote by $M_n(R)$ the set of $n$-by-$n$ matrices with entries in $R$. The trace and the determinant are denoted by $\text{Tr}$ and $\det$ respectively. For $\alpha \in M_{n,m}(R)$, we denote by $^t\alpha$ the transpose of $\alpha$. We denote by $\text{Sym}_n(R)$ the set of symmetric matrices of size $n$ with entries in $R$. If $R$ is an integral domain with $\text{char} R = 0$, we define the set of half integral matrices $\text{Sym}_n^*(R)$ by

$$
\{(a_{ij}) \in \text{Sym}_n(K) \mid a_{ii} \in R \text{ for } 1 \leq i \leq n \text{ and } 2a_{ij} \in R \text{ for } 1 \leq i, j \leq n\},
$$

where $K$ is the fractional field of $R$. For $x \in \mathbb{C}$, we put $e(x) = \exp(2\pi i x)$. For a ring $R$ and an additive character $\varphi$ on $R$, we put $\varphi(X) = \varphi(\text{Tr}(X))$ for $X \in \text{Mat}_n(R)$. For matrices $M$ and $N$, we put $M[N] = ^tMN$, if the right-hand side is defined. For $h \in \text{Sym}_n(\mathbb{R})$, we write $h \geq 0$ (resp. $h > 0$) if $h$ is positive semi-definite (resp. positive definite). For an odd integer $d$, we put $\varepsilon_d = 1$ if $d \equiv 1 \bmod 4$ and $\varepsilon_d = i$ if $d \equiv 3 \bmod 4$. For a nonzero integer $d$, we denote by $\chi_d$ the quadratic Dirichlet character associated with $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$. For an odd prime $p$, we put $p^* = p$ if $p \equiv 1 \bmod 4$ and $p^* = -p$ if $p \equiv 3 \bmod 4$.
is a Dirichlet character, we denote by \( f(\chi) \) the conductor of \( \chi \). For a Dirichlet character \( \chi \) mod \( N \), we assume \( \chi(a) = 0 \) if \( a \) and \( N \) are not co-prime. Let \( \chi \) and \( \psi \) are Dirichlet characters mod \( N \) and mod \( M \) respectively. We define a Dirichlet character \( \chi \cdot \psi \mod \text{lcm}(N, M) \) by \( \chi \cdot \psi(n) = \chi(n)\psi(n) \) for \( n \in \mathbb{Z} \). We denote by \( \chi \psi \) the primitive Dirichlet character induced by \( \chi \cdot \psi \). For a Dirichlet character \( \chi \), we denote by \( G(\chi) \) the Gauss sum; \( G(\chi) = \sum_{x \in \mathbb{Z}/f(\chi)\mathbb{Z}} \chi(x)e(x/f(\chi)) \).

2. Definition of Eisenstein series and the Fourier expansion

2.1. Let \( g \) be a positive integer and \( K \) be a field. The symplectic group of degree \( g \) is defined by

\[
\text{Sp}_g(K) = \{ \alpha \in M_{2g}(K) \mid ^t\alpha \eta \alpha = \eta \}, \quad \eta = \begin{pmatrix} 0_g & -1_g \\ 1_g & 0_g \end{pmatrix}.
\]

Then \( \text{Sp}_g \) is defined over the prime field. For \( \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Sp}_g \) with \( a, b, c, d \in M_g \), we write \( a = a_0, \ b = b_0, \ c = c_0, \ d = d_0 \). The Siegel upper half plane of degree \( g \) is defined by \( \mathcal{H}_g = \{ z \in \text{Sym}_g(\mathbb{C}) \mid \text{Im}(z) > 0 \} \). For \( \alpha \in \text{Sp}_g(\mathbb{R}), \ z \in \mathcal{H}_g \), we define

\[
\alpha \cdot z = (a_0 z + b_0)(c_0 z + d_0)^{-1}, \quad j(\alpha, z) = \det(c_0 z + d_0).
\]

Put \( \text{Sp}_g(\mathbb{Z}) = \text{Sp}_g(\mathbb{Q}) \cap \text{GL}_{2g}(\mathbb{Z}) \). Let \( \Gamma \subset \text{Sp}_g(\mathbb{Z}) \) be a congruence subgroup and \( \chi : \Gamma \to \mathbb{C}^\times \) be a character. For an integer \( k \in \mathbb{Z} \) and a \( \mathbb{C} \)-valued function \( f \) on \( \mathcal{H}_g \), we set

\[
(f|_k \gamma)(z) = f(\gamma \cdot z)^{j(\gamma, z)^{-k}}.
\]

We denote by \( F_k(\Gamma, \chi) \) the space of functions on \( \mathcal{H}_g \) satisfying the following automorphic property:

\[
(f|_k \gamma)(z) = \chi(\gamma) f(z) \quad \text{for} \ \gamma \in \Gamma. \tag{2.1}
\]

Let \( \mathbb{A} \) be the adele ring of \( \mathbb{Q} \). We denote by \( \mathfrak{f} \) (resp. \( \infty \)) the set of finite places of \( \mathbb{Q} \) (resp. the infinite place). The adelization of \( \text{Sp}_g(\mathbb{Q}) \) is denoted by \( \text{Sp}_g(\mathbb{A}) \). For a place \( v \) of \( \mathbb{Q} \), we regard \( \text{Sp}_g(\mathbb{Q}) \), \( \text{Sp}_g(\mathbb{Q}_v) \) as subgroups of \( \text{Sp}_g(\mathbb{A}) \) in a natural fashion. We put \( \text{Sp}_g(\mathbb{A}_f) = \text{Sp}_g(\mathbb{A}) \cap \prod_{v \in \mathfrak{f}} \text{Sp}_g(\mathbb{Q}_v) \). For \( \alpha \in \text{Sp}_g(\mathbb{A}_f) \), we put

\[
\alpha = \alpha_f \alpha_\infty, \quad \alpha_f \in \text{Sp}_g(\mathbb{A}_f), \quad \alpha_\infty \in \text{Sp}_g(\mathbb{A}_\infty). \tag{2.2}
\]

For a place \( v \) of \( \mathbb{Q} \), a maximal compact subgroup \( C_v \) of \( \text{Sp}_g(\mathbb{Q}_v) \) is defined by

\[
C_v = \begin{cases} \{ \alpha \in \text{Sp}_g(\mathbb{R}) \mid \alpha \mathbb{i} = \mathbb{i} \} & \text{if } v = \infty, \\ \text{Sp}_g(\mathbb{Q}_v) \cap \text{GL}_{2g}(\mathbb{Z}_v) & \text{if } v \in \mathfrak{f}. \end{cases}
\]

Here \( \mathbb{i} = i1_g \in \mathcal{H}_g \). Then, a maximal compact subgroup \( C \) of \( \text{Sp}_g(\mathbb{A}) \) is defined by \( C = \prod_{v \in \mathfrak{f}} C_v \). We define algebraic subgroups \( P_g, Q_g, R_g \) of \( \text{Sp}_g \) by

\[
P_g = \{ \alpha \in \text{Sp}_g \mid c_\alpha = 0 \}, \quad Q_g = \left\{ \begin{pmatrix} a & 0 \\ 0 & r^{-1}_a \end{pmatrix} \middle| a \in \text{GL}_g \right\}, \quad R_g = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \text{Sym}_g \right\}.
\]
Then the Iwasawa decomposition holds:

$$\text{Sp}_g(\mathbb{A}) = P_g(\mathbb{A})C_C, \quad \text{Sp}_g(\mathbb{Q}) = P_g(\mathbb{Q})\text{Sp}_g(\mathbb{Z}).$$

For $0 \leq i \leq g$, we put

$$\eta^{(i)} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad j = g - i.$$  

Then the Bruhat decomposition holds:

$$\text{Sp}_g(\mathbb{Q}) = \bigsqcup_{i=0}^{g} P_g(\mathbb{Q})\eta^{(i)}P_g(\mathbb{Q}). \quad (2.3)$$

For an open subgroup $D$ of $C$, we put

$$\Gamma = D \cap \text{Sp}_g(\mathbb{Z}).$$

Then $\Gamma$ is a congruence subgroup of $\text{Sp}_g(\mathbb{Z})$. Conversely, we obtain every congruence subgroup in this way.

Let $\chi : D \to \mathbb{C}^\times$ a group homomorphism. We denote the restriction to $\Gamma$ of $\chi$ by the same letter. For a $\mathbb{C}$-valued function $f$ on $\hat{H}^g$ satisfying (2.1), we define a $\mathbb{C}$-valued function $\phi_f$ on $\text{Sp}_g(\mathbb{A})$ by

$$\phi_f(\xi) = f(g_\infty \cdot i) j(g_\infty, i)^{-k} \chi^{-1}(\delta),$$  

where $\xi = \alpha \delta g_\infty$, $\alpha \in \text{Sp}_g(\mathbb{Q})$, $\delta \in D$, $g_\infty \in \text{Sp}_g(\mathbb{R}). \quad (2.4)$

By strong approximation theorem, we have $\text{Sp}_g(\mathbb{A}) = \text{Sp}_g(\mathbb{Q})D\text{Sp}_g(\mathbb{R})$. Therefore, $\phi_f$ is defined on $\text{Sp}_g(\mathbb{A})$ and is well defined by (2.1). $\phi = \phi_f$ satisfies the following three conditions:

$$\phi(\alpha \xi) = \phi(\xi), \quad \text{for } \alpha \in \text{Sp}_g(\mathbb{Q}). \quad (2.5)$$

$$\phi(\xi \delta) = \chi^{-1}(\delta)\phi(\xi), \quad \text{for } \delta \in D. \quad (2.6)$$

$$\phi(\xi u) = j(u, i)^{-k}\phi(\xi), \quad \text{for } u \in C_\infty. \quad (2.7)$$

We denote the space of $\mathbb{C}$-valued functions on $\text{Sp}_g(\mathbb{A})$ satisfying (2.5),(2.6) and (2.7) by $\mathcal{F}_k(D, \chi)$. For $\phi \in \mathcal{F}_k(D, \chi)$, we put

$$f_{\phi}(z) = \phi(\xi_\infty) j(\xi_\infty, i)^k, \quad \text{where } z = \xi_\infty \cdot i, \quad \xi_\infty \in \text{Sp}_g(\mathbb{R}). \quad (2.8)$$

Then $f \to \phi_f$ is a bijection from $\mathcal{F}_k(\Gamma, \chi)$ to $\mathcal{F}_k(D, \chi)$, and $\phi \to f_{\phi}$ is its inverse. For $\alpha \in \text{Sp}_g(\mathbb{Q})$, $(f_{\phi})_{\alpha}(z)$ corresponds to $\phi(\xi_\alpha r^{-1}).$
2.2. In this subsection, we define Siegel–Eisenstein series.

Let $N = \prod_p p^{n_p}$ be a positive integer and $\psi$ be a Dirichlet character mod $N$. We define a congruence subgroup $\Gamma_0(N)$ as follows

$$\Gamma_0(N) = \{ \gamma \in \text{Sp}_g(\mathbb{Z}) \mid c_\gamma \equiv 0 \mod N \}.$$

Let $k \in \mathbb{Z}$ and $s \in \mathbb{C}$. We assume $\psi(-1) = (-1)^k$. Put $z \in \mathbb{H}$, $z = x + iy$, $x, y \in \text{Sym}_g(\mathbb{R})$. We define Siegel–Eisenstein series $E_{k,\psi}^{(g)}(z, s)$ by

$$E_{k,\psi}^{(g)}(z, s) = |\det(y)|^\frac{s}{2} \sum_{\gamma \in \text{Sp}(\mathbb{Q}) \cap \Gamma_0(N) \setminus \Gamma_0(N)} \overline{\psi}(\gamma) j(\gamma, z)^{-k} |j(\gamma, z)|^{-s}. \quad (2.9)$$

Here we regard $\psi$ as a character of $\Gamma_0(N)$ by

$$\psi(\gamma) = \psi(\det(d_\gamma)), \quad \gamma \in \Gamma_0(N).$$

The right-hand side of (2.9) is absolutely convergent when $\text{Re}(s) + k > g + 1$, and $E_{k,\psi}^{(g)}(z, s)$ satisfies (2.1) for $\Gamma = \Gamma_0(N)$, $\chi = \psi$.

Next we consider the function on $\text{Sp}_g(\mathbb{A})$ which corresponds to $E_{k,\psi}^{(g)}(z, s)$ by (2.4). We define an open subgroup $C_0(N)$ of $C$ by

$$C_0(N) = \prod_{p \notin f} C_0(N)_p, \quad C_0(N)_p = \{ \alpha \in \text{Sp}_g(\mathbb{Z}_p) \mid c_\alpha \equiv 0 \mod N \}.$$ 

Let $\omega$ be the character of $\mathbb{A}_/\mathbb{Q}_\infty$ corresponding to Dirichlet character $\psi$. For $v \in f \cup \{ \infty \}$, the $v$-component $\omega_v$ satisfies the following:

If $v = \infty$, $\omega_\infty(x) = \text{sgn}(k(x))$.

If $v \in f$, $v = p$, $p \mid N$, $\omega_p(p) = \psi(p)$, $\omega_p(u) = 1$, for $u \in \mathbb{Z}_p^\times$.

If $v \in f$, $v = p$, $p \mid N$, $\omega_p(p) = \psi_p^*(p)$, $\omega_p(u) = \overline{\psi}_p(u)$, for $u \in \mathbb{Z}_p^\times$.

Here, $\psi_p$ is the Dirichlet character mod $p^{n_p}$ such that $\psi = \prod_{p \mid N} \psi_p$ and $\psi_p^*$ is

$$\psi_p^* = \prod_{q \mid N, q \neq p} \psi_q. \quad (2.10)$$

If $p \mid N$, then we consider $\omega_p$ a character of $C_0(N)_p$ by

$$\omega_p(\gamma) = \omega_p(\det d_\gamma), \quad \gamma \in C_0(N)_p.$$ 

Then the restriction of $\prod_{p \mid N} \omega_p$ to $\Gamma_0(N) = C_0(N) \cap \text{Sp}_g(\mathbb{Z})$ is $\psi$.

For $v \in f \cup \{ \infty \}$, we define $\mathbb{C}$-valued function $f_v^{(g)}$ on $\text{Sp}_g(\mathbb{Q}_v)$ as follows. Here we note that the Iwasawa decomposition holds:

1. If $v = \infty$,

$$f_{\infty}^{(g)}(\xi) = |j(\xi, i)|^{-s} j(\xi, i)^{-k} = |\det a_\infty|^{\frac{s}{2} + k} \omega_\infty(\det a_\infty) \det(u + iv)^{-k},$$

for $\xi = \alpha \gamma, \quad \alpha \in P_g(\mathbb{Q}_\infty), \quad \gamma = \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \in C_\infty$. 


(2) If \( v = p \in \mathfrak{f} \) and \( p \nmid N \),

\[
f^{(s)}_p(\xi) = |\det a_\alpha|^s \omega_p(\det a_\alpha), \quad \text{for } \xi = \alpha \gamma, \quad \alpha \in P_g(\mathbb{Q}_p), \quad \gamma \in C_p.
\]

(3) If \( v = p \in \mathfrak{f} \) and \( p \mid N \),

\[
f^{(s)}_p(\xi) = \begin{cases} 
0 & \text{if } \xi \notin P_g(\mathbb{Q}_p)C_0(N)_p, \\
|\det a_\alpha|^s \omega_p(\det a_\alpha) \omega_p(\gamma) & \text{if } \xi = \alpha \gamma, \quad \alpha \in P_g(\mathbb{Q}_p), \quad \gamma \in C_0(N)_p.
\end{cases}
\]

We define a function \( f^{(s)}_{k, \psi} \) on \( \text{Sp}_g(\mathbb{A}) \) by

\[
f^{(s)}_{k, \psi}(\xi) = \prod_{v \in \mathfrak{f} \cup \{\infty\}} f^{(s)}_v(\xi_v).
\]

Then \( f^{(s)}_{k, \psi} \) satisfies the following conditions:

\[
f^{(s)}_{k, \psi}(\alpha \xi) = |\det a_\alpha|^s \omega(\det a_\alpha) f^{(s)}_{k, \psi}(\xi), \quad \text{for } \alpha \in P_g(\mathbb{A}), \quad \text{(2.11)}
\]

\[
f^{(s)}_{k, \psi}(\xi \gamma) = \left( \prod_{p \mid N} \omega_p(\gamma_p) \right) f^{(s)}_{k, \psi}(\xi), \quad \text{for } \gamma \in C_0(N), \quad \text{(2.12)}
\]

\[
f^{(s)}_{k, \psi}(\xi u) = j(u, i)^{-k} f^{(s)}_{k, \psi}(\xi), \quad \text{for } u \in C_\infty. \quad \text{(2.13)}
\]

Here \(|x|_\mathbb{A} = \prod_{v \in \mathfrak{f} \cup \{\infty\}} |x|_v\).

We define Eisenstein series \( E^{(g)}_{k, \psi}(\xi, s) \) on \( \text{Sp}_g(\mathbb{A}) \) as follows

\[
E^{(g)}_{k, \psi}(\xi, s) = \sum_{\alpha \in P_g(\mathbb{Q}) \setminus \text{Sp}_g(\mathbb{Q})} f^{(s)}_{k, \psi}(\alpha \xi). \quad \text{(2.14)}
\]

The right-hand side is absolutely convergent when \( \text{Re}(s) + k > g + 1 \) and \( E^{(g)}_{k, \psi}(\xi, s) \) satisfies \( (2.5), (2.6) \) and \( (2.7) \) for \( \Gamma = C_0(N), \chi = \prod_{p \mid N} \omega_p \), namely the following three conditions hold:

\[
E^{(g)}_{k, \psi}(\alpha \xi, s) = E^{(g)}_{k, \psi}(\xi, s), \quad \text{for } \alpha \in \text{Sp}_g(\mathbb{Q}), \quad \text{(2.15)}
\]

\[
E^{(g)}_{k, \psi}(\xi \gamma, s) = \prod_{p \mid N} \omega_p(\gamma_p) E^{(g)}_{k, \psi}(\xi, s), \quad \text{for } \gamma \in C_0(N), \quad \text{(2.16)}
\]

\[
E^{(g)}_{k, \psi}(\xi u, s) = j(u, i)^{-k} E^{(g)}_{k, \psi}(\xi, s), \quad \text{for } u \in C_\infty. \quad \text{(2.17)}
\]

Let us verify that \( E^{(g)}_{k, \psi}(\xi, s) \) corresponds to \( E^{(g)}_{k, \psi}(\zeta, s) \) by \( (2.8) \). For \( z = x + iy \in \mathfrak{H}_g \), we put \( \xi_\infty = \left( \begin{smallmatrix} \frac{1}{2} & xy^{-\frac{1}{2}} \\ 0 & \frac{1}{2} \end{smallmatrix} \right) \in \text{Sp}_g(\mathbb{R}). \) Then we have \( z = \xi_\infty \cdot i. \) By the definition of \( f_{k, \psi} \), we see that \( \alpha \in P_g(\mathbb{A})C_0(N)S_p(\mathbb{R}) \) if \( f_{k, \psi}(\alpha \xi_\infty) \neq 0. \) By the Iwasawa decomposition \( \text{Sp}_g(\mathbb{Q}) = P_g(\mathbb{Q}) \text{Sp}_g(\mathbb{Z}) \), we have \( \text{Sp}_g(\mathbb{Q}) \cap P_g(\mathbb{A})C_0(N) \text{Sp}_g(\mathbb{R}) = P_g(\mathbb{Q})I_0(N). \) Thus we obtain
\[ E_{k,\psi}(\xi, s) j(\xi, i)^k \]
\[ = \sum_{\alpha \in \mathcal{P}_g(Q) \setminus \mathcal{P}_g(Q)} f_{k,\psi}(s, \alpha \xi) j(\xi, i)^k \]
\[ = \sum_{\alpha \in \mathcal{P}_g(Q) \cap \Gamma_0(N) \setminus \Gamma_0(N)} \left( \prod_{p | N} \omega_p(\gamma) \right) |j(\gamma \xi, i)|^{-s} |j(\gamma, z)|^{-k} \]
\[ = |\det(y)|^\frac{1}{2} \sum_{\alpha \in \mathcal{P}_g(Q) \cap \Gamma_0(N) \setminus \Gamma_0(N)} \overline{\psi}(\gamma) j(\gamma, z)^{-k} |j(\gamma, z)|^{-s} \]
\[ = E_{k,\psi}(z, s). \]

Therefore \( E_{k,\psi}(\xi, s) \) corresponds to \( E_{k,\psi}(z, s) \).

2.3. In this subsection, we consider Fourier expansion of Siegel–Eisenstein series. We abbreviate Sym_\(g\) as \( S_g \) in this subsection.

For a place \( v \) of \( Q \), we define a character \( e_v \) of \( Q_v \) by
\[ e_v(x) = \begin{cases} e(-i_v(x)), & v \in \mathfrak{f}, \\ e(x), & v = \infty. \end{cases} \]

Here \( i_v \) is the inclusion \( i_v: Q_v/Z_v \to \bigoplus_{v \in \mathfrak{f}} Q_v/Z_v = Q/Z \) when \( v \) is a finite place. By definition, \( e_v \) is trivial on \( Z_v \) when \( v \in \mathfrak{f} \). Then a character \( e_A \) of \( A/Q \) is defined by
\[ e_A(x) = \prod_{v \in \mathfrak{f} \cup \infty} e_v(x). \]

For \( X \in S_g(A) \) or \( X \in S_g(Q_v) \), we put
\[ e_A(X) = e_A(\text{Tr}(X)), \quad e_v(X) = e_v(\text{Tr}(X)). \]

Next we define Haar measure on \( S_g(A) \). If \( v \in \mathfrak{f} \), we take a Haar measure \( dx_v \) on \( S_g(Q_v) \) such that \( \int_{S_g(Q_v)} dx_v = 1 \). If \( v = \infty \), we take a Haar measure \( dx_\infty \) on \( S_g(\mathbb{R}) \) such that \( dx_\infty = \prod_{i \leq j} dx_{(i)}^{(j)} \). Here \( x_{(i)}^{(j)} \) is the \((i, j)\) component of \( x_\infty \). Then we define a Haar measure \( dx \) on \( S_g(A) \) by
\[ dx = \prod_{v \in \mathfrak{f} \cup \{\infty\}} dx_v. \]

This measure satisfies the following
\[ \int_{S_g(Q) \setminus S_g(A)} dx = 1. \]

In general, if a \( \mathbb{C} \)-valued function \( \lambda \) on \( S_g(Q) \setminus S_g(A) \) satisfies a certain condition, then \( \lambda \) has the following Fourier expansion
\[ \lambda(x) = \sum_{h \in S_g(Q)} a(h)e_A(hx), \]

\[ a(h) = \int_{S_g(Q) \backslash S_g(\mathbb{A})} \lambda(x)e_A(-hx) \, dx. \]

For \( x \in S_g \), we put

\[ \tau(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in Sp_g. \]

For \( \xi_\infty \in Sp_g(\mathbb{R}) \), we see that the function \( E_{k, \psi}(\tau(x)\xi_\infty, s) \) on \( S_g(\mathbb{A}) \) is \( S_g(\mathbb{Q}) \) invariant by (2.15). Therefore, if \( \text{Re}(s) + k > g + 1 \), then the following equations hold

\[ E_{k, \psi}(x) = \sum_{h \in S_g(\mathbb{Q})} b(h, s, \xi_\infty)e_A(hx), \quad x \in S_g(\mathbb{A}), \xi_\infty \in Sp_g(\mathbb{R}). \] (2.18)

\[ b(h, s, \xi_\infty) = \int_{S_g(\mathbb{Q}) \backslash S_g(\mathbb{A})} E_{k, \psi}(\tau(x)\xi_\infty, s)e_A(-hx) \, dx. \] (2.19)

By (2.8), \( E_{k, \psi}(\xi, s) \) corresponds to \( E_{k, \psi}(z, s) \).

Therefore if we put \( \xi_\infty = \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \) in (2.18), we have the following

\[ E_{k, \psi}^{(g)}(z, s) = \sum_{h \in S_g(\mathbb{Q})} a(h, s, y)e(hx), \] (2.20)

\[ a(h, s, y) = \det(y)^{-k/2}b\left(h, s, \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix}\right). \] (2.21)

**Proposition 2.1.** Let \( b(h, s, \xi_\infty) \) be as above. If \( \det h \neq 0 \), then we have

\[ b(h, s, \xi_\infty) = \int_{S_g(\mathbb{A})} f_{k, \psi}^{(g)}(\eta \tau(x)\xi_\infty)e_A(-hx) \, dx. \]

**Proof.** We define \( E_{k, \psi}'(\xi, s) \) by

\[ E_{k, \psi}'(\xi, s) = \sum_{\alpha \in P_g(\mathbb{Q}) \backslash P_g(\mathbb{A}) \eta P_g(\mathbb{Q})} f_{k, \psi}^{(g)}(\alpha \xi) \]

(cf. [1, Kapitel IV, 7.4 Bemerkung]). By definition, \( E_{k, \psi}'(\xi, s) \) satisfies (2.15), (2.16), (2.17) instead of \( E_{k, \psi}^{(g)}(\xi, s) \). Let

\[ E_{k, \psi}'(\tau(x)\xi_\infty, s) = \sum_{h \in S_g(\mathbb{Q})} b'(h, s, \xi_\infty)e_A(hx), \quad x \in S_g(\mathbb{A}), \xi_\infty \in Sp_g(\mathbb{R}) \]

be the Fourier expansion of \( E_{k, \psi}'(\xi, s) \). Since
On the other hand, by Bruhat decomposition (2.3), we have

\[ P_g(\mathbb{Q}) \eta P_g(\mathbb{Q}) = P_g(\mathbb{Q}) \eta Q_g(\mathbb{Q}) R_g(\mathbb{Q}) = P_g(\mathbb{Q}) \eta R_g(\mathbb{Q}), \]

\[ \eta^{-1} P_g(\mathbb{Q}) \eta \cap R_g(\mathbb{Q}) = \{1\}. \]

we have

\[ b'(h, s, \xi_\infty) = \int \sum_{\alpha \in P_g(\mathbb{Q}) \backslash P_g(\mathbb{Q})} f_{k, \psi}^{(s)}(\alpha \tau(x) \xi_\infty) e_h(-hx) \, dx \]

\[ = \int \sum_{\alpha \in R(\mathbb{Q})} f_{k, \psi}^{(s)}(\eta \alpha \tau(x) \xi_\infty) e_h(-hx) \, dx \]

\[ = \int \sum_{b \in S_g(\mathbb{Q})} f_{k, \psi}^{(s)}(\eta \tau(x + b) \xi_\infty) e_h(-hx) \, dx \]

\[ = \int f_{k, \psi}^{(s)}(\eta \tau(x) \xi_\infty) e_h(-hx) \, dx. \tag{2.22} \]

We define a differential operator \( \Delta \) acting on \( C^\infty(S_g(\mathbb{R})) \) by

\[ \Delta = \det \left( \frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial x_{ij}} \right). \]

Here \( x = (x_{ij})_{i,j} \in \text{Sym}_g(\mathbb{R}) \) and \( \delta_{ij} \) is Kronecker's delta. By the definition of \( \Delta \), we have the following

\[ \Delta e_\infty(hx) = (2\pi i)^S (\det h) e_\infty(hx), \quad x \in S_g(\mathbb{R}), \tag{2.23} \]

If \( \det c_\alpha = 0 \), then \( \Delta \left[ \left| j(\alpha, x) \right|^{-s} \right. \}

By (2.23), we have

\[ \Delta \left[ e_{k, \psi}^{(g)}(\tau(x) \xi_\infty, s) - e_{k, \psi}^{(g)}(\tau(x) \xi_\infty, s) \right] \]

\[ = (2\pi i)^S \sum_{h \in S_g(\mathbb{Q})} (\det h) \left\{ b(h, s, \xi_\infty) - b'(h, s, \xi_\infty) \right\} e_h(hx). \tag{2.25} \]

On the other hand, by Bruhat decomposition (2.3), we have

\[ \Delta \left[ e_{k, \psi}^{(g)}(\tau(x) \xi_\infty, s) - e_{k, \psi}^{(g)}(\tau(x) \xi_\infty, s) \right] \]

\[ = \sum_{i=0}^{g-1} \sum_{\alpha \in P_g(\mathbb{Q}) \backslash P_g(\mathbb{Q}) \eta^{(i)} P_g(\mathbb{Q})} \Delta \left[ f_{k, \psi}^{(s)}(\alpha \tau(x) \xi_\infty) \right] \]

\[ = \sum_{i=0}^{g-1} \sum_{\alpha \in P_g(\mathbb{Q}) \backslash P_g(\mathbb{Q}) \eta^{(i)} P_g(\mathbb{Q})} \left| j(\xi_\infty, i) \right|^{-s} j(\xi_\infty, i)^{-k} \]

\[ \times \prod_{v \in \mathfrak{f}} f_{v}^{(g)}(\alpha \tau(x_v) \xi_\infty) \Delta \left[ \left| j(\alpha, x_\infty + \xi_\infty) \right|^{-s} j(\alpha, x_\infty + \xi_\infty)^{-k} \right]. \]
By [11, Lemma 2.12], if \( i < g \) and \( \alpha \in P_g(\mathbb{Q}) \eta(i) P_g(\mathbb{Q}) \), then \( \det c_{\alpha} = 0 \). Thus (2.24) implies \( \Delta(c^{(g)}_{\alpha}(\tau(x)\xi_\infty, s) - c^{(g)}_{\alpha}(\tau(x)\xi_\infty, s)) = 0 \). Therefore, if \( \det h \neq 0 \), then by (2.25) and (2.22), we obtain

\[
b(h, s, \xi_\infty) = b'(h, s, \xi_\infty) = \int_{S_g(A)} f^{(s)}_{k, \psi}(\eta \tau(x)\xi_\infty) e_h(-hx) \, dx.
\]

By Proposition 2.1 and the definition of \( f^{(s)}_{k, \psi} \), we have

\[
b(h, s, \xi_\infty) = \int_{S_g(\mathbb{R})} f^{(s)}_{\infty}(\eta \xi_\infty) e(-hx) \, dx \prod_{p \in \mathbb{Q}_p} \int_{E_p} f^{(s)}_{p}(\eta_p \tau(x)) e_p(-hx) \, dx.
\]

Let \( p \) be a prime. For the computation of the local integral at a finite place \( \nu = p \), we define, for \( x \in S_g(\mathbb{Q}_p) \), \( \nu(x) \in \mathbb{Q}_{>0} \) by

\[
\nu(x) = |\det c|_p^{-1}, \quad x = c^{-1}d,
\]

where \( c \in GL_g(\mathbb{Z}_p) \), \( d \in M_g(\mathbb{Z}_p) \) and \( c, d \) is co-prime. Here \( c, d \) is said to be co-prime if there exist some unimodular matrices \( u \in GL_g(\mathbb{Z}_p) \) and \( v \in GL_{2g}(\mathbb{Z}_p) \) such that \( u(c \, d) v = (1_g, 0_g) \).

For \( x \in \text{Sym}_g(\mathbb{Q}_p) \), \( \eta \tau(x) \) has the following Iwasawa decomposition

\[
\eta \tau(x) = \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} = \begin{pmatrix} c & * \\ a & b \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in C_p.
\]

In particular, there exist \( c, d \in M_g(\mathbb{Z}_p) \) such that \( (c \, d) \) is co-prime and \( x = c^{-1}d \). By the next lemma, \( \nu \) is well defined.

**Lemma 2.1.** (See [11, 3.6 Proposition (3)].) Suppose \( c, c' \in M_g(\mathbb{Z}_p) \cap GL_g(\mathbb{Q}_p) \) and \( d, d' \in M_g(\mathbb{Z}_p) \). Assume \( c^{-1}d = c'^{-1}d' \) and \( (c, d), (c', d') \) are co-prime. Then there exists \( u \in GL_g(\mathbb{Z}_p) \) such that \( c = uc', d = ud' \). Therefore, if we put

\[
\mathfrak{M} = \{(c, d) \in M_{2g}(\mathbb{Z}) \mid \det c \neq 0, \ c \cdot \frac{1}{d} = d \cdot \frac{1}{c}, \ (c, d) \text{ is co-prime}\}.
\]

Then \( (c, d) \to x = c^{-1}d \) is a bijection from \( GL_g(\mathbb{Z}_p) \setminus \mathfrak{M} \) to \( S_g(\mathbb{Q}_p) \).

**Proposition 2.2.** (See [10, Lemma 3.1].) Let \( p \) be a prime. Suppose \( h \in S_g(\mathbb{Q}_p) \). Let \( S^*_g(\mathbb{Z}_p) \subset S_g(\mathbb{Q}_p) \) be the set of half-integral symmetric matrices. Then, the following assertions hold:

1. \( p \mid N \),

\[
\int_{S_g(\mathbb{Q}_p)} f^{(s)}_{p}(\eta \tau(x)) e_p(-hx) \, dx = \begin{cases} \sum_{x \in S^*_g(\mathbb{Q}_p)/S_g(\mathbb{Z}_p)} \omega_p(v(x)) v(x)^{-k_{\tau}^s} e_p(-hx) & h \in S^*_g(\mathbb{Z}_p), \\ 0 & h \not\in S^*_g(\mathbb{Z}_p). \end{cases}
\]

2. \( p \mid N \),
\[
\int_{S_g(\mathbb{Q}_p)} f_p(s)(\eta \tau(x))e_p(-hx) \, dx
\]
\[
= \begin{cases} 
\sum_{x \in S_g(\mathbb{Q}_p)'/S_g(\mathbb{Z}_p)} \overline{\omega}_p(\nu(x))\omega_p(\nu(x) \det x)\nu(x)^{-k-s}e_p(-hx) & \text{ if } h \in S^*_g(\mathbb{Z}_p), \\
0 & \text{ if } h \notin S^*_g(\mathbb{Z}_p). 
\end{cases}
\]

Here \(S_g(\mathbb{Q}_p)' = \{c^{-1}d \mid (c, d) \in \mathcal{M}, \ c \equiv 0 \mod N\}\) and \(\mathcal{M}\) is as in Lemma 2.1.

**Proof.** We only prove (2). We can prove (1) in a similar way. For \(x \in S_g(\mathbb{Q}_p)\), \(x\) has the Iwasawa decomposition as in (2.27). Then \(x = c^{-1}d\) with \((c, d) \in \mathcal{M}\). By the definition of \(f_p(s)\), \(f_p(s)(\eta x) \neq 0\) implies \(c \equiv 0 \mod N\). By \(f_p(s)(\eta x) = \overline{\omega}_p(\det(c)\omega_p(\det(d))\det(c))^{s+k} = \omega_p(\det(c))\nu(x)^{-k-s}\), we have

\[
\int_{S_g(\mathbb{Q}_p)} f_p(s)(\eta \tau(x))e_p(-hx) \, dx = \int_{S_g(\mathbb{Q}_p)'} \overline{\omega}_p(\nu(x))\omega_p(\nu(x) \det x)\nu(x)^{-s-k}e_p(-hx) \, dx.
\]

By definition, \(\nu(x)\) is \(S_g(\mathbb{Z}_p)\) invariant. By \(c \equiv 0 \mod N\), \(d \in \text{GL}_g(\mathbb{Z}_p)\), we see that \(\nu(x) \det x \mod N\mathbb{Z}_p\) is also \(S_g(\mathbb{Z}_p)\) invariant. Therefore, if \(h \notin S^*_g(\mathbb{Z}_p)\) then the integral vanishes. If \(h \in S^*_g(\mathbb{Z}_p)\), then by the choice of Haar measure, we obtain (2). \(\square\)

**Definition 2.1.** Let \(p\) be a prime and \(\psi\) be a Dirichlet character mod \(p^n\). Suppose \(h \in S^*_g(\mathbb{Z}_p)\). We define a Dirichlet series \(S_p(h, \psi, s)\) by

\[
S_p(h, \psi, s) = \begin{cases} 
\sum_{x \in S_g(\mathbb{Q}_p)'/S_g(\mathbb{Z}_p)} \nu(x)^{-s}e_p(-hx) & \text{ if } n = 0, \\
\sum_{x \in S_g(\mathbb{Q}_p)'/S_g(\mathbb{Z}_p)} \overline{\psi}_p(\nu(x)\det(x))\nu(x)^{-s}e_p(-hx) & \text{ if } n \geq 1.
\end{cases}
\]

Here \(S_g(\mathbb{Q}_p)' = \{c^{-1}d \mid (c, d) \in \mathcal{M}, \ c \equiv 0 \mod N\}\) and \(\mathcal{M}\) is as in Lemma 2.1. This Dirichlet series is called the Siegel series. If \(n = 0\), then \(S_p(h, \psi, s)\) does not depend on \(\psi\). Therefore we denote \(S_p(h, \psi, s)\) by \(S_p(h, s)\) when \(n = 0\). We define the formal power series \(A_p(h, \psi; T)\) corresponding to \(S_p(h, \psi, s)\)

\[
A_p(h, \psi; p^{-s}) = S_p(h, \psi, s).
\]

We denote \(A_p(h, \psi; T)\) by \(A_p(h; T)\) when \(n = 0\).

Suppose \(a \in \mathbb{Z}_p^\times\) and \(u \in \text{GL}_g(\mathbb{Z}_p)\). By the definition of the Siegel series, we have

\[
S_p(ah, s, \psi) = \overline{\psi}_p(a^g)S_p(h, \psi, s), \quad S_p(h[u], s, \psi) = \overline{\psi}_p((\det u)^2)S_p(h, \psi, s). \quad (2.28)
\]

By Proposition 2.2, we have

\[
\int_{S_g(\mathbb{Q}_p)} f_p(s)(\eta \tau(x))e_p(-hx) \, dx = A_p(h, \psi;p^{-k-s}). \quad (2.29)
\]

Here \(\psi^*_p\) is defined by (2.10).
Next let us consider the integral of the right-hand side of (2.26) at the infinite place. Let \( h, y \in \text{Sym}_g(\mathbb{R}) \) be symmetric matrices and assume \( y \) is positive definite. For \( \alpha, \beta \in \mathbb{C} \), we define a function \( \xi \) by

\[
\xi(y, h; \alpha, \beta) = \int_{S_g(\mathbb{R})} \det(x + iy)^{-\alpha} \det(x - iy)^{-\beta} e(-hx) \, dx. \tag{2.30}
\]

By the definition of \( f_{\infty}^{(s)} \), we have

\[
\int_{S_g(\mathbb{R})} f_{\infty}^{(s)}(\eta_{\tau}(x)\xi_{\infty}) e(-hx) \, dx = \det(y)^{(k+s)/2} \xi(y, h; k+s/2, s/2). \tag{2.31}
\]

For the Fourier coefficients for \( E^{(g)}_{k, \psi}(z, s) \) we have the following proposition.

**Proposition 2.3.** Suppose \( z = x + iy \in S_g \) with \( x, y \in S_g(\mathbb{R}) \). Let (2.20) be the Fourier expansion of \( E^{(g)}_{k, \psi}(z, s) \). If \( \det h \neq 0 \), then we have

\[
a(h, E^{(g)}_{k, \psi}) = \det(y)^{s/2} \xi(y, h; k+s/2, s/2) \prod_{p \mid N} A_p(h; \overline{\psi}(p) p^{-k-s}) \prod_{p \nmid N} A_p(h, \psi; \overline{\psi}_p(p) p^{-k-s}).
\]

**Proof.** This follows from (2.26), (2.29) and (2.31). \( \square \)

2.4. By Proposition 2.3, the Fourier coefficients of \( E^{(g)}_{k, \psi}(z, s) \) has the Euler product expression. For a place \( v \) of \( \mathbb{Q} \) such that \( v = \infty \) or \( v = p \mid N \), the explicit form of the Euler factor is known. In this subsection, we introduce some of the known results.

First we introduce the result of \( \xi(y, h; \alpha, \beta) \).

**Theorem 2.1.** (See [10, (7.11), (7.12)]; [9, (4.34K), (4.35K)]) Suppose \( y, h \in \text{Sym}_g(\mathbb{R}) \) be symmetric matrices and \( y \) is positive definite. Let \( p \) and \( q \) be the number of positive and negative eigenvalue of \( h \) respectively and put \( t = g - p - q \). We denote by \( \delta_+(hy) \) (resp. \( \delta_-(hy) \)) the product of all positive (resp. negative) eigen values of \( y^{1/2}hy^{1/2} \). For \( m \in \mathbb{Z}_{\geq 0} \), we define by

\[
\Gamma_m(s) = \begin{cases} 1 & \text{if } m = 0, \\ \pi^{m(m-1)/4} \prod_{l=0}^{m-1} \Gamma(s - \frac{l}{2}) & \text{if } m \geq 1. \end{cases}
\]

Then, there exists a function \( \omega(y, h; \alpha, \beta) \) holomorphic with respect to \( \alpha \) and \( \beta \) and satisfies the following equation

\[
\xi(y, h; \alpha, \beta) = i^g(\alpha - \beta) 2^g \pi^\theta \Gamma_1 \left( \alpha + \beta - \frac{m+1}{2} \right) \Gamma_{g+q}(\alpha)^{-1} \Gamma_{g+p}(\beta)^{-1} \times \det(y)^{(g+1)/2 - \alpha - \beta} \delta_+(hy)^\alpha (g+1)/2 + q/4 \delta_-(hy)^\beta (g+1)/2 + p/4 \omega(y, h; \alpha, \beta).
\]

Here \( \tau, \theta \) are

\[
\tau = p\alpha + q\beta + t + \frac{1}{2} \{t(t-1) - pq\},
\]

\[
\theta = (2p - g)\alpha + (2q - g)\beta + g + \frac{t(g+1)}{2} + \frac{pq}{2}.
\]
If $h$ is positive definite, then the following holds

$$\xi(y, h; \alpha, 0) = 2^g(1-\delta)/i^{-g\alpha}(2\pi)^g \Gamma_g(\alpha)^{-1} \det(h)^{(g+1)/2} e(iy)h).$$

Next, we introduce some results of the Siegel series.

**Proposition 2.4.** (See [11, 15.4. Proposition].) If $h = 0$, then $A_p(0; T)$ is as follows

$$A_p(0; T) = \frac{1 - T}{1 - p^2 T^2}\prod_{i=1}^{[n/2]} \frac{1 - p^{2i}T^2}{1 - p^{2g-2i+1}T^2}.$$

By definition, $S_p(h, \psi, s)$ is majorized by $A_p(0; p - \text{Re}(s))$. By this Proposition, we see that the product $\prod_{p|N} A_p(h; \psi(p)p^{k-s}) \prod_{p|N} A_p(h, \psi; \psi(p)p^{k-s})$ in Proposition 2.3 is absolutely convergent when $k + \text{Re}(s) > g + 1$.

**Proposition 2.5.** (See [11, 14.9. Proposition].) Suppose $h \in \text{Sym}^*_g(\mathbb{Z}_p)$ and $\det h \not= 0$.

Then $A_p(h; T)$ is $\mathbb{Z}$-coefficient polynomial with constant term 1 and divisible by a polynomial $\gamma_p(h; T)$ defined as follows

$$\gamma_p(h; T) = \begin{cases} 
\frac{1 - T}{1 - \lambda(h)p^{g/2}T^2}\prod_{i=1}^{g/2} (1 - p^{2i}T^2) & \text{if } g \text{ is even}, \\
(1 - T)\prod_{i=1}^{(g-1)/2} (1 - p^{2i}T^2) & \text{if } g \text{ is odd}.
\end{cases}$$

Here, when $g$ is even, we define $d = (-1)^{g/2} \det(h)$, $K_h = \mathbb{Q}_p(\sqrt{d})$ and

$$\lambda(h) = \begin{cases} 
1 & K_h = \mathbb{Q}_p, \\
-1 & K_h/\mathbb{Q}_p \text{ is unramified quadratic extension}, \\
0 & K_h/\mathbb{Q}_p \text{ is ramified extension}.
\end{cases}$$

Moreover if $h$ satisfies the following condition,

$$\begin{cases} 
\det(2h) \in \mathbb{Z}_p^\times & \text{if } g \text{ is even}, \\
\det(2h) \in 2\mathbb{Z}_p^\times & \text{if } g \text{ is odd},
\end{cases}$$

then

$$A_p(h; T) = \gamma_p(h; T).$$

We define $F_p^{(g)}(h; T)$ by

$$F_p^{(g)}(h; T) = A_p(h; T)/\gamma_p(h; T).$$

(2.32)

By Proposition 2.5, if $h \in \text{Sym}^*_g(\mathbb{Z}_p)$ and $\det h \not= 0$, then $F_p^{(g)}(h; T)$ is $\mathbb{Z}$-coefficient polynomial with constant term 1 and if $h \in \text{Sym}^*_g(\mathbb{Z})$, then

$$F_p^{(g)}(h; T) = 1$$

for all but a finite prime. Kaufhold calculated $F_p^{(g)}(h; T)$ in the case where $g = 2$. 
Remark 2.1. Katsurada proved the explicit formula and the functional equation for \( F_p^{(g)}(h; T) \) for all positive integer \( g \) [3].

Henceforth, we assume the weight \( k \) is larger than \( g + 1 \). Then the left-hand side of (2.9) is absolutely convergent for \( s = 0 \). We denote \( E_{k, \psi}(z) \) by \( E_{k, \psi}^{(g)}(z) \). Since \( E_{k, \psi}^{(g)}(z) \) is a Siegel modular form, \( E_{k, \psi}^{(g)}(z) \) has Fourier expansion of the form

\[
E_{k, \psi}^{(g)}(z) = \sum_{0 \leq h \in S_{k, g}^{(g)}} a(h, E_{k, \psi}^{(g)}) e(hz). \quad (2.34)
\]

By Proposition 2.3, Theorem 2.1, Proposition 2.4, Proposition 2.5 and Proposition 2.6, we have the following theorem.

Theorem 2.2. Let (2.34) be the Fourier expansion of \( E_{k, \psi}^{(g)}(z) \). Suppose \( k > g + 1 \) and \( \det h \neq 0 \). For a Dirichlet \( L \)-function \( L(s, \chi) \) and a positive integer \( N \), we put \( L^{(N)}(s, \chi) = \prod_{p | N} (1 - \chi(p)p^{-s})^{-1} \). Then \( a(h, E_{k, \psi}^{(g)}) \) is as follows.

1. If \( g \) is even,

\[
a(h, E_{k, \psi}^{(g)}) = \xi(y, h; k, 0) e(-iyh) \frac{L^{(N)}(k - g/2, \chi h \psi)}{L(k, \psi) L^{(N)}(2k - g, \psi^2)} \prod_{i=1}^{(g-2)/2} L^{(N)}(2k - 2i, \psi^2)^{-1} \\
\times \prod_{p | N} F_p^{(g)}(h; \psi(p) p^{-k}) \prod_{p | N} A_p(h, \psi p; \psi p^2(p) p^{-k}).
\]
3. Computations of Siegel series

If \( g \) is odd,

\[
a(h, y, F^{(g)}_{\psi_h}) = \xi(y, h; k, 0) e(-iyh) L^{(N)}(k, \psi)^{-1} \prod_{i=1}^{(g-1)/2} L^{(N)}(2k - 2i, \psi^2)^{-1} \times \prod_{p \mid N} \mathcal{E}_p^{(g)}(h; \psi(p)p^{-k}) \prod_{p \mid N} A_p(h, \psi_p; \psi_p^*(p)p^{-k}).
\]

Here \( \xi(y, h; k, 0) \) is defined by (2.30). And by Theorem 2.1, we have

\[
\xi(y, h; \alpha, 0) = 2^{g(1-g)/2} i^{-g\alpha} (2\pi)^{g\alpha} \Gamma_g(\alpha)^{-1} \det(h)^{\alpha-(g+1)/2} e(iyh)
\]

where \( \Gamma_g(s) = \pi^{g(g-1)/4} \prod_{i=0}^{g-1} \Gamma(s - i/2) \).

\( A_p(h, \psi; T) \) is defined in Definition 2.1. \( \chi_h, F^{(g)}_p(h; T) \) is defined in Proposition 2.6.

3. Computations of Siegel series

In this section, we compute Siegel series \( S_p(\psi, h) \) explicitly in the case where degree is two, \( \psi \) is a primitive character mod \( p^n \) and rank \( h \) is two, where \( S_p(\psi, h) \) is defined in Definition 2.1.

3.1. First, we state the results of calculation of \( S_p(\psi, h) \). By (2.28), it is sufficient to calculate \( S_p(\psi, h) \) for each representative \( h \) of \( \text{GL}_2(\mathbb{Z}_p) \)-equivalent class of \( \text{Sym}_2(\mathbb{Z}_p) \). Let \( m \) be the maximum integer which satisfies \( p^{-m}h \in \text{Sym}_2(\mathbb{Z}_p)^* \) and put \( h' = p^{-m}h \).

If \( p \neq 2 \), then \( h' \) is \( \text{GL}_2(\mathbb{Z}_p) \)-equivalent to the matrix of the form

\[
\left( \begin{array}{cc} \alpha & 0 \\ 0 & p^t \beta \end{array} \right), \quad \alpha, \beta \in \mathbb{Z}_p^X.
\]

If \( p = 2 \), then \( h' \) is \( \text{GL}_2(\mathbb{Z}_2) \)-equivalent to the matrix of the form

\[
\left( \begin{array}{cc} \alpha & 0 \\ 0 & 2^t \beta \end{array} \right), \quad \alpha, \beta \in \mathbb{Z}_2^X, \quad \left( \begin{array}{cc} 0 & 1 \\ 1/2 & 0 \end{array} \right), \quad \left( \begin{array}{cc} 1 & 1 \\ 1/2 & 1 \end{array} \right).
\]

If \( h' \) is one of above matrices, then the explicit forms of \( S_p(\psi, h) \) are as follows.

**Proposition 3.1.** Let \( \psi \) be a primitive Dirichlet character mod \( p^n \). Suppose \( p \neq 2 \) and put \( h = p^m \left( \begin{array}{cc} \alpha & 0 \\ 0 & p^r \beta \end{array} \right) \), with \( m \geq 0 \) and \( \alpha, \beta \in \mathbb{Z}_p^X \). We denote the quadratic character mod \( p \) by \( \chi_{p^r}. \) Then the following assertions hold:

1. Suppose \( \psi = \chi_{p^r} \), then

\[
S_p(\psi, h) = \left\{ \begin{array}{ll}
\psi(-1)((p - 1) \sum_{i=1}^{m+t/2} p^{(3-2s)i-2} - p^{(3-2s)(m+t/2+1)-2}) & \text{if } t \text{ is even}, \\
\psi(-1)((p - 1) \sum_{i=1}^{m+t/2+1/2} p^{(3-2s)i-2} + \chi_{p^r}(\alpha \beta) p^{(3-2s)(m+t/2+1)-3/2}) & \text{if } t \text{ is odd}.
\end{array} \right.
\]
(2) Suppose $\psi \neq \chi_p^r$, then
\[
S_p(\psi, h) = \begin{cases}
\psi(\alpha\beta)G(\psi)^2p(3-2s)(m+n+t/2)−5/2n & \text{if } n−t \text{ is even}, \\
\varepsilon_p(\psi \chi_p^r)(\alpha\beta)G(\psi)G(\psi \chi_p^r)p(3-2s)(m+n+t/2)−5/2n & \text{if } n−t \text{ is odd}.
\end{cases}
\]

Here, $\varepsilon_p$ is given by
\[
\varepsilon_p = \begin{cases}
1 & \text{if } p \equiv 1 \mod 4, \\
 i & \text{if } p \equiv 3 \mod 4.
\end{cases}
\]

**Proposition 3.2.** Let $\psi$ be a primitive Dirichlet character mod $2^q$. Put $h = 2^m h'$ with $m \geq 0$ and $h' \in \Sym_2(Z_2)$. Then the following assertions hold:

(a) Suppose $h' = \begin{pmatrix} \alpha & 0 \\ 0 & 2\beta \end{pmatrix}$ with $\alpha, \beta \in \mathbb{Z}_p^\times$.

(1) Suppose $n > 3$, then
\[
S_2(\psi, h) = \begin{cases}
(\psi(\alpha\beta) + \psi(\alpha\beta + 2n-2))G(\psi)^22(3-2s)(n+m+t/2)−5/2n & \text{if } n−t \text{ is even}, \\
\psi(\alpha\beta + 2n-3)G(\psi)^22(3-2s)(n+m+t/2)−5/2n+1/2 & \text{if } n−t \text{ is odd}.
\end{cases}
\]

(2) Suppose $n = 2$, then
\[
S_2(\psi, h) = \begin{cases}
-\left(\sum_{i=2}^{n+m+t/2+1} 2(3-2s)i−3 + \psi(\alpha\beta)2(3-2s)(m+t/2+3/2)−3 \\
+ \psi(1+\alpha\beta)2(3-2s)(m+t/2+5/2)−5/2) & \text{if } t \text{ is even}, \\
-\left(\sum_{i=2}^{n+m+t/2+1} 2(3-2s)i−3 − 2(3-2s)(m+t/2+3/2)−3 \right) & \text{if } t \text{ is odd}.
\end{cases}
\]

(3) Suppose $n = 3$, then
\[
S_2(\psi, h) = 0 \quad \text{if } (m, t) = (0, 0).
\]

For $(m, t) \neq 0$, we have
\[
S_2(\psi, h) = \begin{cases}
\psi(-1)\left(\sum_{i=3}^{n+m+t/2+1} 2(3-2s)i−4 − 2(3-2s)(m+t/2+2)−9/2 \right) & \text{if } t \text{ is even}, \\
\psi(-1)\left(\sum_{i=3}^{n+m+t/2+3/2} 2(3-2s)i−4 \\
+ \psi(1+\alpha\beta)2(3-2s)(m+t/2+5/2)−4 \\
+ (\psi(\alpha\beta) + \psi(\alpha\beta + 2))2(3-2s)(m+t/2+3/2)−4 \right) & \text{if } t \text{ is odd}.
\end{cases}
\]

(b) Suppose $h' = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$.

(1) Suppose $n > 3$, then
\[
S_2(\psi, h) = \begin{cases}
0 & \text{if } m = 0, \\
\psi(-1)G(\psi)^22(3-2s)(m+n−1)−2n+1 & \text{if } m \geq 1.
\end{cases}
\]

(2) Suppose $n = 2$, then
\[
S_2(\psi, h) = \begin{cases}
0 & \text{if } m = 0, \\
-\sum_{i=2}^{m} 2(3-2s)i−3 + 2(3-2s)(m+1)−3 & \text{if } m \geq 1.
\end{cases}
\]
(3) Suppose \( n = 3 \), then
\[
S_2(\psi, h) = \begin{cases}
0 & \text{if } m = 0, 1, \\
\psi(-1) \left( \sum_{i=3}^{m} 2^{(3-2s)i-4} - 2^{(3-2s)(m+1)-4} \right) & \text{if } m \geq 2.
\end{cases}
\]

(c) Suppose \( h' = \left( \begin{array}{cc} 1 & 1 \\ \frac{1}{2} & 1 \end{array} \right) \).

(1) Suppose \( n > 3 \), then
\[
S_2(\psi, h) = \begin{cases}
0 & \text{if } m = 0, \\
\psi(3G(\overline{\psi}^2)2^{(3-2s)(m+n-1)-2n+1} - \psi(3G(\overline{\psi}^2)2^{(3-2s)(m+n-1)-2n+1}) & \text{if } m \geq 1 \text{ and } n \text{ is even}, \\
-\sum_{i=2}^{m} 2^{(3-2s)i-3} + 2^{(3-2s)(m+1)-3} & \text{if } m \geq 1 \text{ and } n \text{ is odd}.
\end{cases}
\]

(2) Suppose \( n = 2 \), then
\[
S_2(\psi, h) = \begin{cases}
0 & \text{if } m = 0, \\
\psi(3) \left( -\sum_{i=2}^{m} 2^{(3-2s)i-4} + 2^{(3-2s)(m+1)-4} \right) & \text{if } m \geq 1.
\end{cases}
\]

(3) Suppose \( n = 3 \), then
\[
S_2(\psi, h) = \begin{cases}
0 & \text{if } m = 0, 1, \\
\psi(3) \left( \sum_{i=3}^{m} 2^{(3-2s)i-4} - 2^{(3-2s)(m+1)-4} \right) & \text{if } m \geq 2.
\end{cases}
\]

3.2. In this subsection, following [2], we calculate \( S_p(\psi, h) \). In the following, we calculate \( S_p(\overline{\psi}, h) \) in stead of \( S_p(\psi, h) \).

We denote by \( \mathfrak{M}(p^n) \) the subset of \( M_2(\mathbb{Z}_p) \times M_2(\mathbb{Z}_p) \) satisfying the following condition.

\((C, D) \in \mathfrak{M}(p^n)\) if and only if

\[
\det C = p^i, \quad i \geq 0, \quad C \equiv 0 \mod p^n, \quad (C, D) \text{ is co-prime and } C \cdot \overset{\text{t}}{D} = D \cdot \overset{\text{t}}{C}.
\]

By Definition 2.1, \( S_p(\overline{\psi}, h) \) is
\[
S_p(\overline{\psi}, h) = \sum_{(C, D) \in \mathfrak{M}(p^n)} \psi(\det D)\psi_p\left(-hC^{-1}D\right)\det(C)^{-5}.
\]

Here, we denote \( D \equiv D' \mod C \) if only if \( D - D' \in CM_2(\mathbb{Z}_p) \). The character \( \psi_p(X) \) is the unique continuous character such that

\[
\psi_p(X) = e(-\text{Tr} X) \quad \text{for } X \in M_2(\mathbb{Z}[1/p]).
\]

We put
\[
\mathfrak{M}'(p^n) = \{(C, D) \in M_2(\mathbb{Z}_p) \times M_2(\mathbb{Z}_p) \mid \det C = p^i, \quad i \geq 0, \quad C \equiv 0 \mod p^n, \quad C \cdot \overset{\text{t}}{D} = D \cdot \overset{\text{t}}{C} \}.
\]

If \((C, D) \in \mathfrak{M}'(p^n)\) and \((C, D)\) is not co-prime, then \( \det D \) is divisible by \( p \). Therefore,
\[
S_p(\overline{\psi}, h) = \sum_{(C, D) \in \mathfrak{M}(p^n) \setminus \mathfrak{M}'(p^n)} \psi(\det D)\psi_p\left(-hC^{-1}D\right)\det(C)^{-5}.
\]
For a complete set representatives of $\text{SL}_2(\mathbb{Z}_p) \setminus \mathcal{M}(p^n)$, we take

$$\left\{ (C, D) \mid C = p^i \begin{pmatrix} 1 & 0 \\ 0 & p^j \end{pmatrix} V, \ V \in \Gamma^0(p^j) \setminus \text{SL}_2(\mathbb{Z}_p), \ i \geq n, \ j \geq 1, \ C^i D = D^i C \right\}.$$ 

Here, $\Gamma^0(p^j)$ is $\{(a \ b) \ c \ d \} \in \text{SL}_2(\mathbb{Z}_p) \ | \ b \equiv 0 \mod p^j$. When $j \geq 1$, we put

$$X = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mid 0 \leq u \leq p^j - 1 \right\},$$

$$Y = \left\{ \begin{pmatrix} p u & 1 \\ -1 & 0 \end{pmatrix} \mid 0 \leq u \leq p^{j-1} - 1 \right\},$$

then $X \cup Y$ is a complete set of representatives of $\Gamma^0(p^j) \setminus \text{SL}_2(\mathbb{Z}_p)$. We put

$$D = \begin{pmatrix} a & b \\ c & d \end{pmatrix} tV^{-1},$$

then

$$C^{-1} D = V^{-1} p^{-i} \begin{pmatrix} a \ p^{-j} c & b \ p^{-j} d \end{pmatrix} tV^{-1}.$$ 

Therefore $C \cdot t D = D \cdot t C$ is equivalent to $c = p^j b$.

We note that for $C = p^i \begin{pmatrix} 1 & 0 \\ 0 & p^j \end{pmatrix} V$ we have

$$\{ D \mid D \mod C \} = \left\{ \begin{pmatrix} a \ b \\ p^j b \ d \end{pmatrix} tV^{-1} \mid a, b \ mod \ p^j, \ d \ mod \ p^{i+j} \right\}.$$ 

Thus, $S_p(\psi, h)$ becomes as follows

$$S_p(\psi, h) = \sum_{i=n}^{\infty} \sum_{j=0}^{\infty} \sum_{a, b \ mod \ p^j}^{\psi(ad - p^j b^2)} \sum_{d \ mod \ p^{i+j}}^{\psi(ad - p^j b^2)} \psi \left( \frac{1}{p_i - m} h' V^{-1} \begin{pmatrix} a & b \\ b & p^{-j} d \end{pmatrix} tV^{-1} \right) 
	\times e \left( \frac{1}{p_i - m} h' V^{-1} \right) \frac{1}{p_i - m} \left( \begin{pmatrix} a & b \\ b & p^{-j} d \end{pmatrix} \right) \psi \left( \frac{1}{p_i - m} \left( V^{-1} \right) \right).$$

(3.1)

Here, we put $h = p^m h'$ and $h'[V^{-1}] = tV^{-1} h' V^{-1}$. 
We define $A$, $B$, $B_1$, $B_2$ and $C$ as follows

\[
A = \sum_{i=n}^{\infty} p^{-2is} \sum_{a,b,d \mod p^i} \psi(ad - b^2) e\left(\frac{1}{p^{i-m}} \left(\begin{array}{c} a \\
 b \\
d \end{array}\right) h'\right). \tag{3.2}
\]

\[
B_1 = \sum_{i=n}^{\infty} \sum_{j=1}^{n-1} p^{-(2i+j)s} \sum_{a \mod p^i} \sum_{u \mod p^i} \psi(ad - p^j b^2) \times e\left(\frac{1}{p^{i-m}} \left(\begin{array}{c} a \\
 b \\
d \end{array}\right) h'\left[\begin{array}{c} 1 \\
-1 \\
u \\
1 \end{array}\right]\right). \tag{3.3}
\]

\[
B_2 = \sum_{i=n}^{\infty} \sum_{j=n}^{\infty} p^{-(2i+j)s} \sum_{a \mod p^i} \sum_{u \mod p^i} \psi(ad) \times e\left(\frac{1}{p^{i-m}} \left(\begin{array}{c} a \\
 b \\
d \end{array}\right) h'\left[\begin{array}{c} 1 \\
0 \\
u \\
1 \end{array}\right]\right). \tag{3.4}
\]

\[
B = \sum_{i=n}^{\infty} \sum_{j=1}^{\infty} p^{-(2i+j)s} \sum_{a \mod p^i} \sum_{u \mod p^i} \psi(ad - p^j b^2) \times e\left(\frac{1}{p^{i-m}} \left(\begin{array}{c} a \\
 b \\
d \end{array}\right) h'\left[\begin{array}{c} 1 \\
0 \\
u \\
1 \end{array}\right]\right) = B_1 + B_2. \tag{3.5}
\]

\[
C = \sum_{i=n}^{\infty} \sum_{j=1}^{\infty} p^{-(2i+j)s} \sum_{a \mod p^i} \sum_{u \mod p^{i-1}} \psi(ad - p^j b^2) \times e\left(\frac{1}{p^{i-m}} \left(\begin{array}{c} a \\
 b \\
d \end{array}\right) h'\left[\begin{array}{c} 0 \\
1 \\
u \\
-p \end{array}\right]\right). \tag{3.6}
\]

By (3.1), we have

\[
S_p(\psi, h) = A + B_1 + B_2 + C = A + B + C. \tag{3.7}
\]

In the following subsections, we express $A$, $B$, $B_1$, $B_2$ and $C$ by Gauss sums. For the computation, we prove some elementary lemmas.

**Lemma 3.1.** Let $\varphi : \mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{C}^\times$ be additive character of $\mathbb{Z}/p^n\mathbb{Z}$ of order $p^n$, and $f$ be a $\mathbb{C}$-valued function on $\mathbb{Z}/p^n\mathbb{Z}$. If $0 \leq l < n$, then

\[
\sum_{i \mod p^n} f(i) \varphi(i) = 0.
\]
Proof.

\[
\sum_{i \mod p^n} f(i) \varphi(i) = \sum_{i \mod p^l} \sum_{j \mod p^{n-i}} f(i + p^j) = \sum_{i \mod p^l} f(i) \varphi(i) \sum_{j \mod p^{n-l}} \varphi(p^j) = 0. \quad \Box
\]

Lemma 3.2. Let \( n \geq 0 \) if \( p \neq 2 \), and \( n \geq 2 \) if \( p = 2 \). Let \( a \in \mathbb{Z}/p^n\mathbb{Z} \), \((a, p) = 1\).

(1) Suppose \( p \neq 2 \), then we have

\[
\sum_{x \mod p^n} e\left(\frac{ax^2}{p^n}\right) = \begin{cases} p\frac{n}{2} & \text{if } n \text{ is even}, \\ \varepsilon_p \chi_{p^*}(a) p\frac{n}{2} & \text{if } n \text{ is odd}. \end{cases}
\]

(2) Suppose \( p = 2 \), then we have

\[
\sum_{x \mod 2^n} e\left(\frac{ax^2}{2^n}\right) = \begin{cases} (1 + e\left(\frac{a}{4}\right))2\frac{n}{2} & \text{if } n \text{ is even}, \\ \left(\frac{a}{8}\right)2^{\frac{n+1}{2}} & \text{if } n \text{ is odd}. \end{cases}
\]

Proof. Let \( n \geq 2 \) if \( p \neq 2 \), and \( n \geq 4 \) if \( p = 2 \), then we show that

\[
\sum_{x \mod p^n} e\left(\frac{ax^2}{p^n}\right) = p \sum_{x \mod p^{n-2}} e\left(\frac{ax^2}{p^{n-2}}\right).
\]

Lemma 3.2 follows from this.

(i) Suppose \( p \neq 2 \), then

\[
\sum_{x \mod p^n} e\left(\frac{ax^2}{p^n}\right) = \sum_{x \in \mathbb{Z}/p^n\mathbb{Z}} e\left(\frac{ax^2}{p^n}\right) + \sum_{x \in (\mathbb{Z}/p^n\mathbb{Z})^\times} e\left(\frac{ax^2}{p^n}\right)
\]

\[
= p \sum_{x \mod p^{n-2}} e\left(\frac{ax^2}{p^{n-2}}\right) + \sum_{x \mod p^n} (1 + \chi_{p^*}(x)) e\left(\frac{ax}{p^n}\right)
\]

\[
= p \sum_{x \mod p^{n-2}} e\left(\frac{ax^2}{p^{n-2}}\right) + \sum_{x \mod p^n} e\left(\frac{ax}{p^n}\right) + \sum_{x \mod p^n} \chi_{p^*}(x) e\left(\frac{ax}{p^n}\right)
\]

\[
= p \sum_{x \mod p^{n-2}} e\left(\frac{ax^2}{p^{n-2}}\right).
\]

(ii) Suppose \( p = 2 \), then

\[
\sum_{x \mod 2^n} e\left(\frac{ax^2}{2^n}\right) = \sum_{x \in 2\mathbb{Z}/2^n\mathbb{Z}} e\left(\frac{ax^2}{2^n}\right) + \sum_{x \in (\mathbb{Z}/2^n\mathbb{Z})^\times} e\left(\frac{ax^2}{2^n}\right)
\]
\[= 2 \sum_{x \mod 2^{n-2}} e^{\frac{ax^2}{2^{n-2}}} + 4 \sum_{x \mod 2^{n-3}} e^{\frac{1 + 8x}{2^{n-3}}} \]

Lemma 3.3. Let \( \alpha \in \mathbb{Z}/p^n\mathbb{Z} \), \( \beta \in (\mathbb{Z}/p^n\mathbb{Z})^\times \), and \( \psi \) be a primitive Dirichlet character mod \( p^n \) (\( n \geq 1 \)).

1. Suppose \( p = 2 \), then we have

\[\sum_{x \mod 2^n} \psi(\alpha + \beta x^2) = \begin{cases} 2^n (\psi(\alpha) + \psi(\alpha + 2^{n-2} \beta)) & \text{if } n \text{ is even,} \\ 2^{n+1} \psi(\alpha + 2^{n-3} \beta) & \text{if } n \text{ is odd.} \end{cases}\]

2. Suppose \( p \neq 2 \), then we have

\[\sum_{x \mod p^n} \psi(\alpha + \beta x^2) = \begin{cases} \psi(\alpha)p^\frac{n}{2} & \text{if } n \text{ is even,} \\ \chi_{p^*}(\beta)\varepsilon_{3/2} \psi(\alpha + 2^{n-3} \beta) & \text{if } \psi \neq \chi_{p^*} \text{ and } n \text{ is odd,} \\ \chi_{p^*}((p - 1) & \text{if } \psi = \chi_{p^*} \text{ and } \alpha = 0, \\ \varepsilon_{3/2} \psi(\alpha) & \text{if } \psi = \chi_{p^*} \text{ and } \alpha \neq 0. \end{cases}\]

Proof. We may assume \( \beta = 1 \).

Since \( \psi \) is a primitive character,

\[
\psi(x) = \frac{1}{p^n} G(\psi) \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \overline{\psi}(a)e\left(-\frac{ax}{p^n}\right). \tag{3.8}
\]

By (3.8), we have

\[\sum_{x \mod p^n} \psi(x^2 + \alpha) = \frac{1}{p^n} G(\psi) \sum_{x, a \mod p^n \atop (a,p) = 1} \overline{\psi}(a)e\left(-\frac{a(x^2 + \alpha)}{p^n}\right)\]

\[= \frac{1}{p^n} G(\psi) \sum_{x, a \mod p^n \atop (a,p) = 1} \overline{\psi}(a)e\left(-\frac{ax^2}{p^n}\right)e\left(-\frac{a\alpha}{p^n}\right).\]

If \( p \neq 2 \) and \( n \) is odd, applying Lemma 3.2, we have

\[\sum_{x \mod p^n} \psi(x^2 + \alpha) = p^{-n} G(\psi) \sum_{a \mod p^n} \varepsilon_p \chi_{p^*}(-a)p^n \overline{\psi}(a)e\left(-\frac{a\alpha}{p^n}\right)\]

\[= \varepsilon_p^3 G(\psi)p^{-2} \sum_{a \mod p^n \atop (a,p) = 1} \overline{\psi}(\chi_{p^*}(a))e\left(-\frac{a\alpha}{p^n}\right).\]

By this and (3.8) for \( \psi \chi_{p^*} \), if \( \psi \neq \chi_{p^*} \), we obtain

\[\sum_x \psi(x^2 + \alpha) = \varepsilon_p^3 (\psi \chi_{p^*})(\alpha) G(\psi) G(\psi \chi_{p^*})^{-1} p^\frac{5}{2}.
\]
If $\psi = \chi_{p^s}$,
\[
\sum_{x} \psi(x^2 + \alpha) = \epsilon_p^3 G(\psi) p^{-\frac{1}{2}} \sum_{(a,p)=1} e\left(-\frac{\alpha a}{p}\right) = \sum_{(a,p)=1} e\left(-\frac{\alpha a}{p}\right)
\]
\[
= \begin{cases} p - 1 & \text{if } \alpha = 0, \\ -1 & \text{if } \alpha \neq 0. \end{cases}
\]

In similar way we can prove the lemma in other cases. \[\square\]

3.3. In this subsection, we prove Proposition 3.1. Let $p \neq 2$, $h' = \begin{pmatrix} \alpha & 0 \\ 0 & p' \beta \end{pmatrix}$, $\alpha, \beta \in \mathbb{Z}_p \times$. We state the results of calculation of $A, B_1, B_2, C$.

**Lemma 3.4.** $C = 0$.

**Lemma 3.5.** Suppose $\psi = \chi_{p^s}$, then we have
\[
A = \begin{cases} \epsilon_p^3 (p - 1) \sum_{i=1}^{m} p^{(3-2s)i-2} - p^{(3-2s)(m+1)-2} & \text{if } t = 0, \\ \epsilon_p^2 (p - 1) \sum_{i=1}^{m+1} p^{(3-2s)i-2} & \text{if } t \geq 1. \end{cases}
\]

Suppose $\psi \neq \chi_{p^s}$, then we have
\[
A = \begin{cases} \overline{\psi}(\alpha \beta) G(\psi)^2 p^{(3-2s)(m+n)-5/2n} & \text{if } n \text{ is even and } t = 0, \\ \epsilon_p (\overline{\psi} \chi_{p^s})(\alpha \beta) G(\psi) G(\psi \chi_{p^s}) p^{(3-2s)(m+n)-5/2n} & \text{if } n \text{ is odd and } t = 0, \\ 0 & \text{if } t \geq 1. \end{cases}
\]

**Lemma 3.6.** Suppose $n = 1$, then we have
\[
B_1 = 0.
\]

Suppose $n \geq 2$, then we have
\[
B_1 = \begin{cases} 0 & \text{if } t = 0 \text{ or } t > n, \\ \psi(\alpha \beta) G(\psi)^2 p^{(3-2s)(m+n+t/2)-5/2n} & \text{if } 1 \leq t < n \text{ and } n - t \text{ is even}, \\ \epsilon_p (\overline{\psi} \chi_{p^s})(\alpha \beta) G(\psi) G(\psi \chi_{p^s}) p^{(3-2s)(m+n+t/2)-5/2n} & \text{if } 1 \leq t < n \text{ and } n - t \text{ is odd}. \end{cases}
\]

**Lemma 3.7.** Suppose $\psi = \chi_{p^s}$, then we have
\[
B_2 = \begin{cases} 0 & \text{if } t = 0, \\ \epsilon_p^3 (p - 1) \sum_{i=1}^{t-2/1} p^{(3-2s)(m+i+1)-2} - p^{(3-2s)(m+t/2+1)-2} & \text{if } t \geq 2 \text{ and even}, \\ \epsilon_p^2 (p - 1) \sum_{i=1}^{t-1} p^{(3-2s)(m+i+1)-2} + \chi_{p^s}(\alpha \beta) p^{(3-2s)(m+t/2+1)-3/2} & \text{if } t \text{ is odd}. \end{cases}
\]

Suppose $\psi \neq \chi_{p^s}$, then we have
Lemma 3.7, \( B_2 = \begin{cases} \frac{\bar{\psi}(\alpha\beta)G(\psi)^2(3-2s)(m+n+t/2)-5/2n}{p(3-2s)(m+n+t/2)-5/2n} & \text{if } t < n, \\ \varepsilon_p(\bar{\psi}\chi_{p^n})(\alpha\beta)G(\psi)G(\psi\chi_{p^n})p(3-2s)(m+n+t/2)-5/2n & \text{if } t \geq n \text{ and } n - t \text{ is even}, \\ \varepsilon_p(\bar{\psi}\chi_{p^n})(\alpha\beta)G(\psi)G(\psi\chi_{p^n})p(3-2s)(m+n+t/2)-5/2n & \text{if } t \geq n \text{ and } n - t \text{ is odd}. \end{cases} \)

Proof of Proposition 3.1. By (3.7), \( S_p(\bar{\psi}, h) = A + B_1 + B_2 + C. \) And by Lemma 3.5, Lemma 3.6 and Lemma 3.7, \( S_p(\psi, h) \) is as stated in the proposition. □

Until the end of this subsection, we prove Lemma 3.5, Lemma 3.6 and Lemma 3.7.

Proof of Lemma 3.4. By (3.6),

\[
C = \sum_{i=n}^{\infty} \sum_{j=1}^{\infty} p^{-(2i+j)s} \sum_{a,b \mod p^j} \sum_{d \mod p^{i+j}} \psi(ad - p^jb^2) \\
\times e\left(\frac{1}{p^{i-m}} \begin{pmatrix} a & b \\ dp^{-j} & 1 \end{pmatrix} h \left[ \begin{pmatrix} 0 & -1 \\ 1 & -pu \end{pmatrix} \right] \right) \\
= \sum_{i=n}^{\infty} \sum_{j=1}^{\infty} p^{-(2i+j)s} \sum_{a,b \mod p^j} \sum_{d \mod p^{i+j}} \psi(ad - p^jb^2) \\
\times e\left(\frac{a\beta}{p^{i-m-t}} \right) e\left(\frac{2b\beta u}{p^{i-m-t-1}} \right) e\left(\frac{(\alpha + \beta u^2p^{t+2})d}{p^{i+j-m}} \right) \\
= \sum_{i=1}^{\infty} \sum_{j=1}^{n-1} p^{-(2i+j)s} \sum_{a,b \mod p^j} \sum_{d \mod p^{i+j}} \psi(ad - p^jb^2) \\
\times e\left(\frac{a\beta}{p^{i-m-t}} \right) e\left(\frac{2b\beta u}{p^{i-m-t-1}} \right) e\left(\frac{(\alpha + \beta u^2p^{t+2})d}{p^{i+j-m}} \right) \\
+ \sum_{i=1}^{\infty} \sum_{j=n}^{\infty} p^{-(2i+j)s} \sum_{a,b \mod p^j} \sum_{d \mod p^{i+j}} \psi(ad) \\
\times e\left(\frac{a\beta}{p^{i-m-t}} \right) e\left(\frac{2b\beta u}{p^{i-m-t-1}} \right) e\left(\frac{(\alpha + \beta u^2p^{t+2})d}{p^{i+j-m}} \right) .
\]

We put

\[
C_1 = \sum_{i=1}^{\infty} \sum_{j=1}^{n-1} p^{-(2i+j)s} \sum_{a,b \mod p^j} \sum_{d \mod p^{i+j}} \psi(ad - p^jb^2) \\
\times e\left(\frac{a\beta}{p^{i-m-t}} \right) e\left(\frac{2b\beta u}{p^{i-m-t-1}} \right) e\left(\frac{(\alpha + \beta u^2p^{t+2})d}{p^{i+j-m}} \right) ,
\] (3.9)
Therefore we have

\[ C_2 = \sum_{i=1}^{\infty} \sum_{j=n}^{\infty} p^{-(2i+j)s} \sum_{a,b \text{ mod } p^i} \sum_{d \text{ mod } p^{i+j}} \psi(ad) \times e\left( \frac{a\beta}{p^{i-m-t}} \right) e\left( \frac{2b\beta u}{p^{i-m-t-1}} \right) e\left( \frac{(\alpha + \beta u^2 p^{i+2})d}{p^{i+j-m}} \right). \] (3.10)

then \( C = C_1 + C_2 \). Since \( \psi \) is a primitive character, the summation for \( a \) and \( d \) of \( C_2 \) vanishes unless \( i - m - t = n \) and \( i + j - m = n \). If \( i - m - t = n \) and \( i + j - m = n \), then we have \( t + j = 0 \) and this contradicts \( j > 0, t \geq 0 \). Therefore \( C_2 = 0 \).

If \( n = 1 \), then \( C_1 = 0 \).

If \( n \geq 2 \), we replace \( a \) by \( a + d^{-1}p^ib^2 \) in (3.9), then we have

\[ C_1 = \sum_{i=n}^{n-1} \sum_{j=1}^{n-1} p^{-(2i+j)s} \sum_{a,d,u} \psi(ad) e\left( \frac{\beta a}{p^{i-m-t}} \right) e\left( \frac{2b\beta u}{p^{i-m-t-1}} \right) \times \sum_{b} e\left( \frac{d^{-1}\beta b^2}{p^{i-j-m-t}} \right) e\left( \frac{2b\beta u}{p^{i-m-t-1}} \right). \]

We apply Lemma 3.1 for the summation for \( b \). If the order of \( e\left( \frac{2b\beta u}{p^{i-m-t-1}} \right) \) is larger than \( p^{i-j-m-t} \), then the summation for \( b \) vanishes. Therefore

\[ \sum_{b} e\left( \frac{d^{-1}\beta b^2}{p^{i-j-m-t}} \right) e\left( \frac{2b\beta u}{p^{i-m-t-1}} \right) = 0 \]

unless \( u \equiv 0 \mod p^{i-1} \). If \( u \equiv 0 \mod p^{i-1} \), then by Lemma 3.2,

\[ \sum_{b} e\left( \frac{d^{-1}\beta b^2}{p^{i-j-m-t}} \right) \text{ or } \chi_{p^*}(d) \sum_{b} e\left( \frac{d^{-1}\beta b^2}{p^{i-j-m-t}} \right) \]

does not depend on \( d \). Since \( n > 1 \), \( \psi \chi_{p^*} \) is primitive, we can show \( C_1 = 0 \) in a similar way to the proof of \( C_2 = 0 \). Thus we have \( C = 0 \). \( \square \)

**Proof of Lemma 3.5.** By (3.2),

\[ A = \sum_{i=n}^{\infty} p^{-2is} \sum_{a,b,d \text{ mod } p^i} \psi(ad - b^2) e\left( \frac{\alpha a + \beta dp^t}{p^{i-m}} \right). \]

Applying Lemma 3.1 for the summation for \( a \), we see that the summation vanishes unless \( i - m \leq n \). Therefore we have

\[ A = \sum_{i=n}^{n+m} p^{-2is+3i-3n} \sum_{a,b,d \text{ mod } p^n} \psi(ad - b^2) e\left( \frac{\alpha a + \beta dp^t}{p^{i-m}} \right). \]

We put

\[ X_i = \sum_{a,b,d \text{ mod } p^n} \psi(ad - b^2) e\left( \frac{\alpha a + \beta dp^t}{p^{i-m}} \right), \]
then
\[ A = \sum_{i=n}^{n+m} p^{-2i\tau + 3i - 3n} X_i. \] (3.11)

By Lemma 3.3,
\[ X_i = \begin{cases} 
  p^{n/2} \sum_{a,d \mod p^n} \psi(ad) e\left(\frac{\alpha a}{p^{i-m}}\right) e\left(\frac{\beta d}{p^{i-m-t}}\right) & \text{if } n \text{ is even,} \\
  \varepsilon_p \frac{G(\psi) G(\psi X_{p^r})^{-1}}{p^{n/2}} \times \sum_{a,d \mod p^n} \psi(ad) e\left(\frac{\alpha a}{p^{i-m}}\right) e\left(\frac{\beta d}{p^{i-m-t}}\right) & \text{if } n \text{ is odd.}
\end{cases} \]

Therefore, we have
\[ X_i = \begin{cases} 
  0 & \text{if } i < m + n \text{ or } t > 0, \\
  \varepsilon_p \frac{G(\psi) G(\psi X_{p^r})^{-1}}{p^{n/2}} & \text{if } n \text{ is even and } t = 0, i = m + n, \\
  \varepsilon_p \frac{G(\psi) G(\psi X_{p^r})^{-1}}{p^{n/2}} & \text{if } n \text{ is odd and } t = 0, i = m + n,
\end{cases} \] (3.12)

where \( \psi \neq \chi_{p^r} \). Next let \( \psi = \chi_{p^r} \). By Lemma 3.3 we have
\[ X_i = \chi_{p^r}(-1) \left\{ (p-1) \sum_{ad=0} e\left(\frac{\alpha a}{p^{i-m}}\right) e\left(\frac{\beta d}{p^{i-m-t}}\right) - \sum_{ad \neq 0} e\left(\frac{\alpha a}{p^{i-m}}\right) e\left(\frac{\beta d}{p^{i-m-t}}\right) \right\} \]
\[ = \chi_{p^r}(-1) \left\{ p \sum_{ad=0} e\left(\frac{\alpha a}{p^{i-m}}\right) e\left(\frac{\beta d}{p^{i-m-t}}\right) - \sum_{ad=0} e\left(\frac{\alpha a}{p^{i-m}}\right) e\left(\frac{\beta d}{p^{i-m-t}}\right) \right\}. \]

By
\[ \sum_{ad=0} e\left(\frac{\alpha a}{p^{i-m}}\right) e\left(\frac{\beta d}{p^{i-m-t}}\right) = \sum_{a} e\left(\frac{\alpha a}{p^{i-m}}\right) + \sum_{d \neq 0} e\left(\frac{\beta d}{p^{i-m-t}}\right) \]
\[ = \begin{cases} 
  -1 & \text{if } i = m + 1 \text{ and } t = 0, \\
  p - 1 & \text{if } i = m + 1 \text{ and } t \geq 1, \\
  2p - 1 & \text{if } i \leq m,
\end{cases} \]
and
\[ \sum_{a,d} e\left(\frac{\alpha a}{p^{i-m}}\right) e\left(\frac{\beta d}{p^{i-m-t}}\right) = \begin{cases} 
  0 & \text{if } i = m + 1, \\
  p^2 & \text{if } i \leq m,
\end{cases} \]
we have
\[ X_i = \chi_{p^r}(-1) \begin{cases} 
  p(p-1) & \text{if } i \leq m, \\
  -p & \text{if } i = m + 1 \text{ and } t = 0, \\
  p(p-1) & \text{if } i = m + 1 \text{ and } t \geq 1.
\end{cases} \] (3.13)

By (3.11), (3.12) and (3.13), we have assertions of the lemma. \( \square \)

**Proof of Lemma 3.6.** If \( n = 1 \), then \( B_1 = 0 \) by definition, therefore we may assume \( n \geq 2 \). By (3.3),
\[
B_1 = \sum_{i=n}^{\infty} \sum_{j=1}^{n-1} p^{-(2i+j)s} \sum_{a,b \mod p^l \atop d \mod p^{i+j}} \psi(\alpha a - p^j b^2) \times e\left(\frac{\alpha a}{p^{i-m}}\right) e\left(-\frac{2u\alpha b}{p^{i-m}}\right) e\left(\frac{(\alpha u^2 + \beta p^t)d}{p^{i+j-m}}\right).
\]

Replacing \(a\) by \(a + d^{-1} p^j b^2\), we have

\[
B_1 = \sum_{i,j} p^{-(2i+j)s} \sum_{a,b,d,u} \psi(ad) e\left(\frac{\alpha a}{p^{i-m}}\right) e\left(\frac{\alpha d^{-1} b^2}{p^{i-j-m}}\right) e\left(-\frac{2u\alpha b}{p^{i-m}}\right) e\left(\frac{(\alpha u^2 + \beta p^t)d}{p^{i+j-m}}\right).
\]

The summation for \(a\) is zero unless \(i = m + n\), therefore

\[
B_1 = \sum_{j=1}^{n-1} p^{-(2m+2n+j)s} \sum_{a,b,d,u \mod p^{m+n} \atop d \mod p^{m+n+j} \atop u \mod p^l} \psi(ad) e\left(\frac{\alpha a}{p^{n}}\right) e\left(\frac{\alpha d^{-1} b^2}{p^{n-j}}\right) e\left(-\frac{2u\alpha b}{p^{n}}\right) e\left(\frac{(\alpha u^2 + \beta p^t)d}{p^{n+j}}\right).
\]

We put

\[
Y_j = \sum_{a,b \mod p^{m+n} \atop d \mod p^{m+n+j} \atop u \mod p^l} \psi(ad) e\left(\frac{\alpha a}{p^{n}}\right) e\left(\frac{\alpha d^{-1} b^2}{p^{n-j}}\right) e\left(-\frac{2u\alpha b}{p^{n}}\right) e\left(\frac{(\alpha u^2 + \beta p^t)d}{p^{n+j}}\right) \tag{3.14}
\]

then

\[
B_1 = \sum_{j=1}^{n-1} p^{-(2m+2n+j)s} Y_j. \tag{3.15}
\]

Computing the summation for \(a\) in (3.14),

\[
Y_j = \overline{\psi}(\alpha) G(\psi) p^m \sum_{u,b,d} \psi(d) e\left(\frac{\alpha d^{-1} b^2}{p^{n-j}}\right) e\left(-\frac{2u\alpha b}{p^m}\right) e\left(\frac{(\alpha u^2 + \beta p^t)d}{p^{n+j}}\right).
\]

Applying Lemma 3.1 for the summation for \(b\), we see that the summation vanishes unless \(u \equiv 0 \mod p^l\), therefore we have

\[
Y_j = \overline{\psi}(\alpha) G(\psi) p^m \sum_{b,d} \psi(d) e\left(\frac{\alpha d^{-1} b^2}{p^{n-j}}\right) e\left(\frac{\beta d}{p^{n+j-l}}\right).
\]

By Lemma 3.2,
We have
\[\sum_{d \text{ mod } p^m+n+j} \psi(d)e\left(\frac{\beta d}{p^{n+j-t}}\right)\]
if \(j \neq t\),
\[\epsilon_p(\overline{\psi} \chi_{p^*})(\alpha \beta)G(\psi) p^{3m+n/2+3j/2} \sum_{d \text{ mod } p^{m+n+j}} \psi d \epsilon\left(\frac{\beta d}{p^{n+j-t}}\right)\]
if \(j = t\) and \(n - t\) is even,
\[\epsilon_p(\overline{\psi} \chi_{p^*})(\alpha \beta)G(\psi) G(\psi \chi_{p^*}) p^{3m+n/2+3j/2}\]
if \(j = t\) and \(n - t\) is odd.

By (3.15) and (3.16), if \(n \geq 2\), we have the assertions of the lemma. □

**Proof of Lemma 3.7.** By (3.4),

\[
B_2 = \sum_{i,j=n}^\infty p^{-(2i+j)s} \sum_{a,b \text{ mod } p^i} \sum_{u \text{ mod } p^j} \psi(a) e\left(\frac{\alpha a}{p^{i-m}}\right) e\left(-\frac{2uab}{p^{i-m}}\right) e\left(\frac{\alpha u^2 + \beta p^j d}{p^{i+j-m}}\right)
\]

\[
= \sum_{i,j=n}^\infty p^{-(2i+j)s} \sum_{a \text{ mod } p^i} \psi(a) e\left(\frac{\alpha a}{p^{i-m}}\right) e\left(-\frac{2uab}{p^{i-m}}\right) \sum_{d \text{ mod } p^{m+n+j}} \psi(d) e\left(\frac{\alpha u^2 + \beta p^j d}{p^{n+j}}\right).
\]

The summation for \(b\) vanishes unless \(u \equiv 0 \text{ mod } p^n\). Replacing \(u\) by \(p^n u\), we have

\[
B_2 = \overline{\psi}(\alpha)G(\psi) \sum_{j=n}^\infty p^{-(2m+2n+j)s+2m+n} \sum_{u \text{ mod } p^{j-n}} e\left(\alpha du^2/p^{j-n}\right) \sum_{d \text{ mod } p^{m+n+j}} \psi(d) e\left(\frac{\beta d}{p^{n+j-t}}\right).
\]

By Lemma 3.2, we have

\[
B_2 = \overline{\psi}(\alpha)G(\psi) \sum_{j=n}^\infty p^{-(2m+2n+j)s+2m+n/2+j/2} \sum_{d \text{ mod } p^{m+n+j}} \psi(d) e\left(\frac{\beta d}{p^{n+j-t}}\right)
\]

\[
+ \epsilon_p(\overline{\psi} \chi_{p^*})(\alpha \beta)G(\psi) \sum_{j=n}^\infty p^{-(2m+2n+j)s+2m+n/2+j/2} \sum_{d \text{ mod } p^{m+n+j}} \psi d \epsilon\left(\frac{\beta d}{p^{n+j-t}}\right).
\]

We put

\[
Z = \overline{\psi}(\alpha)G(\psi) \sum_{j=n}^\infty p^{-(2m+2n+j)s+2m+n/2+j/2} \sum_{d \text{ mod } p^{m+n+j}} \psi(d) e\left(\frac{\beta d}{p^{n+j-t}}\right), \quad (3.17)
\]
\[ W = \varepsilon_p(\overline{\chi} p^*) (\alpha) G(\psi) \sum_{j=n}^{\infty} p^{-(2m+2n+j)s+2m+n/2+j/2} \]
\[ \times \sum_{d \mod p^{n+j}} \psi \chi^*(d) e \left( \frac{\beta d}{p^{n+j-t}} \right), \] (3.18)

then

\[ B_2 = Z + W. \] (3.19)

By (3.17), \( Z \) is as follows

\[ Z = \begin{cases} \overline{\psi}(\alpha \beta) G(\psi)^2 p^{(3-2s)(m+n+t/2)-5/2n} & \text{if } t \geq n \text{ and } n - t \text{ is even,} \\ 0 & \text{otherwise}. \end{cases} \] (3.20)

By (3.18), \( W \) is as follows.

Suppose \( \psi \neq \chi^* p^* \), then

\[ W = \begin{cases} \varepsilon_p(\overline{\psi} \chi^* p^*) (\alpha) G(\psi) G(\psi \chi^* p^*) p^{(3-2s)(m+n+t/2)-5/2n} & \text{if } t \geq n \text{ and } n - t \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases} \] (3.21)

Suppose \( \psi = \chi^* p^* \), then

\[ W = \varepsilon_p G(\chi^* p^*) \sum_{j=1}^{\infty} p^{-(2m+2j)s+2m+1/2+j/2} \sum_{d \mod p^{n+1+j}} \psi \chi^*(d) e \left( \frac{\beta d}{p^{n+j-t}} \right) \]
\[ = \varepsilon^2_p \sum_{j=1}^{\infty} p^{-(2m+2j+2)s+3m+1+j} \sum_{d \mod p^{1+2j}} e \left( \frac{\beta d}{p^{1+2j-t}} \right). \]

The summation for \( d \) is as follows

\[ \sum_{d \mod p^{1+2j}} e \left( \frac{\beta d}{p^{1+2j-t}} \right) = \begin{cases} 0 & \text{if } t < 2j, \\ -p^{2j} & \text{if } t = 2j, \\ p^{2j}(p-1) & \text{if } t > 2j. \end{cases} \]

Therefore we have

\[ W = \begin{cases} 0 & \text{if } t = 0, \\ \varepsilon^2_p ((p-1) \sum_{i=1}^{t/2-1} p^{(3-2s)(m+i+1)-2} - p^{(3-2s)(m+t/2+1)-2}) & \text{if } t \geq 2 \text{ and even}, \\ \varepsilon^2_p (p-1) \sum_{i=1}^{t-1} p^{(3-2s)(m+i+1)-2} & \text{if } t \text{ is odd.} \end{cases} \] (3.22)

By (3.19), (3.20), (3.21), and (3.22), we have the assertions of the lemma. \( \square \)
3.4. In this and the next subsection we prove Proposition 3.2. In this subsection we prove (2) and (3) of Proposition 3.2 and in the next subsection we prove (1) of Proposition 3.2.

**Proof of (2) of Proposition 3.2.** Let \( h' = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \).

By (3.5), \( B \) is as follows
\[
B = \sum_{i=n}^{\infty} \sum_{j=1}^{\infty} 2^{-(2i+j)s} \sum_{u \mod 2^j} \sum_{a \mod 2^j} \sum_{b \mod 2^j} \sum_{d \mod 2^{i+j}} \psi(ad - 2^jb^2) e \left( \frac{b}{2^{i-m}} \right) e \left( -\frac{du}{2^{i-m-j}} \right).
\]

Since the summation for \( a \) vanishes,
\[
B = 0. \tag{3.23}
\]

By (3.6), \( C \) is as follows
\[
C = \sum_{i=n}^{\infty} \sum_{j=1}^{\infty} 2^{-(2i+j)s} \sum_{u \mod 2^{i-1}} \sum_{a \mod 2^i} \sum_{b \mod 2^i} \sum_{d \mod 2^{i+j}} \psi(ad - 2^jb^2) e \left( -\frac{b + ud2^{1-j}}{2^{i-m}} \right).
\]

The summation for \( a \) vanishes, therefore
\[
C = 0. \tag{3.24}
\]

By (3.2),
\[
A = \sum_{i=n}^{\infty} 2^{-2is} \sum_{a,b,d \mod 2^i} \psi((ad - b^2) e \left( \frac{b}{2^{i-m}} \right).
\]

Applying Lemma 3.1 for the summation for \( a \), we see that the summation vanishes if \( i > m + n \), therefore we have
\[
A = \sum_{i=n}^{m+n} 2^{-2is+3i-3n} \sum_{a,b,d \mod 2^i} \psi((ad - b^2) e \left( \frac{b}{2^{i-m}} \right).
\]

If \( d \not\equiv 0 \mod 2^n \), then the summation for \( a \) vanishes, therefore we have
\[
A = \psi(-1) \sum_{i=n}^{m+n} 2^{(3-2s)i-2n} \sum_{b \mod 2^n} \sum_{(b,2)=1} \psi^2(b) e \left( \frac{b}{2^{i-m}} \right). \tag{3.25}
\]

Here the summation for \( b \) is as follows.
Suppose \( n > 3 \), then
\[
\sum_{b \mod 2^n} \psi^2(b) e \left( \frac{b}{2^{i-m}} \right) = \begin{cases} 0 & \text{if } i \neq m + n - 1, \\ 2G(\psi^2) & \text{if } i = m + n - 1. \end{cases} \tag{3.26}
\]

Suppose \( n = 2 \), then
\[
\sum_{\substack{b \mod 2^{n} \atop (b, 2) = 1}} \psi^2(b) e\left(\frac{b}{2^{1-m}}\right) = e\left(\frac{1}{2^{1-m}}\right) + e\left(-\frac{1}{2^{1-m}}\right) = \begin{cases} 
0 & \text{if } i = m + 2, \\
-2 & \text{if } i = m + 1, \\
2 & \text{if } i \leq m.
\end{cases} \tag{3.27}
\]

Suppose \(n = 3\), then

\[
\begin{align*}
\sum_{\substack{b \mod 2^{n} \atop (b, 2) = 1}} \psi^2(b) e\left(\frac{b}{2^{1-m}}\right) &= e\left(\frac{1}{2^{1-m}}\right) + e\left(-\frac{1}{2^{1-m}}\right) + e\left(\frac{3}{2^{1-m}}\right) + e\left(-\frac{3}{2^{1-m}}\right) \\
&= \begin{cases} 
0 & \text{if } i = m + 2, m + 3, \\
-4 & \text{if } i = m + 1, \\
4 & \text{if } i \leq m.
\end{cases} \tag{3.28}
\end{align*}
\]

By (3.7), (3.23) and (3.24), we have \(S_2(\psi, h) = A + B + C = A\). By (3.26), (3.27), (3.28), we obtain the assertion (2) of Proposition 3.2. \(\square\)

**Proof of (3) of Proposition 3.2.** Let \(h' = \left(\frac{1}{2}, \frac{1}{2} \right)\). By (3.5),

\[
B = \sum_{i=n}^{\infty} \sum_{j=1}^{\infty} 2^{-(2i+j)s} \sum_{u, a, b, d} \psi(ad - 2^j b^2) e\left(\frac{a + (1 - 2u)b + 2^{-j}(1 - u + u^2)d}{2^i - m}\right).
\]

Replacing \(a\) by \(a + 2^j d^{-1} b\), we have

\[
B = \sum_{i=n}^{\infty} \sum_{j=1}^{\infty} 2^{-(2i+j)s} \sum_{u, a, b, d} \psi(ad) e\left(\frac{a}{2^{i-m}}\right) e\left(\frac{2^j b^2}{2^{1-m}} + 2^{-j}(1 - u + u^2)d\right) e\left(\frac{(1 - 2u)b}{2^i - m}\right).
\]

If \(i \neq m + n\), then the summation for \(a\) vanishes.

If \(i = m + n\), then \(e\left(\frac{2^j b^2}{2^{1-m}} + 2^{-j}(1 - u + u^2)d\right)\) depends only on \(b \mod 2^{n-1}\). Therefore by Lemma 3.1, we have

\[
B = 0. \tag{3.29}
\]

As for \(C\), by (3.6)

\[
C = \sum_{i=n}^{\infty} \sum_{j=1}^{\infty} 2^{-(2i+j)s} \sum_{u, a, b, d} \psi(ad - 2^j b^2) e\left(\frac{a + (-1 + 4u)b + 2^{-j}(1 - 2u + 4u^2)d}{2^{i-m}}\right)
\]

\[
= \sum_{i=n}^{\infty} \sum_{j=1}^{\infty} 2^{-(2i+j)s} \sum_{u, a, b, d} \psi(ad) e\left(\frac{a}{2^{i-m}}\right) e\left(\frac{2^j b^2}{2^{1-m}} + 2^{-j}(1 - 2u + 4u^2)d\right) e\left(\frac{(-1 + 4u)b}{2^i - m}\right).
\]

We can prove that

\[
C = 0 \tag{3.30}
\]

in a similar way to the proof of \(B = 0\). By (3.2), \(A\) is as follows:
\[
A = \sum_{i=n}^{\infty} 2^{-2i} \sum_{a,b,d \mod 2^i} \psi(ad - b^2) e\left(\frac{a+b+d}{2^{i-m}}\right)
\]

\[
= \sum_{i=n}^{m+n} 2^{-2i} \sum_{a,b,d \mod 2^i} \psi(ad - b^2) e\left(\frac{a+b+d}{2^{i-m}}\right).
\]

We put

\[
Y_i = \sum_{a,b,d \mod 2^n} \psi(ad - b^2) e\left(\frac{a+b+d}{2^{i-m}}\right),
\]

then

\[
A = \sum_{i=n}^{m+n} 2^{-2i} \sum_{a,b,d \mod 2^i} \psi(ad - b^2) e\left(\frac{a+b+d}{2^{i-m}}\right).
\]

For \(Y_i\), we divide the summation for \(d\) into two parts according to the parity of \(d\).

\[
Y_i = \sum_{a,b,d \mod 2^n \atop d \text{ even}} \psi(ad - b^2) e\left(\frac{a+b+d}{2^{i-m}}\right) + \sum_{a,b,d \mod 2^n \atop d \text{ odd}} \psi(ad - b^2) e\left(\frac{a+b+d}{2^{i-m}}\right).
\]

Replacing \(a\) by \(a + d^{-1}b^2\) in the second term, we have

\[
Y_i = \sum_{a,b,d \mod 2^n \atop d \text{ even}} \psi(ad - b^2) e\left(\frac{a+b+d}{2^{i-m}}\right) + \sum_{a,b,d \mod 2^n \atop d \text{ odd}} \psi(ad) e\left(\frac{a}{2^{i-m}}\right) e\left(\frac{d^{-1}b^2 + d}{2^{i-m}}\right) e\left(\frac{b}{2^{i-m}}\right).
\]

By Lemma 3.1 the summation for \(b\) in the second term vanishes, therefore

\[
Y_i = \sum_{a,b \mod 2^n \atop d \mod 2^{n-1}} \psi(2ad - b^2) e\left(\frac{a+b+2d}{2^{i-m}}\right).
\]

Replacing \(a\) by \(-a - b + 2d\) and \(d\) by \(-d + a\), we have

\[
Y_i = \sum_{a,b \mod 2^n \atop d \mod 2^{n-1}} \psi((-b + a - d)^2 - 3\left(d - \frac{2a}{3}\right)^2 + \frac{a^2}{3}) e\left(\frac{a}{2^{i-m}}\right).
\]

Applying Lemma 3.3, we have

\[
Y_i = \begin{cases} 
\overline{\psi}(3) 2^n \sum_{a \mod 2^n \atop (a,2)=1} \psi^2(a) e\left(\frac{a}{2^{i-m}}\right) & \text{if } n \text{ is even}, \\
-\overline{\psi}(3) 2^n \sum_{a \mod 2^n \atop (a,2)=1} \psi^2(a) e\left(\frac{a}{2^{i-m}}\right) & \text{if } n \text{ is odd}.
\end{cases}
\]

As a consequence, \(Y_i\) is as follows.
Suppose $n > 3$, then

\[
Y_i = \begin{cases} 
0 & \text{if } i \neq n + m - 1, \\
\overline{\psi}(3)2^{n+1}G(\psi^2) & \text{if } i = n + m - 1 \text{ and } n \text{ is even}, \\
-\overline{\psi}(3)2^{n+1}G(\psi^2) & \text{if } i = n + m - 1 \text{ and } n \text{ is odd}, 
\end{cases}
\tag{3.33}
\]

Suppose $n = 2$, then

\[
Y_i = \begin{cases} 
0 & \text{if } i = m + 2, \\
2^3 & \text{if } i = m + 1, \\
-2^3 & \text{if } i \leq m. 
\end{cases}
\tag{3.34}
\]

Suppose $n = 3$, then

\[
Y_i = \begin{cases} 
0 & \text{if } i = m + 2, m + 3, \\
\psi(3)2^5 & \text{if } i = m + 1, \\
-\psi(3)2^5 & \text{if } i \leq m. 
\end{cases}
\tag{3.35}
\]

By (3.29) and (3.30), we have $S_2(\overline{\psi}, h) = A + B + C = A$. Therefore by (3.32), (3.33), (3.34) and (3.35), we have the assertion (3) of Proposition 3.2. \qed

3.5. In this subsection we prove (1) of Proposition 3.2. We state the results of calculation of $A$, $B_1$, $B_2$, $C$.

Suppose $p = 2$ and set $h' = \left( \begin{array}{cc} \alpha & 0 \\ 0 & 2^t \beta \end{array} \right)$.

**Lemma 3.8.** $C = 0$.

**Lemma 3.9.** Suppose $n > 3$ and even, then we have

\[
A = \begin{cases} 
(\psi(\alpha \beta) + \overline{\psi}(\alpha \beta + 2^{n-2}))G(\psi)^22^{(3-2s)(m+n)-5/2n} & \text{if } t = 0, \\
0 & \text{if } t \neq 0.
\end{cases}
\]

Suppose $n > 3$ and odd, then we have

\[
A = \begin{cases} 
\overline{\psi}(\alpha \beta + 2^{n-3})G(\psi)^22^{(3-2s)(m+n)-5/2n+1/2} & \text{if } t = 0, \\
0 & \text{if } t > 0.
\end{cases}
\]

Suppose $n = 2$, then we have

\[
A = \begin{cases} 
-\left\{\sum_{i=2}^{m+1} 2^{(3-2s)i-3} + \psi(\alpha \beta)2^{(3-2s)(m+2)-3}\right\} & \text{if } t = 0, \\
-\left\{\sum_{i=2}^{m+1} 2^{(3-2s)i-3} - 2^{(3-2s)(m+2)-3}\right\} & \text{if } t = 1, \\
-\left\{\sum_{i=2}^{m+1} 2^{(3-2s)i-3} + 2^{(3-2s)(m+2)-3}\right\} & \text{if } t \geq 2.
\end{cases}
\]

Suppose $n = 3$, then we have
Lemma 3.9. Let \( n > 3 \). Then we have

\[
B_1 = \begin{cases} 
0 & \text{if } t = 0 \text{ or } t \geq n, \\
\left( \frac{1}{n} \right) (\psi(\alpha\beta) + \psi(\alpha\beta + 2^n)G(\psi)^22^{(3-2s)(m+n+t/2)-5/2n}) & \text{if } 0 < t < n \text{ and } n-t \text{ is odd}, \\
\frac{1}{n} (\psi(\alpha\beta + 2^n)G(\psi)^22^{(3-2s)(m+n+t/2)}-5/2n) & \text{if } 0 < t < n \text{ and } n-t \text{ is even}. 
\end{cases}
\]

Suppose \( n = 2 \), then we have

\[
B_1 = \begin{cases} 
0 & \text{if } t \neq 0, \\
-\psi(\frac{1+\alpha\beta}{2})2^{(3-2s)(m+5/2)}-5/2 & \text{if } t = 0.
\end{cases}
\]

Lemma 3.11. Let \( n > 3 \). Then we have

\[
B_2 = \begin{cases} 
0 & \text{if } t < n, \\
\left( \frac{1}{n} \right) (\psi(\alpha\beta) + \psi(\alpha\beta + 2^n))G(\psi)^22^{(3-2s)(m+n+t/2)-5/2n} & \text{if } t \geq n \text{ and } n-t \text{ is even}, \\
\frac{1}{n} (\psi(\alpha\beta + 2^n)G(\psi)^22^{(3-2s)(m+n+t/2)}-5/2n+1/2) & \text{if } t \geq n \text{ and } n-t \text{ is odd}.
\end{cases}
\]

Suppose \( n = 2 \), then we have

\[
B_2 = \begin{cases} 
0 & \text{if } t = 0, 1, \\
-\left( \sum_{i=1}^{t/2-1} 2^{(3-2s)(m+i+2)}-3 + \psi(\alpha\beta)2^{(3-2s)(m+t/2+2)}-3 \right) & \text{if } t \geq 2 \text{ and even}, \\
-\left( \sum_{i=1}^{t/2} 2^{(3-2s)(m+i+2)}-3 + 2^{(3-2s)(m+t/2+3)}-3 \right) & \text{if } t \geq 2 \text{ and odd}.
\end{cases}
\]

Suppose \( n = 3 \), then we have

\[
B_2 = \begin{cases} 
0 & \text{if } t = 0, 1, 2, 4, 5, \\
\psi(-1) (\psi(\alpha\beta + 2^{n/2} + 2^n)G(\psi)^22^{(3-2s)(m+n+t/2)-9/2}) & \text{if } t = 3.
\end{cases}
\]

Proof of (1) of Proposition 3.2. By (3.7), \( S_2(\psi, h) = A + B_1 + B_2 + C \). Therefore, by Lemma 3.8, Lemma 3.9, Lemma 3.10 and Lemma 3.11, \( S_2(\psi, h) \) is as stated in the proposition. \( \square \)
Proof of Lemma 3.8. By (3.6), C is as follows

\[
C = \sum_{i=n}^{\infty} \sum_{j=1}^{\infty} 2^{-(2i+j)s} \sum_{a, b \mod 2^i \ u \mod 2^{j-1}} \psi(ad - 2^j b^2) \\
\times e \left( \frac{a\beta}{2^{i-m-t}} \right) e \left( \frac{2b\beta u}{2^{i-m-t-1}} \right) e \left( \frac{(\alpha + 2^t \beta u^2) d}{2^{i+j-m}} \right)
\]

\[
= \sum_{i=n}^{\infty} \sum_{j=1}^{n-1} 2^{-(2i+j)s} \sum_{a, b \mod 2^i \ u \mod 2^{j-1}} \psi(ad - 2^j b^2) \\
\times e \left( \frac{a\beta}{2^{i-m-t}} \right) e \left( \frac{b\beta u}{2^{i-m-t-2}} \right) e \left( \frac{(\alpha + 2^t \beta u^2) d}{2^{i+j-m}} \right)
\]

\[
+ \sum_{i=n}^{\infty} \sum_{j=n}^{\infty} 2^{-(2i+j)s} \sum_{a, b \mod 2^i \ u \mod 2^{j-1}} \psi(ad) \\
\times e \left( \frac{a\beta}{2^{i-m-t}} \right) e \left( \frac{b\beta u}{2^{i-m-t-2}} \right) e \left( \frac{(\alpha + 2^t \beta u^2) d}{2^{i+j-m}} \right).
\]

We put

\[
C_1 = \sum_{i=n}^{\infty} \sum_{j=1}^{n-1} 2^{-(2i+j)s} \sum_{a, b \mod 2^i \ u \mod 2^{j-1}} \psi(ad - 2^j b^2) \\
\times e \left( \frac{a\beta}{2^{i-m-t}} \right) e \left( \frac{b\beta u}{2^{i-m-t-2}} \right) e \left( \frac{(\alpha + 2^t \beta u^2) d}{2^{i+j-m}} \right).
\]

\[
C_2 = \sum_{i=n}^{\infty} \sum_{j=n}^{\infty} 2^{-(2i+j)s} \sum_{a, b \mod 2^i \ u \mod 2^{j-1}} \psi(ad) \\
\times e \left( \frac{a\beta}{2^{i-m-t}} \right) e \left( \frac{b\beta u}{2^{i-m-t-2}} \right) e \left( \frac{(\alpha + 2^t \beta u^2) d}{2^{i+j-m}} \right).
\]

The summation for \(a\) and \(d\) in \(C_2\) vanishes unless \(i - m - t = n\) and \(i + j - m = n\) since \(\psi\) is a primitive character. If \(i - m - t = n\) and \(i + j - m = n\), then we have \(t + j=0\). This contradicts \(j > 0\) and \(t \geq 0\). Therefore \(C_2 = 0\).

As for \(C_1\), replacing \(a\) by \(a + 2^j d^{-1} b^2\), we have

\[
C_1 = \sum_{i=n}^{\infty} \sum_{j=1}^{n-1} 2^{-(2i+j)s} \sum_{a, d, u} \psi(ad) e \left( \frac{\beta a}{2^{i-m-t}} \right) e \left( \frac{(\alpha + \beta u^2 2^{t+2}) d}{2^{i+j-m}} \right)
\]

\[
\times \sum_b e \left( \frac{d^{-1} \beta b^2}{2^{i-j-m-t}} \right) e \left( \frac{b\beta u}{2^{i-m-t-2}} \right).
\]
The summation for \( a \) vanishes unless \( i - m - t = n \). Therefore we have

\[
C_1 = \sum_{j=1}^{n-1} \sum_{a,d} 2^{-(2n+2m+2t+j)s} \sum_{u \mod 2^{j-1}} \psi(ad) e\left( \beta a \frac{2^n}{2n+t+j} \right) e\left( \frac{\alpha + 2^{t+2} \beta u^2 d}{2n+t+j} \right) e\left( \frac{d-1 \beta b^2}{2n-j} \right) e\left( b \beta u \frac{2^n}{2n-j} \right).
\]

We put

\[
X_j = \sum_{a,d} \sum_{u \mod 2^{j-1}} \psi(ad) e\left( \beta a \frac{2^n}{2n+t+j} \right) e\left( \frac{\alpha + 2^{t+2} \beta u^2 d}{2n+t+j} \right) e\left( \frac{d-1 \beta b^2}{2n-j} \right) e\left( b \beta u \frac{2^n}{2n-j} \right),
\]

then

\[
C_1 = \sum_{j=1}^{n-1} 2^{-(2n+2m+2t+j)s} X_j.
\]

When \( j = n - 1 \), \( e\left( \frac{d-1 \beta b^2}{2^{n-j}} \right) = e\left( \frac{d-1 \beta b^2}{2} \right) \) does not depend on \( d \). We can prove \( X_{n-1} = 0 \) in a similar way to the case of \( C_2 \). In particular, \( C_1 = 0 \) if \( n = 2 \). When \( j < n - 1 \), for all \( k \geq 2 \), \( b \equiv b' \mod 2^{k-1} \Rightarrow b^2 \equiv b'^2 \mod 2^k \) therefore \( e\left( \frac{d-1 \beta b^2}{2^{n-j}} \right) \) depends only on \( b \mod 2^{n-j-1} \). Applying Lemma 3.1 for the summation for \( b \) in \( X_j \), we have that the summation for \( b \) vanishes unless \( u \equiv 0 \mod 2^{j-1} \). If \( n > 3 \), by Lemma 3.2, we can show that the summation for \( d \) in \( X_j \) vanishes. Therefore \( C_1 = 0 \) if \( n > 3 \). If \( n = 3 \) and \( j = 1 \), then by \( e\left( \frac{d-1 \beta b^2}{2^{n-j}} \right) = e\left( \frac{b^2 \beta d}{4} \right) \), the summation for \( d \) in \( X_j \) vanishes. Therefore \( C_1 = 0 \) for all cases. \( \square \)

**Proof of Lemma 3.9.** By (3.2),

\[
A = \sum_{i=n}^{\infty} 2^{-2is} \sum_{a,b,d \mod 2^n} \psi(ad - b^2) e\left( \frac{\alpha a}{2^{i-m}} \right) e\left( \frac{\beta d}{2^{i-m-t}} \right).
\]

Applying Lemma 3.1, we see that the summation for \( a \) vanishes if \( i - m > n \), therefore

\[
A = \sum_{i=n}^{m+n} 2^{(3-2s)i-3n} \sum_{a,b,d \mod 2^n} \psi(ad - b^2) e\left( \frac{\alpha a}{2^{i-m}} \right) e\left( \frac{\beta d}{2^{i-m-t}} \right).
\]

By Lemma 3.3

\[
A = \begin{cases} 
\sum_{i=n}^{m+n} 2^{(3-2s)i-5/2n} \sum_{a,d \mod 2^n} (\psi(ad) + \psi(ad - 2^{n-2})) e\left( \frac{\alpha a}{2^{i-m}} \right) e\left( \frac{\beta d}{2^{i-m-t}} \right) & \text{if } n \text{ is even}, \\
\sum_{i=n}^{m+n} 2^{(3-2s)i-5/2n+1/2} \sum_{a,d \mod 2^n} \psi(ad - 2^{n-3}) e\left( \frac{\alpha a}{2^{i-m}} \right) e\left( \frac{\beta d}{2^{i-m-t}} \right) & \text{if } n \text{ is odd}.
\end{cases}
\]

We suppose that \( n \) is even. Then we have
We put

\[ A_1 = \sum_{i=n}^{m+n} 2^{(3-2s)i-5/2n} \sum_{a,d \mod 2^n} \psi(ad) e\left(\frac{\alpha a}{2^{i-m}}\right) e\left(\frac{\beta d}{2^{i-m-i}}\right), \]  

(3.36)

\[ A_2 = \sum_{i=n}^{m+n} 2^{(3-2s)i-5/2n} \sum_{a,d \mod 2^n} \psi(ad - 2^{n-2}) e\left(\frac{\alpha a}{2^{i-m}}\right) e\left(\frac{\beta d}{2^{i-m-i}}\right). \]  

(3.37)

then

\[ A = A_1 + A_2. \]  

(3.38)

By (3.36), \( A_1 \) is as follows

\[ A_1 = \begin{cases} 0 & \text{if } t > 0, \\ \frac{\psi(\alpha \beta)}{\psi(\alpha \beta) G(\psi) 2^{(3-2s)(m+n)-5/2n}} & \text{if } t = 0. \end{cases} \]  

(3.39)

As for \( A_2 \), if \( n > 3 \), we replace \( a \) by \( a + 2^{n-2}d^{-1} \) then we have

\[ A_2 = \sum_{i=n}^{m+n} 2^{(3-2s)i-5/2n} \sum_{a,b \mod 2^n} \psi(ad) e\left(\frac{\alpha a}{2^{i-m}}\right) e\left(\frac{2^{n-2}ad^{-1}}{2^{i-m-i}}\right) e\left(\frac{\beta d}{2^{i-m-i}}\right). \]

It follows that

\[ A_2 = \begin{cases} 0 & \text{if } t > 0, \\ \frac{\psi(\alpha \beta + 2^{n-2})G(\psi) 2^{(3-2s)(m+n)-5/2n}}{\psi(\alpha \beta) G(\psi) 2^{(3-2s)(m+n)-5/2n}} & \text{if } t = 0. \end{cases} \]  

(3.40)

For \( 0 \leq r \leq n, t \geq 0 \), we consider the following sum

\[ \sigma_n(r, t, \alpha, \beta) = \sum_{a,d \mod 2^n} \psi(ad - 1) e\left(\frac{\alpha a}{2^r}\right) e\left(\frac{\beta d}{2^{r-i}}\right). \]  

(3.41)

We put \( d = 2^k x, 0 \leq k \leq n, x \in (\mathbb{Z}/2^n \mathbb{Z})^* \) then

\[ \sigma_n(r, t, \alpha, \beta) = \sum_{k=0}^{n} \sum_{x \in (\mathbb{Z}/2^n \mathbb{Z})^*} \sum_{a \mod 2^n} \psi(2^k xa - 1) e\left(\frac{\alpha a}{2^r}\right) e\left(\frac{\beta x}{2^{r-k-i}}\right). \]
We can prove the summation for \( a \) vanishes if \( r + k \neq n \) in a similar way to the proof of Lemma 3.1. A direct calculation shows that

\[
\sigma_2(i - m, t, \alpha, \beta) = \begin{cases} 
-4 & \text{if } i \leq m + 1, \\
0 & \text{if } i = m + 2 \text{ and } t = 0, \\
4 & \text{if } i = m + 2 \text{ and } t = 1, \\
-4 & \text{if } i = m + 2 \text{ and } t \geq 2.
\end{cases}
\] (3.42)

By (3.42) and (3.37),

\[
A_2 = \begin{cases} 
-\sum_{i=2}^{m+1} \frac{2(3-2s)i-3}{2(3-2s)i-3} & \text{if } t = 0, \\
-\{\sum_{i=2}^{m+1} \frac{2(3-2s)i-3 - 2(3-2s)(m+2-3)}{2(3-2s)i-3} \} & \text{if } t = 1, \\
-\{\sum_{i=2}^{m+1} \frac{2(3-2s)i-3 + 2(3-2s)(m+2-3)}{2(3-2s)i-3} \} & \text{if } t \geq 2,
\end{cases}
\] (3.43)

where \( n = 2 \). By (3.38), (3.39), (3.40) and (3.43), we have the following.

Suppose \( n > 3 \) and even, then

\[
A = \begin{cases} 
(\overline{\psi}(\alpha \beta) + \overline{\psi}(\alpha \beta + 2^{n-2}))G(\psi)^22^{(3-2s)(m+n)-5/2n} & \text{if } t = 0, \\
0 & \text{if } t \neq 0.
\end{cases}
\]

Suppose \( n = 2 \), then

\[
A = \begin{cases} 
-\{\sum_{i=2}^{m+1} \frac{2(3-2s)i-3}{2(3-2s)i-3} + \psi(\alpha \beta)2^{(3-2s)(m+2-3)} \} & \text{if } t = 0, \\
-\{\sum_{i=2}^{m+1} \frac{2(3-2s)i-3}{2(3-2s)i-3} - 2^{(3-2s)(m+2-3)} \} & \text{if } t = 1, \\
-\{\sum_{i=2}^{m+1} \frac{2(3-2s)i-3 + 2(3-2s)(m+2-3)}{2(3-2s)i-3} \} & \text{if } t \geq 2.
\end{cases}
\]

Next we consider the case where \( n \) is odd.

\[
A = \sum_{i=n}^{m+n} 2^{(3-2s)i-5/2n+1/2} \sum_{a,d \text{ mod } 2^n} \psi(ad - 2^{n-3})e\left(\frac{\alpha a}{2^{i-m}}\right)e\left(\frac{-\beta d}{2^{i-m-t}}\right).
\]

Suppose \( n > 3 \). Replacing \( a \) by \( a + 2^{n-3}d^{-1} \), we have

\[
A = \begin{cases} 
\overline{\psi}(\alpha \beta + 2^{n-3})G(\psi)^22^{(3-2s)(m+n)-5/2n+1/2} & \text{if } t = 0, \\
0 & \text{if } t > 0.
\end{cases}
\]

Let \( n = 3 \) and \( \sigma_n(r, t, \alpha, \beta) \) be the sum defined in (3.41), then a direct calculation shows that

\[
\sigma_3(r, t, \alpha, \beta) = \begin{cases} 
\psi(-1)2^3 & \text{if } r \leq 1, \\
-\psi(-1)2^3 & \text{if } t = 0, \\
\psi(-1)2^3 & \text{if } t \geq 1.
\end{cases}
\] (3.44)

\[
\sigma_3(2, t, \alpha, \beta) = \begin{cases} 
-\psi(-1)2^3 & \text{if } t = 0, \\
\psi(-1)2^3 & \text{if } t \geq 1.
\end{cases}
\] (3.45)

\[
\sigma_3(3, t, \alpha, \beta) = \begin{cases} 
0 & \text{if } t = 0, \\
e\left(\frac{\alpha + \beta}{4}\right)2^3 & \text{if } t = 1, \\
-\psi(-1)2^3 & \text{if } t = 2, \\
\psi(-1)2^3 & \text{if } t \geq 3.
\end{cases}
\] (3.46)
Noting that $\psi(-1 - 2\alpha \beta) = e^{(\alpha \beta \beta \gamma)}$, by (3.44), (3.45) and (3.46), $A$ becomes as follows

$$A = \begin{cases} 0 & \text{if } m = 0 \text{ and } t = 0, \\ \psi(-1) \sum_{i=3}^{m+1} 2^{(3-2s)i - 4} - 2^{(3-2s)(m+2) - 4} & \text{if } m \geq 1 \text{ and } t = 0, \\ \psi(-1) \sum_{i=3}^{m+2} 2^{(3-2s)i - 4} + \psi(1 + 2\alpha \beta) 2^{(3-2s)(m+3) - 4} & \text{if } t = 1, \\ \psi(-1) \sum_{i=3}^{m+2} 2^{(3-2s)i - 4} - 2^{(3-2s)(m+3) - 4} & \text{if } t = 2, \\ \psi(-1) \sum_{i=3}^{m+3} 2^{(3-2s)-4} & \text{if } t \geq 3. \end{cases}$$

Proof of Lemma 3.10. By (3.3), $B_1$ is as follows

$$B_1 = \sum_{i=n}^{n-1} \sum_{j=1}^{2^{(2i+j)s}} \sum_{u \mod 2^i} \sum_{a \mod 2^i} \sum_{b \mod 2^i} \sum_{d \equiv 2^{i+j}} \psi(ad - 2^{i+j})$$

$$\times e\left(\frac{\alpha a}{2^{i-m}}\right) e\left(\frac{\alpha b^2 d^{-1}}{2^{i-m}}\right) e\left(\frac{\alpha u_2 + 2^{i+j}}{2^{i+j-m}}\right).$$

Replacing $a$ by $a + 2^{i-j}b^2$, we have

$$B_1 = \sum_{i=n}^{n-1} \sum_{j=1}^{2^{(2i+j)s}} \sum_{u \mod 2^i} \sum_{a \mod 2^i} \sum_{b \mod 2^i} \sum_{d \mod 2^{i+j}} \psi(ad)$$

$$\times e\left(\frac{\alpha a}{2^{i-m}}\right) e\left(\frac{\alpha b^2 d^{-1}}{2^{i-m}}\right) e\left(\frac{\alpha u_2 + 2^{i+j}}{2^{i+j-m}}\right)$$

$$= \overline{\psi}(\alpha) G(\psi) \sum_{j=1}^{2^{(2m+2n+j)s+m}} \sum_{u \mod 2^j} \sum_{b \mod 2^{m+n}} \sum_{d \mod 2^{m+n+j}} \psi(d)$$

$$\times e\left(\frac{\alpha b^2 d^{-1}}{2^{n-j}}\right) e\left(\frac{\alpha u_2 + 2^{i+j}}{2^{n+j}}\right).$$

Applying Lemma 3.1 for the summation for $b$, we see that the summation vanishes unless $u \equiv 0$, $2^{j-1} \equiv 0$ mod $2^j$, therefore we have

$$B_1 = \overline{\psi}(\alpha) G(\psi) \sum_{j=1}^{2^{(2m+2n+j)s+m}} \sum_{b \mod 2^{m+n}} \sum_{d \mod 2^{m+n+j}} \psi(d) e\left(\frac{\alpha b^2 d^{-1}}{2^{n-j}}\right) e\left(\beta d \frac{\beta d}{2^{n+j-t}}\right)$$

$$+ \sum_{b \mod 2^{m+n}} \sum_{d \mod 2^{m+n+j}} \psi(d) e\left(\frac{\alpha b^2 d^{-1}}{2^{n-j}}\right) e\left(\alpha b \frac{\beta d}{2^{n-j}}\right) e\left(2^{j-2} \alpha + 2^{i+j} \beta d\right) e\left(2^{j-2} \alpha + 2^{i+j} \beta d\right).$$

We put

$$U_j = \sum_{b \mod 2^{m+n}} \sum_{d \mod 2^{m+n+j}} \psi(d) e\left(\frac{\alpha b^2 d^{-1}}{2^{n-j}}\right) e\left(\frac{\beta d}{2^{n+j-t}}\right),$$

(3.47)
\[ V_j = \sum_{\substack{d \mod 2^{m+n} \\ b \mod 2^{m+n+j}}} \psi(d) e \left( \frac{\alpha b^2 d^{-1}}{2^{n-j}} \right) e \left( \frac{-\alpha b}{2^n} \right) e \left( \frac{(2^{j-2} \alpha + 2^t \beta)d}{2^{n+j}} \right). \] (3.48)

then we have

\[ B_1 = \overline{\psi}(\alpha) G(\psi) \sum_{j=1}^{n-1} 2^{-(2m+2n+j)x+m} (U_j + V_j). \] (3.49)

If \( j = n - 1 \), then the summation for \( b \) in \( U_{n-1} \) vanishes, therefore

\[ U_{n-1} = 0. \] (3.50)

If \( j = n - 1 \), then by \( e \left( \frac{\alpha b^2 d^{-1}}{2^{n-j}} \right) e \left( -\frac{\alpha b}{2^n} \right) = 1 \),

\[ V_{n-1} = 2^{m+n} \sum_{d \mod 2^{m+2n-1}} \psi(d) e \left( \frac{(2^{2n-4} \alpha + 2^t \beta)d}{2^{2n-1}} \right). \]

By this, the following assertions hold.

Suppose \( n > 3 \), then

\[ V_{n-1} = \begin{cases} 0 & \text{if } t \neq n - 1, \\ \overline{\psi}(2^{n-3} \alpha + \beta)G(\psi)2^{2m+2n-1} & \text{if } t = n - 1. \end{cases} \] (3.51)

Suppose \( n = 2 \), then

\[ V_{n-1} = \begin{cases} 0 & \text{if } t \neq 0, \\ \psi(\frac{\alpha + \beta}{2^n})G(\psi)2^{2m+2n-1} & \text{if } t = 0. \end{cases} \] (3.52)

Suppose \( n = 3 \), then

\[ V_{n-1} = \begin{cases} 0 & \text{if } t \leq 2, \\ \psi(\alpha + 2^{t-2} \beta)G(\psi)2^{2m+2n-1} & \text{if } t \geq 3. \end{cases} \] (3.53)

Next we consider the case where \( j < n - 1 \). Applying Lemma 3.1 for the summation for \( b \), we have

\[ \text{if } j < n - 1, \quad V_j = 0. \] (3.54)

By (3.47), \( U_j \) is as follows

\[ U_j = 2^{m+j} \sum_{d \mod 2^{m+n+j}} \psi(d) e \left( \frac{\beta d}{2^{n+j-t}} \right) \sum_{b \mod 2^{n-j}} e \left( \frac{\alpha d^{-1} b^2}{2^n} \right). \] (3.55)

If \( n - j \) is even, then by Lemma 3.2 we have

\[ U_j = 2^{2m+n/2+3/2} \left\{ \sum_{d \mod 2^n} \psi(d) e \left( \frac{\beta d}{2^{n+j-t}} \right) + \sum_{d \mod 2^n} \psi(d) e \left( \frac{\beta d}{2^{n+j-t}} \right) e \left( \frac{\alpha d}{4} \right) \right\}. \]
Therefore, if $n - j$ is even and $n \geq 3$,

$$U_j = \begin{cases} 0 & \text{if } j \neq t, \\ (\psi(\beta) + \psi(\beta + 2^{n-2}\alpha))G(\psi)2^{2m+n/2+3/2t} & \text{if } j = t. \end{cases} \quad (3.56)$$

If $n - j$ is odd, by (3.55) and Lemma 3.2 we have

$$U_j = 2^{2m+n/2+3/2j+1/2} \sum_{d \mod 2^n} \psi(d) e\left(\frac{\beta d}{2^n+j-1}\right) e\left(\frac{\alpha d}{8}\right).$$

As a consequence, if $j < n - 1$, $n > 3$ and $n - j$ is odd, $U_j$ is as follows

$$U_j = \begin{cases} 0 & \text{if } j \neq t, \\ \psi(\beta + 2^{n-3}\alpha)G(\psi)2^{2m+n/2+3/2j+1/2} & \text{if } j = t. \end{cases} \quad (3.57)$$

Suppose $n > 3$. By (3.50), (3.51), (3.54), (3.56) and (3.57), we have

$$U_j + V_j = \begin{cases} 0 & \text{if } j \neq t, \\ (\psi(\beta) + \psi(\beta + 2^{n-2}\alpha))G(\psi)2^{2m+n/2+3/2t} & \text{if } j = t \text{ and } n - t \text{ is even}. \end{cases} \quad (3.58)$$

Suppose $n = 2$. By (3.50) and (3.52), we have

$$U_1 + V_1 = \begin{cases} 0 & \text{if } t \neq 0, \\ \psi(\alpha + \beta)G(\psi)2^{2m+2n-1} & \text{if } t = 0. \end{cases} \quad (3.59)$$

Suppose $n = 3$. By (3.54) and (3.56), we have

$$U_1 + V_1 = \begin{cases} 0 & \text{if } t \neq 1, \\ (\psi(\beta) + \psi(\beta + 2^{n-2}\alpha))G(\psi)2^{2m+n/2+3/2} & \text{if } t = 1. \end{cases} \quad (3.60)$$

Suppose $n = 3$. By (3.50) and (3.53), we have

$$U_2 + V_2 = \begin{cases} 0 & \text{if } t \leq 2, \\ \psi(\alpha + 2^{l-2}\beta)G(\psi)2^{2m+2n-1} & \text{if } t \geq 3. \end{cases} \quad (3.61)$$

By (3.49), (3.58), (3.59), (3.60) and (3.61), we obtain the assertions of the lemma. □

**Proof of Lemma 3.11.** By (3.4), $B_2$ is as follows

$$B_2 = \sum_{i,j=n}^{\infty} 2^{-(2i+j)s} \sum_{u \mod 2^i} \sum_{a,b \mod 2^j \atop d \mod 2^{i+j}} \psi(ad)e\left(\frac{\alpha a}{2^{i-m}}\right)e\left(-u\alpha b\frac{2^{i-m-1}}{2^{i-m-1}}\right)e\left(\frac{\alpha u^2 + 2^i \beta d}{2^{i+j-m}}\right).$$

The summation for $a$ vanishes unless $i = m + n$. If $i = m + n$, the summation for $b$ vanishes unless $2^{n-1} | u$. Replacing $u$ by $2^{n-1}u$, we have

$$B_2 = \overline{\psi(\alpha)}G(\psi) \sum_{j=n}^{\infty} 2^{-(2m+2n+j)s+2m+n} \sum_{u \mod 2^{j-n+1}} \sum_{d \mod 2^{m+n+j}} \psi(d)e\left(\frac{\alpha u^2}{2^{j-n+2}}\right)e\left(\frac{\beta d}{2^{n+j-1}}\right).$$
We put
\[
W_j = \sum_{u \mod 2^{j-n+1}} \sum_{d \mod 2^{m+n+j}} \psi(d) e\left(\frac{\alpha d u^2}{2^{j-n+2}}\right) e\left(\frac{\beta d}{2^{n+j-t}}\right),
\]  
(3.62)
then we have
\[
B_2 = \psi(\alpha) G(\psi) \sum_{j=n}^{\infty} 2^{-(2m+2n+j)x+2m+n} W_j.
\]  
(3.63)

If \( j - n \) is even, by Lemma 3.2, \( W_j \) is as follows
\[
W_j = 2^{j-n} \left\{ \sum_{d \mod 2^{m+n+j}} \psi(d) e\left(\frac{\beta d}{2^{n+j-t}}\right) + \sum_{d \mod 2^{m+n+j}} \psi(d) e\left(\frac{\beta d}{2^{n+j-t}}\right) e\left(\frac{\alpha d}{4}\right) \right\}.
\]

Suppose \( j - n \) is even and \( n \geq 3 \). We have
\[
W_j = \begin{cases} 
0 & \text{if } j \neq t, \\
(\overline{\psi}(\beta) + \overline{\psi}(\beta + 2^{n-2} \alpha)) G(\psi) 2^{m-n/2+3/2j} & \text{if } j = t.
\end{cases}
\]  
(3.64)

Suppose \( n = 2 \) and is even. We have
\[
W_j = \begin{cases} 
\psi(\alpha + 2^{j-n} \beta) G(\psi) 2^{m+3/2j-1} & \text{if } j < t, \\
\psi(\beta) G(\psi) 2^{n+3/2j-1} & \text{if } j = t, \\
0 & \text{if } j > t.
\end{cases}
\]  
(3.65)

If \( j - n \) is odd, by Lemma 3.2, \( W_j \) is as follows
\[
W_j = 2^{j-n} \sum_{d \mod 2^{m+n+j}} \psi(d) e\left(\frac{\beta d}{2^{n+j-t}}\right) e\left(\frac{\alpha d}{8}\right).
\]

Therefore, \( W_j \) becomes as follows.

Suppose \( j - n \) is odd and \( n > 3 \), then
\[
W_j = \begin{cases} 
0 & \text{if } j \neq t, \\
(\overline{\psi}(\beta + 2^{n-3} \alpha) G(\psi) 2^{m-n/2+3/2j+1/2} & \text{if } j = t.
\end{cases}
\]  
(3.66)

Suppose \( n = 2 \) and \( j \) is odd, then
\[
W_j = \begin{cases} 
0 & \text{if } j \neq t+1, \\
\psi(\frac{\alpha + \beta}{2}) G(\psi) 2^{m+3/2j-1/2} & \text{if } j = t+1.
\end{cases}
\]  
(3.67)

Suppose \( n = 3 \) and \( j \) is even, then
\[
W_j = \begin{cases} 
0 & \text{if } j \geq t, \\
\psi(\alpha + 2^{j-n} \beta) G(\psi) 2^{m+3/2j-1} & \text{if } j < t.
\end{cases}
\]  
(3.68)

By (3.63), (3.64), (3.66), (3.65), (3.67), (3.64), (3.68), we obtain the assertions of the lemma. \( \Box \)
4. An explicit formula for Fourier coefficients of Siegel–Eisenstein series of degree two

In this section we prove an explicit formula for Fourier coefficients for Siegel–Eisenstein series of degree two, where the weight $k$ is larger than three, $\psi$ is a primitive character and the level $N$ is arbitrary.

4.1. In this subsection, by using the local factors from the local functional equation in Tate’s thesis, we express Euler factor of Fourier coefficients for $E_{k,\psi}^{(2)}(z)$, at infinite place and at finite place $p$ ($p \mid N$).

For a place $v$ of $\mathbb{Q}$ and a quasi character $\chi : \mathbb{Q}_v^\times \to \mathbb{C}^\times$, we denote $\varepsilon$-factor and $\gamma$-factor from the local functional equation in Tate’s thesis by $\varepsilon_v(\chi, s)$ and $\gamma_v(\chi, s)$ for fixed additive character $e_v$. The explicit forms of $\varepsilon_v(\chi, s)$ and $\gamma_v(\chi, s)$ are as follows.

Suppose that $v = \infty$ is infinite place and $\chi = \text{sgn}^k$,

$$\gamma_\infty(\chi, s) = \begin{cases} \pi^{s+1} & \text{if } k \text{ is even}, \\ i\pi^{s+1} & \text{if } k \text{ is odd}. \end{cases}$$

Suppose that $v = p$ is a finite place

$$\varepsilon_p(\chi, s) = \begin{cases} 1 & \chi \text{ is unramified,} \\ \chi(-f(\chi))\frac{\varepsilon_p(\chi)}{\pi^{1-s}} & \chi \text{ is ramified,} \end{cases}$$

$$\gamma_p(\chi, s) = \begin{cases} \frac{1-\chi(p)p^{-1}G(\chi)}{\varepsilon_p(\chi)} & \chi \text{ is unramified,} \\ \frac{1-\chi(p)p^{-1}G(\chi)}{\varepsilon_p(\chi)} & \chi \text{ is ramified}. \end{cases} \quad (4.1)$$

Here, $f(\chi)$ is the power of $p$ such that the ideal $(f(\chi))$ is the conductor of $\chi$ and $G(\chi)$ is Gauss sum,

$$G(\chi) = \sum_{a \in \mathbb{Z}_p^\times/(1+p\mathbb{Z}_p)} \chi(a)e_p(-a/p^n),$$

where $p^n = f(\chi)$.

Let $0 < h \in \text{Sym}_2(\mathbb{Z})$ be a half integral, positive-definite, symmetric matrix and $a(h, y, E_{k,\psi}^{(2)})$ be the $h$-th Fourier coefficient of $E_{k,\psi}^{(2)}(z)$. By Theorem 2.2,

$$a(h, y, E_{k,\psi}^{(2)}) = \xi(y, h; k, 0)e(-iyh) \frac{L(N)(k-1, \chi_h\overline{\psi})}{L(k, \overline{\psi})L(N)(2k-2, \overline{\psi}^2)} \times \prod_{p \mid N} F_p^{(2)}(h; \overline{\psi}(p)p^{-k}) \prod_{p \mid N} A_p(h, \psi_p; \overline{\psi}_p(p)p^{-k}). \quad (4.2)$$

Here $\xi(y, h; \alpha, \beta)$ is a function of $(2.30)$, $F_p^{(2)}(h; T)$ is a polynomial of $(2.32)$ and $A_p(h, \psi_p; T)$ is the formal power series which corresponds to Siegel series.

By the explicit form for $\xi(y, h; k, 0)$ in Theorem 2.1, and the duplication formula of the Gamma function: $\Gamma(s)\Gamma(s + 1/2) = 2^{1-2s}\pi^{1/2}\Gamma(2s)$, we can prove the following proposition.

**Proposition 4.1.** Suppose that $h \in \text{Sym}_g(\mathbb{R})$ is positive definite and put $\chi = \text{sgn}^k$, $\rho = \text{sgn}$.

1. If $g$ is even,

$$\xi(y, h, k, 0)e(-iyh) = 2^{k/2}g^{1/4}(\det 2h)^{k-(g+1)/2} \frac{\gamma_\infty(\chi, k)}{\gamma_\infty(\rho\chi, k - g/2)} \prod_{i=1}^{g/2} \gamma_\infty(\chi, 2k - 2i).$$
(2) If \( g \) is odd,

\[
ξ(y, h, k, 0) e(-iyh) = 2^{(g+1)/2}(-1)^{g^2/2} 2^{-1 \det 2h} k^{-(g+1)/2} \\
\times γ_∞(χ, k) \prod_{i=1}^{(g-1)/2} γ_∞(χ, 2k - 2i).
\]

Next let us consider the Euler factor at a finite place. For a Dirichlet character \( χ \mod p^n \), we define \( A'_p(h, χ; T) \) as follows

\[
A'_p(h, χ; T) = \begin{cases} 
A_p(h, χ; T) & \text{if } χ^2 \neq 1, \\
A_p(h, χ; T) - χ(-1) \frac{(p-1)p^{-n-1}(p^3T^2)^n}{1-pT^2} & \text{if } χ^2 = 1.
\end{cases}
\]

**Proposition 4.2.** Let \( ψ \) be a primitive Dirichlet character mod \( N \). Let \( ω \) denote the character of \( \mathbb{A}^×/\mathbb{Q}^× \) corresponding to \( ψ \) and \( ρ_h \) denote the character of \( \mathbb{Q}^×_p \) corresponding to \( \mathbb{Q}_p(√-\det(h))/\mathbb{Q}_p \) by local class field theory.

Suppose \( p \mid N, h ∈ \text{Sym}_2(\mathbb{Z}/p) \cap \text{GL}_2(\mathbb{Q}_p) \) and \( A_p(h, ω; T) \neq 0 \). Then the following equation holds

\[
A'_p(h, ψ; \overline{ψ}_p(p)p^{-s}) = \overline{ω}_p(\det(2h)) \frac{γ_p(\overline{ω}_p, s)γ_p(\overline{ω}_p^2, 2s - 2)}{γ_p(\overline{ω}_p, s - 1)} \\
\times ϵ_p(ρ_h, s - 1)p^{(3-2s)α_p}.
\]

Furthermore if \( ψ_p^2 = 1 \),

\[
A'_p(h, ψ; \overline{ψ}_p(p)p^{-s}) = ω_p(-1)p^{-n_p}(\overline{ω}_p(p^2)p^{3-2s})^{β_p} \\
\times L(ω_p^2, 3-2s)L(ρ_h, ω^p, s - 1) \\
L(\overline{ω}_p^2, 2s - 2)L(ρ_h, ω^p, 2 - s).
\]

Here \( n_p = \text{ord}_p(\overline{f}(ω_p)) \) and \( α_p, β_p \) is

\[
α_p = \frac{1}{2} \text{ord}_p(\det(2h)/f(ρ_h)),
\]

\[
β_p = \frac{1}{2} \text{ord}_p(\overline{f}(ω_p)f(ω_p^2)/f(ω_pρ_h)) + \frac{1}{2} \text{ord}_p(\det(2h)).
\]

For the proof of Proposition 4.2, we prove some lemmas for Gauss sums.

**Lemma 4.1.** Let \( p \) be a prime and \( χ, \psi \) be primitive Dirichlet characters mod \( p^n, \mod p^m \) respectively. Put \( l = \text{ord}_p(f(χψ)) \). We define the Jacobi sum by

\[
J(χ, ψ) = \begin{cases} 
\sum_{x \mod p^n} χ(p^{n-l} - x)ψ(x) & \text{if } n = m, \\
\sum_{x \mod p^m} χ(1 - p^{n-m}x)ψ(x) & \text{if } n > m.
\end{cases}
\]

Then we have \( J(χ, ψ) = \frac{G(χ)G(ψ)}{G(χψ)} \), where \( G(χψ) = \sum_{x \mod p^f} χψ(x)e(x/p^f) \).

**Remark 4.1.** This lemma seems well known, but for the sake of completeness we give a proof.
Proof. We prove the lemma in the case where \( n = m \). If \( n > m \), we can prove the lemma in a similar way.

By the definition of Gauss sum,

\[
G(\chi)G(\psi) - 1 \sum_{x,y \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(x)\psi(y)e\left(\frac{x}{p^n}\right)e\left(\frac{y}{p^n}\right).
\]

Replacing \( x \) by \( xy \), we have

\[
G(\chi)G(\psi) - 1 \sum_{x,y \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(x)\chi\psi(y)e\left(\frac{(x+1)y}{p^n}\right).
\]

By Lemma 3.1 the summation for \( y \) vanishes if \( p^n - l \nmid x + 1 \). If \( p^n - l \mid x + 1 \), then put \( x + 1 = p^n - lx' \).

Then we have

\[
G(\chi)G(\psi) = p^n - 1 \sum_{x' \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(-1 + p^{n-1}x')\chi\psi(x')
\]

\[
= \sum_{x \in (\mathbb{Z}/p^n\mathbb{Z})^\times} \chi(p^{n-1} - x^{-1})\psi(x^{-1}). \quad \square
\]

Lemma 4.2. Let \( \psi \) be a primitive Dirichlet character mod \( p^n \). For \( d \in \mathbb{Q}^\times \), we denote the primitive Dirichlet character associated with \( \mathbb{Q}(\sqrt{d})/\mathbb{Q} \) by \( \chi_d \).

(1) Suppose \( p \neq 2 \). If \( n \) is even,

\[
G(\psi)^2 = \overline{\psi}(4)G(\psi^2)p^{n/2}.
\]

If \( n \) is odd and \( \psi^2 \neq 1 \),

\[
G(\psi)G(\psi \chi_{p^n}) = \varepsilon_p \overline{\psi}(4)G(\psi^2)p^{n/2}.
\]

(II) Suppose \( p = 2 \). Let \( n \geq 4 \). Then we have

\[
J(\psi, \psi) = \begin{cases} 2^{n/2}(1 + \psi(0)) & \text{if } n \text{ is even}, \\ 2^{(n+1)/2}\psi(0) & \text{if } n \text{ is odd}, \end{cases}
\]

\[
J(\chi_{-1}, \psi) = -2\psi(1 + 2^{n-2}).
\]

Suppose \( \psi(1 + 2^{n-3}) = (1 + i)\sqrt{2} \) if \( n \geq 5 \) and \( \psi(5) = i \) if \( n = 4 \) and \( \chi = \chi_2 \) or \( \chi_{-2} \). Then we have

\[
J(\chi, \psi) = \begin{cases} \sqrt{2}(1 + i)(\chi(-1) - i) & \text{if } n \geq 6, \\ \sqrt{2}(1 + i)(\chi(-1) + i) & \text{if } n = 5, \\ 2\psi(-1)(1 + \chi(-1)i) & \text{if } n = 4. \end{cases}
\]

Remark 4.2. If \( p \) is odd and \( n = 1 \), then this lemma is the special case of Davenport–Hasse’s product formula.
We only prove the formula for \( J(\psi, \psi) \) for case where \( p = 2 \). In a similar way, we can prove the others. Since \( J(\psi^2) = 2^{n-1} \), applying Lemma 4.1, we have

\[
J(\psi, \psi) = \sum_{x \mod 2^n} \psi(2x - x^2) = \sum_{x \mod 2^n} \psi(1 - x^2).
\]

By Lemma 3.3 we have

\[
J(\chi, \psi) = \begin{cases} 2^{n/2}(1 + \psi(1 - 2^{n-2})) & \text{if } n \text{ is even}, \\ 2^{(n+1)/2} \psi(1 - 2^{n-3}) & \text{if } n \text{ is odd}. \end{cases}
\]

Now we prove Proposition 4.2.

**Proof of Proposition 4.2.** We suppose that \( S_p(\psi, h) \neq 0 \). By definition of Siegel series, we see that Siegel series does not depend on \( \omega_p(p) \). Replacing \( \overline{\psi}_p^*(p) p^{-s} \) by \( p^{-s} \) in both sides of Proposition 4.2, we may assume that \( N = p^n \) and \( \psi_p^*(p) = \omega_p(p) = 1 \). We denote \( \psi \) by \( \psi_p \) and \( n \) by \( n_p \) for simplicity of notation. If \( p \neq 2 \) or \( \psi^2 = 1 \), then we see that \( S_p(\psi, h) \) depends only on \( \det h \) by Proposition 3.1 and Proposition 3.2. If \( p = 2 \) and \( \psi^2 \neq 1 \), then by Lemma 4.2, we have

\[
G(\overline{\psi})^2 / G(\overline{\psi}^2) = \begin{cases} 2^{n/2}(1 + \overline{\psi}(1 - 2^{n-2})) & \text{if } n \text{ is even}, \\ 2^{(n+1)/2} \overline{\psi}(1 - 2^{n-3}) & \text{if } n \text{ is odd}. \end{cases}
\]

From this and Proposition 3.2, we see that \( S_p(\psi, h) \) depends only on \( \det h \). Obviously, the right-hand side of Proposition 4.2 depends only on \( \det h \). We put \( \det h = p^t u \) with \( u \in \mathbb{Z}^n_p \) and \( t \in \mathbb{Z} \).

(I) Suppose \( p \neq 2 \) and \( \psi^2 \neq 1 \). By Proposition 3.1, we have

\[
S_p(\psi, h) = \begin{cases} \psi(u) G(\overline{\psi}^2) p^{(3/2)(n/2) - 5/2} & \text{if } n - t \text{ is even}, \\ \epsilon_p(\psi_x p^*) \psi(u) G(\overline{\psi}) G(\overline{\psi}_x p^*) p^{(3/2)(n/2) - 5/2} & \text{if } n - t \text{ is odd}. \end{cases}
\]

Put

\[
X = S_p(\psi, h) \omega_p(\det(2h)) = \frac{\gamma_p(\rho_h \overline{\omega}_p, s - 1)}{\gamma_p(\omega_p, s) \gamma_p(\overline{\omega}_p p^2, 2s - 2)} \epsilon_p(\rho_h, s - 1)^{-1} p^{(2s - 3)\alpha_p}.
\]

We need to prove \( X = 1 \). By (4.1), we have

\[
X = S_p(\psi, h) \omega_p(4u) \rho_h \left( p^{n/2} \overline{\psi}(\rho_h) \right) \frac{G((\chi_h)_p \overline{\psi})}{G(\overline{\psi}_x p^*)} G((\chi_h)_p)^{-1} p^{(3/2)(n/2) - 5/2} \epsilon_p(\psi_x p^*, p^{n-1} \rho_h, \overline{\psi}((\chi_h)_p)^{-1} p^{(3/2)(n/2) - 5/2} + 2n + \delta_p/2). \tag{4.5}
\]

Here, \( \delta_p = \operatorname{ord}_p(f(\chi_h)) \). By (4.4) and (4.5), we have

\[
X = \begin{cases} \frac{\overline{\psi}(4) \overline{G(\overline{\psi}^2)} p^{-n/2}}{G(\overline{\psi}_x p^*)} & \text{if } n \text{ is even and } t \text{ is even}, \\ \frac{\epsilon_p(\psi_x p^*(u) \rho_h(p) \overline{\psi}(4) G(\overline{\psi}_x p^*)^2 p^{-n/2 + 1/2}}{G(\overline{\psi}_x p^*)^2} & \text{if } n \text{ is even and } t \text{ is odd}, \\ \frac{\epsilon_p(\psi_x p^*(u) \rho_h(p) \overline{\psi}(4) G(\overline{\psi}_x p^*)^2 p^{-n/2}}{G(\overline{\psi}_x p^*)^2} & \text{if } n \text{ is odd and } t \text{ is even}, \\ \frac{\overline{\psi}(4) G(\overline{\psi} G(\overline{\psi}_x p^*) p^{-n/2 + 1/2}}{G(\overline{\psi}_x p^*)^2} & \text{if } n \text{ is odd and } t \text{ is odd}. \end{cases}
\]
Note that the following holds

$$\rho_h(p) = \begin{cases} \chi_{p^*}(-u) & \text{if } t \text{ is even,} \\ \chi_{p^*}(u) & \text{if } t \text{ is odd.} \end{cases}$$

By Lemma 4.2, we obtain $X = 1$.

(II) Suppose that $p = 2$ and $\psi^2 \neq 1$. We put

$$\eta(\psi, h) = \begin{cases} 1 + \psi(1 + 2n^{-2}u) & \text{if } n - t \text{ is even,} \\ \psi(1 + 2n^{-3}u) & \text{if } n - t \text{ is odd.} \end{cases}$$

If $S_2(\psi, h) \neq 0$ and $n \geq 4$, then by Proposition 3.2, we see that

$$S_2(\psi, h) = \psi(u)\eta(\psi, h)G(\psi)^22^{(3-2s)(n+t/2)-5/2n+1/2}.$$ We put

$$X = S_2(\psi, h)\omega(\det(2h))\frac{\gamma_2(\rho_h\bar{\omega}_2, s - 1)}{\gamma_2(\bar{\omega}_2, s)\gamma_2(\bar{\omega}_2, 2s - 2)}\epsilon_2(\rho_h, s - 1)^{-1}2^{(2s-3)\alpha_2}.$$ We need to prove $X = 1$. By Lemma 4.1 and the explicit form of $\gamma_p(\chi, s)$ and $\epsilon_p(\chi, s)$, we have

$$X = \eta(\psi, h)\frac{J(\bar{\psi}, \bar{\psi})}{J(\psi, (\chi_h)2)}\rho_h\left(\bar{\psi}(\rho_h\omega_p)/\bar{\psi}(\rho_h)\right)2^{(3-2s)(n+t/2-\alpha_2)+\lambda}.$$ Here $\lambda = (2n + \delta_2 - 2)s - 7/2n - \delta_2 + 5/2$ and $\delta_2 = \text{ord}_2(\bar{\psi}(\rho_h))$. We put

$$\theta(\psi) = J(\bar{\psi}, \bar{\psi})2^{-(n+1)/2},$$

$$\iota(\psi, h) = J(\bar{\psi}, (\chi_h)2)2^{-\delta_2/2}.$$ By definition of $\alpha_2$, we have $t + 2 = \delta_2 + 2\alpha_2$. Therefore

$$X = \eta(\psi, h)\frac{\theta(\psi)}{\iota(\psi, h)}\rho_h(2^{n-t}).$$ Here, we note that $\rho_h(\bar{\psi}(\rho_h\omega_p)/\bar{\psi}(\rho_h)) = \rho_h(2^{n-t})$ if $n \geq 4$. Henceforth we assume that $\psi(1 + 2n^{-3}) = (1 + i)/\sqrt{2}$ if $n \geq 5$ and $\psi(5) = i$ if $n = 4$. By Lemma 4.2, we have

$$\theta(\psi) = \begin{cases} \zeta_8 & \text{if } n \text{ is even,} \\ \bar{\psi}(1 - 2n^{-3}) & \text{if } n \text{ is odd,} \end{cases}$$

$$\iota(\psi, h) = \begin{cases} 1 & \text{if } \delta_2 = 0, \\ i & \text{if } \delta_2 = 2 \text{ and } n \neq 5, \\ -i & \text{if } \delta_2 = 2 \text{ and } n = 5, \\ \zeta_8^{-1}(\chi_h)2^{(1)+i} & \text{if } \delta_2 = 3 \text{ and } n \geq 6, \\ \zeta_8^{-1}(\chi_h)2^{(1)-i} & \text{if } \delta_2 = 3 \text{ and } n = 5, \\ \psi(-1)\frac{1-(\chi_h)2}{\sqrt{2}} & \text{if } \delta_2 = 3 \text{ and } n = 4. \end{cases}$$
Here we put $\zeta_8 = (1 + i)/\sqrt{2}$. Note that the following holds.

<table>
<thead>
<tr>
<th>$-\det(2h) \mod (\mathbb{Q}^\times)^2$</th>
<th>1</th>
<th>5</th>
<th>$-1$</th>
<th>$-5$</th>
<th>2</th>
<th>$5 \cdot 2$</th>
<th>$-2$</th>
<th>$-5 \cdot 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\chi_8)_2$</td>
<td>$X_1$</td>
<td>$X_1$</td>
<td>$X_8$</td>
<td>$X_8$</td>
<td>$X_{-8}$</td>
<td>$X_{-8}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho_h(2)$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>

By the explicit forms above, we can verify that $X = \eta(\psi, h)\frac{\vartheta(\psi)}{\eta(\psi, h)}\rho_h(2^{n-t}) = 1$ when $n \geq 4$.

(III) Suppose that $\psi^2 = 1$.

Put

$$S'_p(\psi, h) = A'_p(h, \psi_p; \overline{\psi}_p(p)p^{-s}),$$

$$T_p(\psi, h) = A'_p(h, \psi_p; \psi_p(p)p^{-s})L(\overline{\sigma}_p^2, 2s - 2)L(\rho_h\omega_p, 2 - s)$$

We have to show

$$\overline{\sigma}_p(\det(2h))\frac{e_p(\overline{\sigma}_p^2, s)}{e_p(\rho_h\overline{\sigma}_p, s - 1)}e_p(\rho_h, s - 1)p^{(3 - 2s)\alpha_p} = \omega_p(-1)p^{(3 - 2s)\beta_p - n}, \quad (4.6)$$

$$T_p(\psi, h) = \omega_p(-1)p^{(3 - 2s)\beta_p - n}. \quad (4.7)$$

By definition of $\beta_p$, the left-hand side of (4.6) is a form of $c p^{(3 - 2s)\beta_p - n}$, where $c$ is independent of $s$. Substituting $s = 3/2$, we see that $|c| = 1$. Therefore, (4.6) is equivalent to

$$\arg \left\{ \psi(-u)\rho_h(\overline{\psi}((\chi_h)_{p})/\overline{\psi}((\chi_h)_{p}\psi)) \frac{G(\psi)G((\chi_h)_{p})}{G((\chi_h)_{p}\psi)} \right\} = 0.$$

We can verify this equality directly. Next let us show (4.7). For simplicity, we treat the case where $\overline{\psi} = p$ or 4 with $p \neq 2$.

First suppose $\overline{\psi} = p$. By Proposition 3.1, we have

$$S_p(\psi, h) = \left\{ \begin{array}{ll} \psi(-1)(p - 1)^{t/2}p^{(3 - 2s)(t - 2) - 2} + \chi_p(u)p^{(3 - 2s)(t/2 - 1) - 2} & \text{if } t \text{ is even,} \\ \psi(-1)(p - 1)^{t/2+1/2}p^{(3 - 2s)(t - 2) + 1} & \text{if } t \text{ is odd.} \end{array} \right.$$  

By definition of $S'_p(\psi, h)$, we have

$$S'_p(\psi, h) = \left\{ \begin{array}{ll} \psi(-1)p^{(3 - 2s)(t/2 - 1)} & \text{if } t \text{ is even,} \\ \psi(-1)p^{(3 - 2s)(t + 1)/2 - 1} & \text{if } t \text{ is odd.} \end{array} \right.$$  

Note that $\omega\rho_h(p) = \psi(u)$ if $t$ is odd. Thus, we obtain

$$T_p(\psi, h) = \left\{ \begin{array}{ll} \psi(-1)p^{(3 - 2s)(t/2 - 1)} & \text{if } t \text{ is even,} \\ \psi(-1)p^{(3 - 2s)(t + 1)/2 - 1} & \text{if } t \text{ is odd.} \end{array} \right.$$  

Therefore, we have (4.7).
Next suppose \( f(\psi) = 4 \). By Proposition 3.2, we have

\[
S_2(\psi, h) = \begin{cases} 
-\sum_{i=2}^{t/2+1} 2(3-2s)i-3 - \psi(u)2^{3-2s}(t/2+2) - 3 - \psi\left(\frac{1+u}{2}\right)2^{(3-2s)(t/2+5/2)-5/2} & \text{if } t \text{ is even}, \\
-\sum_{i=2}^{t+1/2} 2(3-2s)i-3 + 2^{3-2s}(t/2+3/2) - 3 & \text{if } t \text{ is odd}.
\end{cases}
\]

By definition of \( S'_2(\psi, h) \), we have

\[
S'_2(\psi, h) = \begin{cases} 
\frac{2^{(3-2s)(t/2+1) - 3}}{1 - 2^{2s-3}} ((1 - \psi(u)) - \psi\left(\frac{1+u}{2}\right)2^{2-s} + \psi(u)2^{3-2s} + \psi\left(\frac{1+u}{2}\right)2^{5-3s}) & \text{if } t \text{ is even}, \\
-2^{(3-2s)(t/2+1/2) - 2} 2^{2-s} & \text{if } t \text{ is odd}.
\end{cases}
\]

(i) If \( t \) is odd, then

\[
T_2(\psi, h) = S'_2(\psi, h) \frac{1 - 2^{2s-3}}{1 - 2^{2s+2}} = -2^{(3-2s)(t/2+1/2)-2}.
\]

(ii) If \( t \) is even and \( u \equiv -1 \mod 4 \), then

\[
T_2(\psi, h) = S'_2(\psi, h) \frac{1 - 2^{2s-3}}{1 - 2^{2s+2}} = -2^{(3-2s)(t/2-1/2)-2}.
\]

(iii) If \( t \) is even and \( u \equiv 1 \mod 4 \), then

\[
S'_2(\psi, h) = -\frac{2^{(3-2s)(t/2+2) - 2}}{1 - 2^{2s-3}} \frac{(1 - \psi\left(\frac{1+u}{2}\right)2^{2-s})(1 - 2^{2s+2})}{1 - \psi\left(\frac{1+u}{2}\right)2^{s-1}}.
\]

Noting \( \psi\left(\frac{1+u}{2}\right) = \psi \chi_h(2) = \rho_h(2) \), we have,

\[
T_2(\psi, h) = S'_2(\psi, h) \frac{1 - 2^{2s-3})(1 - \rho_h(2)2^{s-1})}{(1 - 2^{2s+2})(1 - \rho_h(2)2^{s-2})} = -2^{(3-2s)(t/2+2) - 2}.
\]

By the definition of \( \beta_p \),

\[
\beta_2 = \frac{1}{2} \operatorname{ord}_2(\overline{f(\psi)} + \overline{f(\psi \chi_h)}) + \frac{1}{2} \operatorname{ord}_2(\det 2h) = t/2 + 2 - \frac{1}{2} \operatorname{ord}_2(\overline{f(\psi \chi_h)}).
\]

Therefore, (4.7) holds. \( \square \)

4.2. By Proposition 4.1, Proposition 4.2 and the functional equations of Dirichlet \( L \)-functions and \( F_p^{(2)}(h; T) \), we can prove the following explicit formula for the Fourier coefficients for Siegel–Eisenstein series of degree two.

**Theorem 4.1.** Let \( \psi \) be a primitive Dirichlet character mod \( N \) and \( h \) be a half integral, positive-definite, symmetric matrix. We denote the \( h \)-th Fourier coefficient for Siegel–Eisenstein series of degree two by \( a(h, E_k^{(2)}) \).

Suppose \( k > 3 \). For a prime \( p \mid N \), we define \( c_p(h, \psi; T) \in \mathbb{Q}(\psi)(T) \) as follows.

1. If \( (p, \psi_p, h) \) satisfies the condition (i) or (iii) below, then we define \( c_p(h, \psi; T) = 0 \).
2. If \( (p, \psi_p, h) \) satisfies the condition neither (i) nor (ii), and \( \psi_p^2 \neq 1 \), then we define \( c_p(h, \psi; T) = 1 \).
(3) If \((p, \psi_p, h)\) satisfies the condition neither (i) nor (ii), and \(\psi_p^2 = 1\), then we define \(c_p(h, \psi; T)\) by

\[
c_p(h, \psi; T) = 1 + p^{-1}(1 - p) \frac{1 - \chi_h(\psi)(p)p^{-2}T^{-1}}{(1 - \psi^2(p)p^{-4}T^{-2})(1 - \chi_h(\psi)(p)pT)}(p^3\psi^2(p)T^2)^{\beta_p - n_p + 1}.
\]

Here \(n_p\) and \(\beta_p = \beta_p(h)\) are given by

\[
n_p = \text{ord}_p(f(\psi)),
\]

\[
2\beta_p = 2\beta_p(h) = \text{ord}_p \left( \frac{f(\psi)\overline{f}(\psi^2)}{f(\psi)\overline{\chi}_h} \right) + \text{ord}_p(\text{det} 2h).
\]

The conditions (i) and (ii) are as follows

(i) \(p = 2, f(\psi_p) \geq 4\) and \(f(\psi_p) \neq 8\) and \(h \in \text{Sym}_2^d(\mathbb{Z}) \setminus \text{Sym}_2^d(\mathbb{Z})\);

(ii) \(p = 2, f(\psi_p) = 8\) and \(h\) is \(\text{GL}_2(\mathbb{Z})\)-equivalent to a matrix of the form

\[
\begin{pmatrix}
\alpha & 0 \\
0 & \beta
\end{pmatrix},
\]

\[2^m \begin{pmatrix}
0 & 1/2 \\
1/2 & 0
\end{pmatrix}
\]

or \[2^m \begin{pmatrix}
1 & 1/2 \\
1/2 & 1
\end{pmatrix},
\]

with \(\alpha, \beta \in \mathbb{Z}_2^\times\) and \(m \in \{0, 1\}\).

Then we have

\[
a(h, E_{k, \psi}^{(2)}) = 2\frac{L(N)(2 - k, \chi_h\psi)}{L(1 - k, \psi)L(N)(3 - 2k, \psi^2)} \times \prod_{p: \text{prime}} F^{(2)}_p(h, \psi_p)^{\chi_h(\psi)(p)p^{-k} - 3} \prod_{p: \text{prime}} c_p(h, \psi; p^{k-3}).
\]

Here \(F^{(2)}_p(h; T)\) is a polynomial of (2.33), and \(\chi_h\) is the primitive Dirichlet character associated with \(\mathbb{Q}(\sqrt{-\text{det}(2h)})\).

**Proof.** If \((p, \psi_p, h)\) satisfies the condition (i) or (ii), then we see that \(a(h, E_{k, \psi}^{(2)}) = 0\) by (4.2), Proposition 3.1 and Proposition 3.2.

Suppose \((p, \psi_p, h)\) does not satisfy the condition (i) nor (ii). Let \(\omega\) be the character of \(\mathbb{A}^\times / \mathbb{Q}^\times\) corresponding to \(\psi\). We put

\[
a(h, E_{k, \psi}^{(2)})' = \xi(y, h; k, 0)e(-iyh) \frac{L(N)(k - 1, \chi_h\psi)}{L(k, \psi)L(N)(2k - 2, \psi^2)}
\]

\[
\times \prod_{p: \text{prime}} F^{(2)}_p(h, \psi_p)^{\chi_h(\psi)(p)p^{-k}} \prod_{p: \text{prime}} A'_p(h, \psi_p; \overline{\psi}_p^n(p)p^{-k}),
\]

with \(A'_p(h, p; T)\) of (4.3), then by (4.2) we have

\[
a(h, E_{k, \psi}^{(2)}) = a(h, E_{k, \psi}^{(2)})' \prod_{p: \text{prime}} A_p(h, \psi_p; \overline{\psi}_p^n(p)p^{-k})/A'_p(h, \psi_p; \overline{\psi}_p^n(p)p^{-k}).
\]
By Proposition 4.2, we obtain

\[
A_p(h, \psi_p; \overline{\psi}_p(p) p^{-k}) / A_p'(h, \psi_p; \overline{\psi}_p(p) p^{-k}) = 1 + \psi_p(-1) \frac{(p - 1)p^{-n_p-1}(\overline{\psi}_p(p)^2) p^{3-2k} \nu_p}{1 - \overline{\psi}_p(p)^2 p^{3-2k}} A_p'(h, \psi_p; \overline{\psi}_p(p) p^{-k})^{-1}
\]

\[
= 1 - p(p - 1)(\overline{\psi}_p(p)^2 p^{3-2k})^{-1} \frac{1 - \chi h \overline{\psi}_p(p) p^{1-k}}{(1 - \overline{\psi}_p(p) p^{2-2k})(1 - \chi h \overline{\psi}_p(p) p^{k-2})} = c_p(h, \psi; p^{k-3}).
\]

By the functional equation of the Dirichlet L-function, we have

\[
a(h, E_k^{(2)})' = \xi(y, h; k, 0) e(-iyh)
\]

\[
\times \prod_{v=\infty \text{ or } v=p|N} \frac{\gamma_v(\rho h, v \overline{\omega}_v, k - 1)}{\gamma_v(\overline{\omega}_v, k) \gamma_v(\overline{\omega}_v^2, 2k - 2)} \prod_{p|N} \varepsilon_p(\rho h, p \overline{\omega}_p, k - 1)
\]

\[
\times \frac{L(N)(2 - k, \chi h \psi)}{L(1 - k, \psi) L(N)(3 - 2k, \psi^2)} \prod_{p|N} F_p^{(2)}(\overline{\psi}(p) p^{k-3}) \prod_{p|N} A_p'(h, \psi_p; \overline{\psi}_p(p) p^{-k}).
\]

Here \(\rho h, v\) is the \(v\)-component of the character corresponding to \(\chi h\). By Proposition 4.1 and Proposition 4.2, we have

\[
a(h, E_k^{(2)})' = 2i \det(2h)^{k-3/2} \prod_{p|N} \overline{\omega}_p(\det(2h)) \varepsilon_p(\rho h, p, k - 1) p^{(3-2k)\alpha_p}
\]

\[
\times \prod_{p|N} \varepsilon_p(\rho h, p \overline{\omega}_p, k - 1) \frac{L(N)(2 - k, \chi h \psi)}{L(1 - k, \psi) L(N)(3 - 2k, \psi^2)} \prod_{p|N} F_p^{(2)}(\overline{\psi}(p) p^{k-3}).
\]

If \(p \nmid N\), by Proposition 2.6, \(F_p^{(2)}(h; T)\) has the following functional equation

\[
F_p^{(2)}(h; \overline{\psi}(p) p^{k-3}) = \overline{\omega}_p(p^{2\alpha_p}) p^{(3-2k)\alpha_p} F_p^{(2)}(h; \psi(p) p^{k-3}).
\]

Thus,

\[
a(h, E_k^{(2)})' = 2i \det(2h)^{k-3/2} \prod_{p|N} \overline{\omega}_p(\det(2h)) \varepsilon_p(\rho h, p, k - 1) p^{(3-2k)\alpha_p}
\]

\[
\times \prod_{p|N} \varepsilon_p(\rho h, p \overline{\omega}_p, k - 1) \overline{\omega}_p(p^{2\alpha_p}) p^{(3-2k)\alpha_p}
\]

\[
\times \frac{L(N)(2 - k, \chi h \psi)}{L(1 - k, \psi) L(N)(3 - 2k, \psi^2)} \prod_{p|N} F_p^{(2)}(\psi(p) p^{k-3}).
\]

Put
\[ A = i \det(2h)^{k-3/2} \prod_{p \mid N} \bar{\omega}_p(\det 2h) \varepsilon_p(\rho_{h, p}, k - 1) p^{(3-2k)\alpha_p} \]
\[
\times \prod_{p \mid N} \varepsilon_p(\rho_{h, p} \bar{\omega}_p, k - 1) \bar{\omega}_p(p^{2\alpha_p}) p^{(3-2k)\alpha_p}.
\]

Then
\[
a(h, E_{k, \psi})' = 2A \frac{L(N)(2-k, \chi_h, \psi)}{L(1-k, \psi)L(N)(3-2k, \psi^2)} \prod_{p \mid N} F_p^{(2)}(h; \bar{\psi}(p)p^{-k}).
\]

Note that the following equations hold
\[
\varepsilon_p(\rho_{h, p} \bar{\omega}_p, k - 1) = \bar{\omega}_p(\bar{f}(\rho_{h, p})) \varepsilon_p(\rho_{h, p}, k - 1),
\]
\[
\prod_{p\text{ prime}} \varepsilon_p(\rho_{h, p}, k - 1) = -\bar{f}(\chi_h)^{1-k} G(\chi_h) = -i \bar{f}(\chi_h)^{3/2-k},
\]
\[
\det 2h = \bar{f}(\chi_h) \prod_{p\text{ prime}} p^{2\alpha_p}.
\]

Thus we obtain,
\[
A = i \det(2h)^{k-3/2} \prod_{p\text{ prime}} p^{(3-2k)\alpha_p} \prod_{p\text{ prime}} \varepsilon_p(\rho_{h, p}, k - 1)
\]
\[
\times \prod_{p \mid N} \bar{\omega}_p(\det 2h) \prod_{p \mid N} \bar{\omega}_p(\bar{f}(\rho_{h, p})) \bar{\omega}_p(p^{2\alpha_p})
\]
\[
= i \det(2h)^{k-3/2} \prod_{p\text{ prime}} p^{(3-2k)\alpha_p} (-i \bar{f}(\chi_h)^{3/2-k}) \prod_{p\text{ prime}} \bar{\omega}_p(\det(2h)) = 1.
\]

As a consequence, we have
\[
a(h, E_{k, \psi}') = a(h, E_{k, \psi})' \prod_{p \mid N} c_p(h, \psi; p^{k-3})
\]
\[
= 2 \frac{L(N)(2-k, \chi_h, \psi)}{L(1-k, \psi)L(N)(3-2k, \psi^2)} \prod_{p \mid N} F_p^{(2)}(h; \psi(p)p^{k-3}) \prod_{p \mid N} c_p(h, \psi; p^{k-3}).
\]

5. The \(p\)-adic analytic family of Siegel–Eisenstein series of degree two and a \(p\)-adic Siegel–Eisenstein series of degree two

In this section, applying Proposition 5.2, we prove that there exists the \(p\)-adic analytic family which consists of Siegel–Eisenstein series of degree two and that a \(p\)-adic Siegel–Eisenstein series of degree two is also Siegel modular form. We fix a prime \(p\) and embeddings \(\mathbb{Q} \hookrightarrow \mathbb{C}, \mathbb{Q} \hookrightarrow \mathbb{Q}_p\).
5.1. For expressing a $p$-adic Eisenstein series with an Eisenstein series and the $p$-adic interpolation of Eisenstein series, we need to remove the Euler factor of $a(h, E_{k,\psi}^{(2)})$ at $p$. In this subsection, we remove the Euler factor at $p$ by Hecke operators.

Let $f \in M_k^g(\Gamma_0(N), \psi)$ be a Siegel modular form of degree $g$, weight $k$, level $N$ and character $\psi$. Then $f$ has the following Fourier expansion.

$$f(z) = \sum_{0 \leq h \in \text{Sym}^*_g(\mathbb{Z})} a(h, f)e(hz).$$

We define a Hecke operator $U(p)$ as follows

$$(f \mid U(p))(z) = \sum_{0 \leq h \in \text{Sym}^*_g(\mathbb{Z})} a(ph, f)e(hz).$$

By the definition of $U(p)$, we have

$$f \mid U(p) \in \begin{cases} M_k^g(\Gamma_0(pN), \psi) & \text{if } p \nmid N, \\
M_k^g(\Gamma_0(N), \psi) & \text{if } p \mid N. \end{cases}$$

We define Hecke operators $V(q)$ and $W(p)$ as follows

$$V_{\psi}(q) = \begin{cases} \frac{1 - \overline{\psi}(q)^2q^{3-2k}U(q)}{1 - \overline{\psi}(q)q^{3-2k}} & \text{if } q \neq 2, \\
\frac{U(q)^3 - \overline{\psi}(q)^3q^{3-2k}U(q)^3}{1 - \overline{\psi}(q)^3q^{3-2k}} & \text{if } q = 2, \end{cases}$$

$$W_{\psi}(p) = \frac{(U(p) - \psi(p)p^{k-1})(U(p) - \psi^2(p)p^{2k-3})}{(1 - \psi(p)p^{k-1})(1 - \psi^2(p)p^{2k-3})}.$$ 

**Proposition 5.1.** Let $\psi$ be a primitive Dirichlet character mod $N$ and $0 \leq h \in \text{Sym}^*_g(\mathbb{Z})$ be a half integral positive semi-definite symmetric matrix and suppose $k > 3$. We put

$$E_{k,\psi}^{'} = E_{k,\psi}^{(2)} \prod_{q \nmid N} V_{\psi}(q).$$

We denote $h$-th Fourier coefficient of $E_{k,\psi}^{'}$ by $a(h, E_{k,\psi}^{'})$. Then the following assertions hold:

1. If $\text{rank } h \leq 1,$

$$a(h, E_{k,\psi}^{'}) = a(h, E_{k,\psi}^{(2)}).$$

2. If $\text{rank } h = 2,$

$$a(h, E_{k,\psi}^{'}) = 2 \frac{L^{(N)}(2-k, \chi_h \psi)}{L(1-k, \psi)L^{(N)}(3-2k, \psi^2)} \prod_{q \text{ prime } q \mid N} F_q^{(2)}(h; \psi(q)q^{k-3}).$$
Proof. Let \( q \) be a prime such that \( q \mid N \) and \( \psi_q^2 = 1 \). Put \( N_0 = N/(\psi_q)^{-1} \). Suppose \( \text{rank } h \leq 1 \).

Then \( a(h, E_{k, \psi}^{(2)}) = \frac{1}{2} L(1 - k, \psi) a(\varepsilon(h), E_{k, \psi}^{(1)}) \), where \( E_{k, \psi}^{(1)} \) is Eisenstein series of degree one. Since \( E_{k, \psi}^{(1)} \mid U(q) = E_{k, \psi}^{(1)} \), the assertion (1) holds. Suppose \( \text{rank } h = 2 \). Let \( i \) be an integer such that \( i \geq 0 \) if \( q \neq 2 \), \( i \geq 2 \) if \( q = 2 \). By Theorem 4.1, we have

\[
a(q^ih, E_{k, \psi}^{(2)}) = 2c_q(q^ih, \psi; q^{k-3}) \frac{L(N)(2 - k, \chi_{h\psi})}{L(1 - k, \psi)L(N)(3 - 2k, \psi^2)} \\
\times \prod_{r \mid N} F_r^{(2)}(h; \psi(r)r^{k-3}) \prod_{r \mid N_0, \psi_r = 1} c_r(h, \psi; r^{k-3}),
\]

where \( r \) is a prime. Since \( \beta_q(q^ih) = \beta_q(h) + i \), we see that

\[
c_q(q^ih, \psi; T) - \psi^2(q)q^{-3}T^{-2}c_q(q^{i+1}h, \psi; T) = 1 - \psi^2(q)q^{-3}T^{-2}.
\]

It follows that

\[
a(h, E_{k, \psi}^{(2)} \mid V_\psi(q)) = \frac{a(q^ih, E_{k, \psi}^{(2)}) - \psi^2(q)q^{-2k}a(q^{i+1}h, E_{k, \psi}^{(2)})}{1 - \psi^2(q)q^{-2k}}
\]

\[
= \frac{L(N)(2 - k, \chi_{h\psi})}{L(1 - k, \psi)L(N)(3 - 2k, \psi^2)} \prod_{r \mid N} F_r^{(2)}(h; \psi(r)r^{k-3}) \prod_{r \mid N_0, \psi_r = 1} c_r(h, \psi; r^{k-3}).
\]

Therefore, assertion (2) holds. \( \square \)

**Proposition 5.2.** Let \( N \) be a positive integer and \( \psi \) be a Dirichlet character mod \( N \). Put \( N = N_0 p^r \) with \( p \nmid N_0 \) and \( r \geq 0 \). Suppose that \( \psi_0 \) is primitive for all \( q \mid N_0 \) and \( \psi_p \) is primitive if \( r > 1 \). We define \( G_{k, \psi}^{(2)} \) as follows.

(i) If \( \psi_p \) is primitive, then we define

\[
G_{k, \psi}^{(2)} = \frac{1}{2} L(1 - k, \psi)L(N)(3 - 2k, \psi^2)E'_{k, \psi}.
\]

(ii) If \( \psi_p \) is the trivial character mod \( p \), then we define

\[
G_{k, \psi}^{(2)} = \frac{1}{2} L(1 - k, \psi)L(N)(3 - 2k, \psi^2)E'_{k, \psi} \mid W_\xi(p).
\]

Here \( \xi = \prod_{q \mid N_0} \psi_q \).

Let \( 0 \leq h \in \text{Sym}_2(\mathbb{Z}) \) be a half integral positive semi-definite symmetric matrix and suppose that \( k > 3 \). We denote \( h \)-th Fourier coefficient of \( G_{k, \psi}^{(2)} \) by \( a(h, G_{k, \psi}^{(2)}) \). Then the following assertions hold:

1. If \( \text{rank } h = 0 \),

\[
a(h, G_{k, \psi}^{(2)}) = \frac{1}{2} L(1 - k, \psi)L(N)(3 - 2k, \psi^2).
\]
(2) If rank $h = 1$,
\[ a(h, G_{k, \psi}^{(2)}) = L^{(N)}(3 - 2k, \psi^2) \prod_{q \text{prime}} F_q^{(1)}(\varepsilon(h); \psi(q)q^{k-2}). \]

Here $F_q^{(1)}(m; T)$ is $1 + qT + \cdots + (qT)^{\text{ord}_q(m)}$ and $\varepsilon(h)$ is defined as follows
\[ \varepsilon(h) = \max \{ m \in \mathbb{Z}_{\geq 0} \mid m^{-1}h \in \text{Sym}_2^*(\mathbb{Z}) \}. \]

(3) If rank $h = 2$,
\[ a(h, G_{k, \psi}^{(2)}) = L^{(N)}(2 - k, \chi_h \psi) \prod_{q \text{prime}} F_q^{(2)}(h; \psi(q)q^{k-3}). \]

**Proof.** If $\psi_p$ is primitive, then the assertions follows from Proposition 5.1. Suppose that $\psi_p$ is the trivial character mod $p$. Put
\[ P(X) = \frac{(X - \xi(p)p^{k-1})(X - \xi^2(p)p^{2k-3})}{(1 - \xi(p)p^{k-1})(1 - \xi^2(p)p^{2k-3})}. \]

By definition, $W_\xi(p) = P(U(p))$. If rank $h = 2$, by Proposition 5.1, we have
\[ a(h, E'_{k, \xi}) = 2 \frac{L^{(N_0)}(2 - k, \chi_h \xi)}{L(1 - k, \xi)L^{(N_0)}(3 - 2k, \xi^2)} \prod_{q \text{prime}} F_q^{(2)}(h; \xi(q)q^{k-3}). \]

We denote the Euler factor at $p$ by $a_p(h, E'_{k, \xi}).$ Then
\[ a_p(h, E'_{k, \xi}) = \frac{(1 - \xi(p)p^{k-1})(1 - \xi^2(p)p^{2k-3})}{1 - \chi_h \xi(p)p^{k-2}} \frac{F_p^{(2)}(h; \xi(p)p^{k-3})}{F_p^{(2)}(h; \xi(p)p^{k-3})}. \]

We need to prove $a_p(h, E'_{k, \xi} \mid P(U(p))) = 1$. We put
\[ \varepsilon(h) = \max \{ m \in \mathbb{Z}_{\geq 1} \mid m^{-1}h \in \text{Sym}_2^*(\mathbb{Z}) \}, \]
and
\[ \alpha_1(h) = \text{ord}_p(\varepsilon(h)), \quad \alpha(h) = \frac{1}{2} \text{ord}_p(\det 2h/\xi(\chi_h)). \]

Then by (2.33), $F_p^{(2)}(h; T)$ has the following explicit expression.
\[ F_p^{(2)}(h; T) = \frac{\alpha(h)}{\sum_{i=0}^{\alpha(h)} (p^2 T)^i} \left\{ \sum_{j=0}^{\alpha(h) - i} (p^2 T)^j - \chi_h(p)T \sum_{j=0}^{\alpha(h) - i - 1} (p^2 T)^j \right\}. \]

A simple calculation shows that there exist $c_i(\chi_h; T) \in \mathbb{Q}(T)$ ($i = 0, 1, 2$) that depend only on $p$ and $\chi_h$, and satisfies the following equation:
Here for any finite order character \( \chi \) of \( \mathbb{Z}/p\mathbb{Z} \) with \( \chi(p) \neq 1 \), we have
\[
F_p^{(2)}(h; T) = \frac{1 - \chi(h)(pT)}{(1 - p^3T^2)(1 - pT)} + c_0(\chi(h); T)(p^2T)^{\alpha_1(h)} + c_1(\chi(h); T)(p^3T^2)^{\alpha(h)}.
\]

Put \( P(X) = \sum_i a_i X^i \). Since \( \alpha(p^i) = \alpha(h) + i \) and \( \alpha_1(p^i) = \alpha_1(h) + i \), we have
\[
\sum_i a_i F_p(p^i h; T) = \frac{1 - \chi(h)(pT)}{(1 - p^3T^2)(1 - pT)} P(1)
+ c_0(\chi(h); T) P(p^2T) + c_1(\chi(h); T) P(p^3T^2).
\]

Since \( P(1) = 1 \) and \( P(\xi(p)p^{k-3}) = P(\xi^2(p)p^{2k-3}) = 0 \), we obtain
\[
a_p(h, E_{k, \xi} | P(U(p))) = \frac{(1 - \xi(p)p^{k-1})(1 - \xi^2(p)p^{2k-3})}{1 - \chi(h)(p)p^{k-2}} \sum_i a_i F_p(p^i h; \xi(p)p^{k-3}) = 1.
\]

Therefore, assertion (3) holds. We can prove assertions (1) and (2) in a similar way. \( \square \)

5.2. In this subsection, we prove there exists the \( p \)-adic analytic family which interpolates \( G_{k, \psi}^{(2)} \).

Let us recall some properties of \( p \)-adic Dirichlet \( L \)-functions. Let \( N \) be a positive integer and \( \chi \) be a Dirichlet character mod \( N \). Put \( N = N_0p^r \) with \( r > 0 \) and \( (N_0, p) = 1 \). Then \( \chi \) can be regarded as a character of \( \lim_n (\mathbb{Z}/N_0p^n\mathbb{Z})^\times = (\mathbb{Z}/pN_0\mathbb{Z})^\times \times (1 + p\mathbb{Z}_p) \). Here \( p \) is
\[
p = \begin{cases} p & \text{if } p \neq 2, \\ 4 & \text{if } p = 2. \end{cases} \tag{5.1}
\]

We decompose
\[
\chi = \chi_1\chi_2, \tag{5.2}
\]
where \( \chi_1 \) is the character of \( (\mathbb{Z}/pN_0\mathbb{Z})^\times \) and \( \chi_2 \) is the character of \( (1 + p\mathbb{Z}_p) \).

**Theorem 5.1** *(Kubota, Leopoldt, Iwasawa).* Let \( \chi \) be an even Dirichlet character mod \( N \). Fix a topological generator \( u \) of \( 1 + p\mathbb{Z}_p \). We denote Teichmüller character by \( \omega \). There exists \( \Phi(\chi; T) \in \mathrm{Frac}(\mathbb{Z}_p[\chi][T]) \) which satisfies the following interpolation property
\[
\Phi(\chi; \varepsilon(u)u^k - 1) = L^{(p)}(1 - k, \chi\varepsilon\omega^{-k}),
\]
for any finite order character \( \varepsilon \) of \( 1 + p\mathbb{Z}_p \) and \( k \in \mathbb{Z}_{\geq 1} \). Put
\[
P(\chi; T) = \begin{cases} 1 & \text{if } \chi_1 \neq 1, \\ 1 - \chi_2(u)u(1 + T)^{-1} & \text{if } \chi_1 = 1. \end{cases}
\]

Here \( \chi_1 \) and \( \chi_2 \) are as in (5.2). Then \( \frac{1}{2} P(\chi; T)\Phi(\chi; T) \in \mathbb{Z}_p[\chi][T] \).

By Proposition 5.2 and Theorem 5.1, we can prove the following theorem.
Theorem 5.2. Let \( N \) be a positive integer divisible by \( p \) and \( \psi \) be a Dirichlet character mod \( N \). Put \( N = N_0p^r \) with \( p \mid N_0 \) and \( r \geq 1 \). Suppose that \( \psi_q \) is primitive for all \( q \mid N_0 \) and \( \psi_p \) is primitive if \( r > 1 \). For half integral positive semi-definite symmetric matrix \( h \in \text{Sym}_2(\mathbb{Z}) \), there exists \( a(h, \psi; T) \in \text{Frac}(\mathbb{Z}_p[\psi][T]) \) which satisfies the following interpolation property

\[
a(h, \psi; \varepsilon(u)u^k - 1) = a(h, \mathcal{G}_{k, \varepsilon}^{(2)}),
\]

for any finite order character \( \varepsilon \) of \( 1 + p\mathbb{Z}_p \) and \( k \in \mathbb{Z} \) with \( k > 3 \). Put

\[
Q(\psi; T) = P(\psi; T)P(\psi^2 \omega^{-2}; u^{-2}(1 + T)^2 - 1)P'(\psi; u^{-1}(1 + T) - 1),
\]

where \( P(\psi; T) \) is as in Theorem 5.1 and \( P'(\psi; T) \) is

\[
P'(\psi; T) = \begin{cases} 
1 & \text{if } \psi_1^2 \neq \omega^2, \\
1 - \psi_2(u)u(1 + T)^{-1} & \text{if } \psi_1^2 = \omega^2 \text{ and } p \neq 2, \\
(1 - \psi_2(u)u(1 + T)^{-1})(1 + \psi_2(u)u(1 + T)^{-1}) & \text{if } \psi_1^2 = \omega^2 \text{ and } p = 2.
\end{cases}
\]

Here \( \psi_1 \) and \( \psi_2 \) are as in (5.2). Then \( Q(\psi; T)a(h, \psi; T) \in \mathbb{Z}_p[\psi][T] \).

Proof. For a Dirichlet character \( \chi \) mod \( M \) and a positive integer \( N \), we define \( \Phi^{(N)}(\chi; T) \) as follows

\[
\Phi^{(N)}(\chi; T) = \prod_{q \mid N, q \neq p} (1 - \chi(q)q^{-1}(1 + T)^{\log_p(q)}) \Phi(\chi; T).
\]

Here \( (q) = q\omega^{-1}(q) \) and \( \log_p(q) \) is the unique \( p \)-adic integer such that \( u^{\log_p(q)} = (q) \). By Theorem 5.1, we have

\[
\Phi^{(N)}(\chi; \varepsilon(u)u^k - 1) = L^{(Np)}(1 - k, \varepsilon \psi \omega^{-k}).
\]  

(5.3)

for any finite order character \( \varepsilon \) of \( 1 + p\mathbb{Z}_p \) and \( k \in \mathbb{Z}_{\geq 1} \).

We define \( a(h, \psi; T) \) as follows.

If \( \text{rank } h = 0 \), then

\[
a(h, \psi; T) = \frac{1}{2} \Phi(\psi; T) \Phi^{(N)}(\psi^2 \omega^{-2}; u^{-2}(1 + T)^2 - 1).
\]

If \( \text{rank } h = 1 \), then

\[
a(h, \psi; T) = \Phi^{(N)}(\psi^2 \omega^{-2}; u^{-2}(1 + T)^2 - 1) \prod_{q \text{ prime } q \mid Np} F_q^{(1)}(\varepsilon(h); \psi(q)q^{-2}(1 + T)^{\log_p(q)}).
\]

If \( \text{rank } h = 2 \), then

\[
a(h, \psi; T) = \Phi^{(N)}(\chi_h \psi \omega^{-1}; u^{-1}(1 + T) - 1) \prod_{q \text{ prime } q \mid Np} F_q^{(2)}(h; \psi(q)q^{-3}(1 + T)^{\log_p(q)}).
\]

By Proposition 5.2 and (5.3), we see that \( a(h, \psi; T) \) has the interpolation property stated in the theorem. The last assertion of the theorem follows from Theorem 5.1. \( \square \)
5.3. In this subsection, we prove that a $p$-adic Siegel–Eisenstein series of degree two is also Siegel–Eisenstein series of degree two.

We define $X$ and $X_\psi$ by

$$X = \mathbb{Z}_p \times \mathbb{Z}/\phi(p)\mathbb{Z} \cong \text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times),$$

(5.4)

$$X_\psi = \{(s, a) \in X \mid (-1)^a = \psi(-1)\}.$$  

(5.5)

Here $p$ is as in (5.1), $\phi$ is Euler's phi function and $\text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ is the set of continuous group homomorphisms from $\mathbb{Z}_p^\times$ to $\mathbb{Z}_p^\times$. $X$ is equipped with the $p$-adic topology. We embed $\mathbb{Z}$ in $X$ by $\mathbb{Z} \ni m \mapsto (m, m \mod \phi(p)) \in X$. For $v \in \mathbb{Z}_p^\times$ and $x = (s, a) \in X$, we put $v^x = \omega(v)^a (v)^x$. Then the isomorphism $X \cong \text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$ is given by $X \ni x \mapsto (v \mapsto v^x) \in \text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$. Let

$$\mathbb{C}_p[\mathfrak{a}] = \left\{ f = \sum_{0 \leq h \in \text{Sym}_k^\times(\mathbb{Z})} a(h, f) e(hz) \mid a(h) \in \mathbb{C}_p \right\},$$

be the space of formal Fourier expansions, where $\mathbb{C}_p$ is the completion of $\overline{\mathbb{Q}}_p$. We put $|f|_p = \sup_{0 \leq h \in \text{Sym}_k^\times(\mathbb{Z})} |a(h, f)|_p$.

**Theorem 5.3.** Let $N$ be a positive integer such that $p \nmid N$ and $\psi$ be a primitive Dirichlet character mod $N$. Suppose $(k, a) \in X_\psi$ and let $k$ be an integer such that $k > 3$. For any sequence $\{l_m\}_m \subset X_\psi$ such that $l_m \rightarrow \infty$, $\lim_{m \rightarrow \infty} l_m = +\infty$ in usual topology of $\mathbb{R}$ and $\lim_{m \rightarrow \infty} l_m = (k, a) \in X_\psi$, we have

$$\lim_{m \rightarrow \infty} \left| G_{lm, \psi}^{(2)} - G_{k, \psi, \psi, \omega^a - k}^{(2)} \right|_p = 0.$$

Thus, the $p$-adic Siegel–Eisenstein series of level $N$, weight $k$, character $\psi$ and degree two is Siegel–Eisenstein series of level $N \mathfrak{p}$, weight $k$, character $\psi \cdot \omega^a - k$ and degree two.

**Proof.** Put $X_e = \{(s, a) \in X \mid a \text{ is even}\}$. For an even Dirichlet character $\chi$ and $x = (s, a) \in X_e$, we put

$$A(h, \chi, x) = a(h, \chi \omega^a; u^s - 1).$$

Here $a(h, \chi \omega^a; T)$ is as in Theorem 5.2. By definition, $A(h, \chi, x)$ is a $p$-adic analytic function on $X_e$. We put

$$u(l, \psi) = \left\{ \begin{array}{ll} (1 - \psi(p)p^{l-1})^{-1}(1 - \psi^2(p)p^{2l-3})^{-1} & \text{if rank } h = 0, \\ (1 - \psi^2(p)p^{2l-3})^{-1}F_p^{(1)}(\chi(h); \psi(p)p^{l-2}) & \text{if rank } h = 1, \\ (1 - \chi_h \psi(p)p^{l-2})^{-1}F_p^{(2)}(\psi(p)p^{l-3}) & \text{if rank } h = 2. \end{array} \right.$$  

Proportion 2.5 and (2.32) imply that $\lim_{l \rightarrow +\infty} u(l, \psi) \rightarrow 1$ and $u(l, \psi)$ is a $p$-adic unit if $l > 3$. By Theorem 5.2, we have

$$a(h, G_{lm, \psi}^{(2)}) = u(l_m, \psi)A(h, \psi, l_m), \quad a(h, G_{k, \psi, \omega^a - k}^{(2)}) = A(h, \psi, (a, k)).$$

By Theorem 5.2, $Q(\psi \omega^a; u^s - 1)A(h, \psi, (a, s))$ is an Iwasawa function on $\mathbb{Z}_p$, which means that there exists $\Psi(T) \in \mathbb{Z}_p[\psi][T]$ such that

$$\Psi(u^s - 1) = Q(\psi \omega^a; u^s - 1)A(h, \psi, (s, a)).$$
Since $\Psi(u^{s_1} - 1) \equiv \Psi(u^{s_2} - 1) \mod p(s_1 - s_2)$ for $s_1, s_2 \in \mathbb{Z}_p$, we easily see that $a(h, G^{(2)}_{km, \psi})$ is convergent to $a(h, G^{(2)}_{km, \psi, a-k})$ uniformly on $h$. □

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References