The Lasker–Noether theorem for commutative and noetherian module algebras over a pointed Hopf algebra

Andrzej Tyc* 1 and Piotr Wiśniewski

Faculty of Mathematics and Computer Science, N. Copernicus University in Toruń, ul. Chopina 12/18, 87-100 Toruń, Poland

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Abstract

Let $H$ be a pointed Hopf algebra over a field, let $A$ be a commutative noetherian $H$-module algebra, and let $I$ be an invariant ideal in $A$ such that $g(P) \subset P$ for any group-like element $g \in H$ and any associated prime $P \in \text{Ass}(I)$. We prove that $I$ admits an irredundant primary decomposition $I = Q_1 \cap \cdots \cap Q_n$ such that each $Q_i$ is invariant. Moreover, we introduce the concept of a convolutionally Hopf algebra and show that each associated prime of the ideal $I$ is invariant, provided the Hopf algebra $H$ is convolutionally reduced. Also it will be proved that in characteristic 0 every connected Hopf algebra is convolutionally reduced.

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1. Introduction

Let $K$ be a field and let $A$ be a fixed commutative, noetherian $K$-algebra. It is well known that each ideal $I$ in $A$ possesses an irredundant primary decomposition, that is, there are primary ideals $Q_1, \ldots, Q_n$ such that $I = Q_1 \cap \cdots \cap Q_n$, $Q_j \not\subset \bigcap_{i \neq j} Q_i$ for all $j$, $1 \leq j \leq n$.  

* Corresponding author.

E-mail addresses: atyc@mat.uni.torun.pl (A. Tyc), pikonrad@mat.uni.torun.pl (P. Wiśniewski).

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and the associated prime ideals \( P_1 = \sqrt{Q_1}, \ldots, P_n = \sqrt{Q_n} \) are different. In general, such a decomposition is not unique, but the prime ideals \( P_1, \ldots, P_n \) are uniquely defined by \( I \).

As usual we write \( \text{Ass}(I) = \{P_1, \ldots, P_n\} \).

Now suppose that \( H \) is a Hopf algebra over \( K \) and that an action \( H \otimes A \to A, h \otimes a \mapsto h(a) \), of \( H \) on the algebra \( A \) is given; in other words, \( A \) together with the map \( H \otimes A \to A \) is an \( H \)-module algebra \([S, M]\). An ideal \( I \) in \( A \) is called invariant if \( h(a) \in I \) for all \( h \in H \) and \( a \in I \). Let \( I \) be an invariant ideal in \( A \). Then the following natural questions arise.

**Question 1.** Does there exist an irredundant primary decomposition \( I = Q_1 \cap \cdots \cap Q_n \) such that \( Q_1, \ldots, Q_n \) are invariant ideals in \( A \)?

**Question 2.** Is every prime ideal \( P \in \text{Ass}(I) \) invariant?

The main objective of this paper is to investigate these questions.

Let \( G \) denote the set of all group-like elements in \( H \), i.e., \( G = \{g \in H; \Delta(g) = g \otimes g\} \). Observe that if Questions 1, 2 admit positive answers, then clearly \( g(P) \subset P \) for any \( g \in G \) and \( P \in \text{Ass}(I) \). However, the latter condition need not be always satisfied, so that, in general, the answer to both questions is negative.

**Example 1.** Let \( A = K[x, y] \) and let \( g : A \to A \) be the automorphism of \( A \) determined by \( g(x) = y, g(y) = x \). Moreover, let \( H \) be the group algebra of the group \([\text{Id}, g] \subset \text{Aut}(A)\). Obviously, \( g \) makes \( A \) an \( H \)-module algebra, and \( (xy) \) is an invariant ideal in \( A \) with \( \text{Ass}(xy) = \{(x), (y)\} \). But neither \( (x) \) nor \( (y) \) are preserved by the group-like element \( g \) (notice that \( (xy) = (x) \cap (y) \) is the unique irredundant primary decomposition of the ideal \( (xy) \)).

Therefore, the hypothesis that \( g(P) \subset P \) for \( g \in G \) and \( P \in \text{Ass}(I) \) will be a natural assumption in the main theorems concerning Questions 1, 2.

To present our results, let us recall that the Hopf algebra \( H \) is said to be connected (respectively pointed) if \( K1_H \) is the unique simple subcoalgebra of \( H \) (respectively if each simple subcoalgebra of \( H \) is one-dimensional). Also recall that if \( C \) is a coalgebra (over \( K \)), then for any \( K \)-algebra \( B \) we have the convolution algebra \( \text{Hom}(C, B) \)[S, M]. We say that a \( K \)-algebra \( D \) is reduced if it has no nonzero nilpotent elements. The following concept plays an important role in the paper.

**Definition.** The Hopf algebra \( H \) is called convolutionally reduced if there exists a subcoalgebra \( C \) in \( H \) such that \( C \), as a set, generates \( H \) as an algebra, and the convolution algebra \( \text{Hom}(C, B) \) is reduced for any commutative and reduced algebra \( B \).

The main results of the paper are the following.

**Theorem 3.** Suppose the Hopf algebra \( H \) is pointed and \( I \) is an invariant ideal in \( A \) with \( g(P) \subset P \) for \( g \in G \) and \( P \in \text{Ass}(I) \). Then there exists an irredundant primary decomposition \( I = Q_1 \cap \cdots \cap Q_n \) such that the ideals \( Q_1, \ldots, Q_n \) are invariant.
Theorem 4. Assume $H$ is pointed and convolutionally reduced. If $I$ is an invariant ideal in $A$ such that $g(P) \subset P$ for all $g \in G$, $P \in \text{Ass}(I)$, then each $P \in \text{Ass}(I)$ is invariant.

Below Theorems 1 and 2 will be proved in a more general context, for the so called $(H, A)$-modules.

Notice that the hypothesis “$g(P) \subset P$ for any $g \in G$ and $P \in \text{Ass}(I)$” trivially holds, provided the Hopf algebra $H$ is connected.

Theorem 5. If $\text{char}(K) = 0$, then every connected Hopf algebra is convolutionally reduced.

If $H = K[t]$ with $t$ primitive, then Theorems 1, 2 say that if $d$ is a derivation of the algebra $A$ and $I$ is a $d$-invariant ideal in $A$, then there exists an irredundant primary decomposition $I = Q_1 \cap \cdots \cap Q_s$ such that the ideals $Q_1, \ldots, Q_s$ are $d$-invariant. Moreover, each $P \in \text{Ass}(I)$ is $d$-invariant, provided $\text{char}(K) = 0$. This is nothing else than the well-known Seidenberg result [Se, Theorem 1] (the fact that in characteristic 0 the minimal primes associated with any $d$-invariant ideal are $d$-invariant is also proved in the Dixmier book [D, Lemma 3.3.3]). When we fix an $n \in \mathbb{N} \cup \{\infty\}$ and apply Theorems 1–3 to the connected Hopf algebra $H(n) = K \langle t_0 = 1, t_1, \ldots, t_n \rangle$ with $\Delta(t_m) = \sum_{i+j=m} t_i \otimes t_j$, $\epsilon(t_m) = \delta_m$, and $S(t_m) = -\sum_{i=1}^m t_i S(t_{m-i})$ for $m > 0$, then we get (for algebras over a field) the known results [Br, Theorem 1] and [Sa, Theorem and Proposition 1], which are the natural generalization of the Seidenberg result for higher derivation of degree $n$. In fact, the mentioned results were the principal motivation for Theorems 1, 2. Now assume that the field $K$ is finite and $P^*$ is the Steenrod algebra over $K$ [Mi,Sm]. L. Smith and M. D. Neusel proved in [NS, Theorem 3.5] that if $A$ is a connected graded and unstable algebra over $P^*$ and $I$ is a $P^*$-invariant ideal in $A$, then each $P \in \text{Ass}(I)$ is $P^*$-invariant, and $I$ admits an irredundant primary decomposition consisting of $P^*$-invariant ideals. This theorem is a consequence of our Theorems 1, 2 applied to the (connected) Steenrod Hopf algebra $H = P^*$ with $\Delta(P^n) = \sum_{i+j=n} P^i \otimes P^j$ for the Steenrod operations $P^n$, because it is easily seen that $H$ is convolutionally reduced. Also the main results of [Ne] can be deduced from the analog of Theorems 1, 2 for the $(H, A)$-modules.

Example 2 below shows that in positive characteristic Theorem 3 is no longer true.

Example 2. Let $\text{char} K = p > 0$ and let $H = K[t]$ with $t$ primitive. Moreover, let $A = K[X]/(X^p)$, and let $x = X + (X^p)$. Then the derivation $\partial/\partial X$ makes $A$ an $H$-module algebra such that $(0)$ is an $(x)$-primary invariant ideal in $A$, but $(x)$ itself is not invariant. In particular, this means that the (connected) Hopf algebra $H$ is not convolutionally reduced. It also shows that the assertion of Theorem 2 need not to hold when we remove the assumption that $H$ is convolutionally reduced.

In the last part of the paper, for a given action of $H$ on the algebra $A$ and an invariant ideal $I$ in $A$ we find a sufficient condition for every $P \in \text{Ass}(I)$ to be invariant. To formulate this condition, we need the following.
Definition. Let $Q$ be a $P$-primary invariant ideal in $A$ and let $s$ be the natural number such that $P^{s+1} \subset Q$ and $P^s \not\subset Q$. We say that $Q$ satisfies condition $(\star)$ if for each $g \in G$ there exists an $a \in P$ such that
\[
\sum_{i=0}^s g(a)^i a^{s-i} \not\in Q.
\]

Theorem 6. Assume $H$ is pointed and $I$ is an invariant ideal in $A$ with an irredundant primary decomposition $I = Q_1 \cap \cdots \cap Q_n$ such that $Q_i$ is invariant and satisfies condition $(\star)$ for all $i = 1, \ldots, n$. If $\text{char } K = 0$ and $g(P) \subset P$ for all $g \in G$, $P \in \text{Ass}(I)$, then each $P \in \text{Ass}(I)$ is invariant.

Example 3. Let $H = K \langle t, g \rangle / (g^2 - 1)$, $\Delta(g) = g \otimes g$, $\Delta(t) = t \otimes 1 + g \otimes t$, $\epsilon(g) = 1$, $\epsilon(t) = 0$, $S(g) = g$, $S(t) = -gt$. Let $A = K[X]/(X^2)$, and let $x = X + (X^2)$. Then the formulas $g(x) = -x$ and $t(x) = 1$ determine an action of $I$ on $A$. Observe that $(0)$ is an invariant $(x)$-primary ideal in $A$, but $(x)$ itself is not invariant. According to Theorem 4, this implies that $(0)$ does not satisfy condition $(\star)$.

Let $\sigma : A \to A$ be an algebra automorphism. Recall that a $\sigma$-derivation of $A$ is a linear map $d : A \to A$ such that $d(ab) = d(a)b + \sigma(a)d(b)$ for all $a, b \in A$.

As a consequence of Theorem 3 we obtain the following.

Theorem 7. Assume that $\text{char } K = 0$ and that $\sigma : A \to A$ is an automorphism of $A$ satisfying the conditions:

(a) $A = \bigoplus_{\tau \in T} A_{\tau}$, where $T \subset K$ is the set of eigenvalues of the automorphism $\sigma$ and $A_{\tau} = \{a \in A ; \sigma(a) = \tau a\}$.

(b) The set $T$ of eigenvalues of $\sigma$ is $\mathbb{N}$-independent, that is, if $t_1, \ldots, t_s \in T$, $n_1, \ldots, n_s \in \mathbb{N}$, and $n_1 t_1 + \cdots + n_s t_s = 0$, then $n_1 = \cdots = n_s = 0$.

Furthermore, let $d : A \to A$ be a $\sigma$-derivation of $A$ and let $I$ be an ideal of $A$ such that $d(I) \subset I$ and $\sigma(P) = P$ for each $P \in \text{Ass}(I)$. Then $d(P) \subset P$ for each $P \in \text{Ass}(I)$.

The proofs of Theorems 1, 2 heavily depend on essential properties of the pointed Hopf algebras. The authors know nothing about Questions 1, 2 for an arbitrary Hopf algebra $H$.

The content of the paper can be summarized as follows. Preliminaries are presented in Section 2. In Section 3 it is shown that there are interesting classes of commutative, noetherian module algebras over pointed Hopf algebras. In Section 4 we recall the definition of an $(H, A)$-module and prove a generalization of Theorem 1 for submodules of those $(H, A)$-modules which are finitely generated as $A$-modules. Finally, in Section 5 the concept of a convolutionally reduced Hopf algebra is introduced and Theorems 2–5 are proved.
2. Preliminaries

Throughout the paper $K$ denotes a fixed field which will serve as the ground field for all vector spaces, algebras, Lie algebras, coalgebras, bialgebras, and Hopf algebras under consideration. All tensor products are defined (unless otherwise stated) over $K$. For each set $B$ we denote by $KB$ the vector space with $B$ as a basis. Given vector spaces $V$ and $W$, $\text{Hom}(V, W)$ stands for the vector space $\text{Hom}_K(V, W)$, and $T(V)$ (respectively $S(V)$) stands for the tensor algebra of $V$ (respectively for the symmetric algebra of $V$). If $n \in \mathbb{N} \cup \{\infty\}$ and $\{v_1, v_2, \ldots, v_n\}$ are arbitrary symbols, then $K\langle v_1, v_2, \ldots, v_n \rangle$ will denote the free algebra generated by these symbols. By a ring we mean a ring with unity, and by a module over a ring $A$ we mean a left $A$-module. As usual $\mathbb{Z}$ is the ring of integers and $\mathbb{N}$ is the set of all non-negative integers. If $A$ is an algebra and $n \in \mathbb{N} \cup \{\infty\}$, then a higher derivation of degree $n$ is a sequence of linear maps $D = \{D_i: A \to A; \ i = 0, 1, \ldots, n\}$ such that $D_0 = \text{Id}$ and

$$D_m(xy) = \sum_{i+j=m} D_i(x)D_j(y)$$

for all $m \leq n$ and $x, y \in A$. Higher derivations of degree $\infty$ are called the Hasse–Schmidt derivations. Obviously, a higher derivation of degree 1 is nothing else than an ordinary derivation. Given an automorphism $\sigma$ of an algebra $A$, a $\sigma$-derivation of $A$ is meant to be a linear map $d: A \to A$ such that $d(xy) = d(x)y + \sigma(x)d(y)$ for all $x, y \in A$.

Now we recall some facts on coalgebras, bialgebras, and Hopf algebras that will be needed in the sequel. They come from the books [S,M].

If $C$ is a coalgebra, then $C_0$ denotes its coradical, i.e., the sum of all simple subcoalgebras in $C$. The coalgebra $C$ is called connected if $C_0$ is one-dimensional. If every simple subcoalgebra is one-dimensional, then the coalgebra $C$ is called pointed. In this case $C_0 = KG(C)$, where $G(C) = \{g \in C; \Delta(g) = g \otimes g\}$ (the set of group-like elements in $C$). For $c \in C$ we use the following notation: $\sum c_1 \otimes c_2 = \Delta(c)$, and inductively $\sum c_1 \otimes c_2 \otimes \cdots \otimes c_{n+1} = \sum c_1 \otimes \cdots \otimes c_{n-1} \otimes \Delta(c_n)$.

If $C_1, \ldots, C_n$ are coalgebras, then the vector space $\bigotimes_{i=1}^n C_i$, with

$$\Delta(c_1 \otimes \cdots \otimes c_n) = \sum (c_{1_1} \otimes \cdots \otimes c_{n_1}) \otimes (c_{1_2} \otimes \cdots \otimes c_{n_2}),$$

where $\Delta(c_i) = \sum c_{i_1} \otimes c_{i_2}$ for $c_i \in C_i$ and $\epsilon(c_1 \otimes \cdots \otimes c_n) = \prod_{i=1}^n \epsilon(c_i)$, is a coalgebra called the tensor product of coalgebras $C_1, \ldots, C_n$. In particular, for a given coalgebra $C$ and $n \in \mathbb{N}$ we have the coalgebra $C^{\otimes n} = C \otimes \cdots \otimes C$ ($n$-times).

Below we will frequently use the following well-known properties of the pointed coalgebras.

**Theorem 2.1.** Let $C$ be a pointed coalgebra and let $C_i$, $i = 1, 2, \ldots$, be arbitrary coalgebras.

(a) If $f: C \to D$ is a surjective homomorphism of coalgebras, then $D_0 = f(C_0)$ and the coalgebra $D$ is also pointed. In particular, if $C$ is connected, then so is $D$. 


(b) The coradical of the coalgebra $D = \bigoplus_{i \geq 1} C_i$ equals $\bigoplus_{i \geq 1} C_0$, where $C_0$ is the coradical of $C_i$. In particular, if the coalgebras $C_i$ are pointed, then the coalgebra $D$ is also pointed and $G(D) = \bigcup_{i=1}^n G(C_i)$.

(c) If the coalgebras $C_1, \ldots, C_n$ are pointed, then the coalgebra $E = \bigotimes_{i=1}^n C_i$ is pointed and $G(E) = \bigotimes_{i=1}^n G(C_i)$. In particular, if the $C_i$ are connected for $i = 1, \ldots, n$, then $E$ is connected.

**Proof.** Properties (a) and (c) follow from [M, 5.3.5 and 5.1.10], respectively. As for (b), if $D'$ is a simple subcoalgebra of $D$, then $D'$ is finitely dimensional, and hence $D' \subset C_i$ for some $i$, by [M, 5.6.2]. This means that $D_0 \subset \bigoplus_{i=1}^n C_0$. Since obviously $\bigoplus_{i=1}^n C_0 \subset D_0$, we are done. The theorem is proved. □

A bialgebra $B$ is called connected (respectively pointed) if $B$ is connected (respectively pointed) as a coalgebra. If $B$ is a pointed bialgebra, then $G(B)$ with the multiplication from $B$ is a monoid. By [M, 5.2.10], a pointed bialgebra $B$ is a Hopf algebra if and only if the monoid $G(B)$ is a group. An element $x$ of a bialgebra $B$ is said to be primitive if $\Delta(x) = 1 \otimes x + x \otimes 1$.

Now let $C$ be a coalgebra and let $T(C)$ be the tensor algebra on $C$ (as a vector space). Then $T(C)$ together with the algebra homomorphisms $\Delta: T(C) \to T(C) \otimes T(C)$ and $\varepsilon: T(C) \to K$ determined by the linear maps $\Delta: C \to C \otimes C \subset T(C) \otimes T(C)$ and $\varepsilon: C \to K$ is a bialgebra. As a coalgebra $T(C)$ is the direct sum of the coalgebras $C^\otimes n$, $n \geq 0$, where $C_0 = K$. Similarly, we have the symmetric bialgebra $S(C) = \bigoplus_{i \geq 0} S^i(C)$.

Now assume that $C$ is pointed with $G = G(C)$, and that $c_0$ is a fixed group-like element in $C$. Moreover, let $G' = G - c_0$. Then $K G'$ is a subcoalgebra in $C$, so that we have the coalgebra $C' = C \oplus K G'$. Consider the bialgebras $T(C, c_0) = T(C')/(c_0 - 1)$ and $S(C, c_0) = S(C')/(c_0 - 1)$. It is easy to see that the ideal $I$ in the algebra $T(C, c_0)$ (respectively in $S(C, c_0)$) generated by the set $\{(g, 0)(0, g) - 1, (0, g)(g, 0) - 1; g \in G'\}$ is a biideal. Let, by definition,

$$H(C, c_0) = T(C', c_0)/I, \quad HS(C, c_0) = S(C, c_0)/I.$$  

Then we have

**Theorem 2.2.** (1) $H = H(C, c_0)$ is a pointed Hopf algebra and $G(H)$ is the free group generated by the set $G'$. In particular, if $C$ is connected, then $H = T(C)/(c_0 - 1)$ is connected, too.

(2) $HS(C, c_0)$ is a pointed Hopf algebra and $G(HS(C, c_0))$ is the free Abelian group generated by the set $G'$. In particular, if $C$ is connected, then $HS(C, c_0) = S(C)/(c_0 - 1)$ is connected.

**Proof.** In view of Theorem 2.1(a), it suffices to prove part (1). By Theorem 2.1(b,c), $T(C, c_0)$ is a pointed bialgebra and $G(T(C, c_0))$ is the free monoid generated by the set $G'$. The conclusion now follows, by Theorem 2.1(a). □
Example 2.3. Let \( n \in \mathbb{N} \cup \{\infty\} \) and let \( H(n) = K \langle t_0 = 1, t_1, \ldots, t_n \rangle \) with \( \Delta(t_m) = \sum_{i+j=m} t_i \otimes t_j \), \( \varepsilon(t_i) = \delta_{i0} \), and antipode \( S \) defined inductively by \( S(t_0) = t_0 \), \( S(t_m) = -\sum_{m=0}^{m-1} S(t_i) t_{m-i} \). By Theorem 2.2, \( H(n) \) is a connected Hopf algebra. Notice that \( H(1) = K[t] \), where \( t \) is primitive.

Example 2.4. Let \( H = K \langle g, g^{-1}, t \rangle \) defined as \( K \langle g, g', t \rangle / \langle gg' - 1, g'g - 1 \rangle \) with

\[
\Delta(g) = g \otimes g, \quad S(g) = g', \quad \varepsilon(g) = 1,
\]

\[
\Delta(g') = g' \otimes g', \quad S(g') = g, \quad \varepsilon(g') = 1,
\]

\[
\Delta(t) = t \otimes 1 + g \otimes t, \quad S(t) = -g't, \quad \varepsilon(t) = 0.
\]

Again by Theorem 2.2, \( H \) is a pointed Hopf algebra and \( G(H) \) is the free group generated by \( g \).

Example 2.5. The Taft Hopf algebra \( T(n) = K \langle g, t \rangle / \langle g^n - 1, t^n, tg - \zeta gt \rangle \), where \( \zeta \) is a primitive root of unity of degree \( n \) and

\[
\Delta(g) = g \otimes g, \quad S(g) = g^{n-1}, \quad \varepsilon(g) = 1,
\]

\[
\Delta(t) = t \otimes 1 + g \otimes t, \quad S(t) = -g^{n-1}t, \quad \varepsilon(t) = 0.
\]

[Ta] is a pointed Hopf algebra with \( G(H) = \{1, g, \ldots, g^{n-1}\} \). Note that \( T(n) = K \langle g, g^{-1}, t \rangle / \langle g^n - 1, t^n, tg - \zeta gt \rangle \).

Example 2.6. Let \( H = U(L) \) be the universal enveloping Hopf algebra of a Lie algebra \( L \). Then \( H \) is connected, because \( H \) is a homomorphic image of \( T(C)/(c_0 - 1) \), where \( C = K \oplus L \) with all \( x \in L \) primitive and \( c_0 = (1, 0) \). Similarly, one shows that \( H = U_q(sl(2, K)) \), the quantum enveloping Hopf algebra of the Lie algebra \( sl(2, K) \) (see [K]), is pointed and \( G(H) \) is the free group generated by \( k \).

An important tool we shall need below is the coradical filtration \( \{C_i; \ i \geq 0\} \) of a given coalgebra \( C \). Recall that \( C_0 \) is the coradical of \( C \) and inductively \( C_n = \Delta^{-1}(C \otimes C_{n-1} + C_0 \otimes C) \) for \( n > 0 \). In view of [M, 5.2.2 and 5.4.1], the following holds.

**Theorem 2.7.** Let \( C \) be a coalgebra, and let \( \{C_i\} \) be the coradical filtration of \( C \).

1. Each \( C_i \) is a subcoalgebra, \( C_0 \subset C_1 \subset \cdots \), and \( C = \bigcup_{i \geq 0} C_i \).
2. If \( C \) is pointed with \( G = G(C) \), then for any \( n > 0 \) and \( c \in C_n \)

\[
c = \sum_{f, g \in G} c_{f, g}, \quad \text{where} \quad \Delta(c_{f, g}) = f \otimes c_{f, g} + c_{f, g} \otimes g + w_{f, g}
\]
for some \( w_{f,g} \in C_{n-1} \otimes C_{n-1} \). In particular, if \( C \) is connected, then
\[
\Delta(c) = 1 \otimes c + c \otimes 1 + w \quad \text{for some} \ w \in C_{n-1} \otimes C_{n-1},
\]
where 1 is the unique group-like element in \( C \).

**Lemma 2.8.** Let \( C \) be a finite dimensional connected coalgebra. Then there exists a basis \( c_0, c_1, \ldots, c_n \) of \( C \) such that
\[
\Delta(c_0) = c_0 \otimes c_0 \quad \text{and} \quad \Delta(c_i) = c_i \otimes c_0 + c_0 \otimes c_i + \sum_{j,m=0}^{i-1} \alpha_{ijm}c_j \otimes c_m
\]
for some \( \alpha_{ijm} \in K, \ i = 1, \ldots, n \).

**Proof.** Let \( c_0 \) be the unique group-like element in \( C \) and let \( C_0 \subset C_1 \subset \cdots \subset C_k = C \) be the coradical filtration in \( C \). Moreover, let \( d_i = \dim C_i - 1 \). Then clearly there exists a basis \( c_0, \ldots, c_n \) of \( C \) such that \( c_0, \ldots, c_{d_i} \) is a basis of \( C_i \) for all \( i = 0, \ldots, n \). By Theorem 2.7(2), this basis satisfies the required condition. The lemma follows. \( \square \)

Finally recall [S,M] that if \( C \) is a coalgebra and \( A \) is an algebra, then the vector space \( \text{Hom}(C, A) \) together with the convolution product “\( \ast \)" given by \( (f \ast g)(c) = \sum f(c_1)g(c_2) \) is an algebra. This algebra is called the convolution algebra. If \( F : C \to D \) is a homomorphism of coalgebras, then \( F^* : \text{Hom}(D, A) \to \text{Hom}(C, A), F^*(f) = f \circ F, \) is a homomorphism of algebras, and similarly if \( F' : A \to B \) is a homomorphism of algebras, then \( F'_* : \text{Hom}(C, A) \to \text{Hom}(C, B), F'_*(g) = F' \circ g, \) is a homomorphism of algebras.

### 3. Commutative noetherian module algebras over pointed Hopf algebras

**Definition 3.1** [S,M]. Let \( A \) be an algebra. We say that a Hopf algebra \( H \) measures \( A \) to \( A \) if a linear map \( H \otimes A \to A, \ (h, a) \mapsto h(a), \) is given such that \( h(ab) = \sum h_1(a)h_2(b) \) and \( h(1_A) = \epsilon(h)1_A \) for all \( a, b \in A \) and \( h \in H \) (the map \( H \otimes A \to A \) is then called a measuring).

If \( H \otimes A \to A \) is a measuring, then \( \Psi : A \to \text{Hom}(H, A) \) will denote the map given by \( \Psi(a)(h) = h(a) \). It is easily seen that \( \Psi \) is a homomorphism of algebras and that every homomorphism of algebras \( \Psi : A \to \text{Hom}(H, A) \) is of this form. So, a measuring is nothing else than a homomorphism of algebras \( A \to \text{Hom}(H, A) \). If \( H \) measures an algebra \( A \) to \( A \) and \( h \in H \), then \( h : A \to A \) will denote the map \( a \mapsto h(a) \). If \( h \) is a group-like element, then clearly \( h : A \to A \) is an automorphism of algebras, and if \( h \) is primitive, then \( h : A \to A \) is a derivation of \( A \).

Now recall that an action of a Hopf algebra \( H \) on an algebra \( A \) is a measuring \( H \otimes A \to A \) which makes \( A \), as a vector space, an \( H \)-module. An algebra \( A \) together with an action of \( H \) on \( A \) is called an \( H \)-module algebra. By a homomorphism of \( H \)-module algebras we mean a homomorphism of algebras which is also a homomorphism of
\(H\)-modules. Note that if \(A\) is an \(H\)-module algebra, then \(H \otimes \text{Hom}(H, A) \to \text{Hom}(H, A)\), where \(h(f)(h') = f(h'h)\), makes the convolution algebra \(\text{Hom}(H, A)\) an \(H\)-module algebra, and the corresponding \(\Psi: A \to \text{Hom}(H, A)\) is a homomorphism of \(H\)-module algebras.

**Example 3.2.** Let \(n \in \mathbb{N} \cup \{\infty\}\), and let \(H^{\langle n\rangle}\) be the Hopf algebra from Example 2.3. Then an action of \(H^{\langle n\rangle}\) on an algebra \(A\) is nothing else than a higher derivation \(D = \{D_i: A \to A; 0 \leq i \leq n\}\) of rank \(n\) on \(A\) (\(D_i(a) = t_i(a)\)).

**Example 3.3.** Consider the Hopf algebra \(H = K \langle g, g^{-1}, t \rangle\) from Example 2.4. Then an action of \(H\) on an algebra \(A\) is given by an automorphism \(\sigma: A \to A\) and a \(\sigma\)-derivation \(d: A \to A\), where \(\sigma(a) = g(a)\) and \(d(a) = t(a)\).

**Example 3.4.** Consider the Taft Hopf algebra \(T^{\langle n\rangle}\) as in Example 2.4. Then an action of \(T^{\langle n\rangle}\) on an algebra \(A\) is clearly an automorphism \(\sigma: A \to A\) and a \(\sigma\)-derivation \(d: A \to A\) such that \(\sigma^n = \text{Id}, d^n = 0,\) and \(d\sigma = \zeta \sigma d\).

If \(H = U(L)\) for some Lie algebra \(L\), then an action of \(H\) on an algebra \(A\) is simply an action of \(L\) on \(A\).

The main results of the paper are proved for commutative, noetherian module algebras over a Hopf algebra. Therefore, we are now going to show that there exist interesting classes of such algebras, especially for pointed Hopf algebras. Let \(H\) be a Hopf algebra. If \(V\) is an \(H\)-module, then the tensor algebra \(T(V)\) is an \(H\)-module algebra, via

\[
h(v_1 \otimes \cdots \otimes v_n) = \sum h_1(v_1) \otimes \cdots \otimes h_n(v_n).\]

It is obvious that the action of \(H\) on \(T(V)\) preserves the natural grading of \(T(V)\). Let \(I = I(V)\) denote the ideal in \(T(V)\) generated by the set

\[
\{h(v \otimes v' - v' \otimes v); h \in H, v, v' \in V\}.
\]

Then \(I\) is an invariant homogeneous ideal in \(T(V)\). Set

\[S_H(V) = T(V)/I\]

(the definition of \(S_H(V)\) comes from [Zh]). Recall that a graded algebra \(A = \bigoplus_{i \geq 0} A_i\) is called connected if \(A_0 = K\). With the above notation, one simply verifies that the following statements are true.

1. \(S_H(V)\) is a graded, connected, commutative \(H\)-module algebra such that all its homogeneous components \(S_H(V)_i, i \geq 0\), are \(H\)-submodules of \(A\) and \(S_H(V)_1 = V\). Furthermore, if \(H\) is cocommutative, then \(S_H(V)\) is the symmetric algebra \(S(V)\).

2. If \(V\) is finite dimensional, then the algebra \(S_H(V)\) is finitely generated.

So we see that every finite dimensional \(H\)-module \(V\) gives us a commutative, noetherian \(H\)-module algebra \(S_H(V)\).

Now let \(A\) be an \(H\)-module algebra and let \(I\) be an invariant ideal in \(A\). Then the completion \(\widehat{A} = \lim A/I^n\) of \(A\) in the \(I\)-adic topology is an \(H\)-module algebra, via
If \( A \) is commutative and noetherian, then \( \hat{A} \) is also (commutative) noetherian, by \([B, 2.6.24]\).

Another type of noetherian \( H \)-module algebra we obtain by means of localizations, but only for pointed Hopf algebras. Namely, we have the following.

**Theorem 3.4.** Let \( H \) be a pointed Hopf algebra with \( G = G(H) \), and let \( A \) be a commutative \( H \)-module algebra. Moreover, let \( T \) be a \( G \)-invariant multiplicative system in \( A \). Then there exists a unique action of \( H \) on the localization \( A_T \) (called the induced action) such that the natural homomorphism of algebras \( j : A \to A_T, \ j(a) = a/1 \), is a homomorphism of \( H \)-module algebras.

**Proof.** First, we show that there exists a unique measuring \( H \otimes A_T \to A_T \) such that \( h(j(a)) = j(h(a)) \) for all \( h \in H \) and \( a \in A \). Then it will be proved that this measuring makes \( A_T \) an \( H \)-module.

The action of \( H \) on \( A \) defines the homomorphism of algebras \( \Psi : A \to \text{Hom}(H, A) \) \((\Psi(a)(h) = h(a))\). Let \( \Psi' = j_1 \Psi : A \to \text{Hom}(H, A_T) \), that is, \( \Psi'(a)(h) = j(\Psi(a)(h)) \).

We are going to show that there is a unique algebra homomorphism \( \Phi : A_T \to \text{Hom}(H, A_T) \) such that \( \Phi(j(a)) = \Psi'(a) \) for \( a \in A \). It suffices to prove that the element \( \Psi'(t) \) is invertible in \( \text{Hom}(H, A_T) \) for each \( t \in T \). So, let \( t \in T \) and let \( f = \Psi'(t) \). First, we show that \( f \) is invertible in the convolution algebra \( \text{Hom}(H_0, A_T) \). We know that \( H_0 = KG \), because the Hopf algebra \( H \) is pointed. Consider the linear map \( f' : KG \to A_T \) given by \( f'(g) = 1/g(t) \). The map \( f' \) is well-defined, because \( g(t) \in T \), by the assumption. Further, \( f \star f'(g) = f(g)f'(g) = 1 = \epsilon(g) \) and \( f' \star f(g) = f'(g)f(g) = 1 = \epsilon(g) \). This means that \( f \) is invertible in \( \text{Hom}(H_0, A_T) \), which in turn implies that \( \Psi'(t) \) is invertible in \( \text{Hom}(H, A_T) \), by \([M, 5.2.10]\). The result is that we have an algebra homomorphism \( \Phi : A_T \to \text{Hom}(H, A_T) \) such that \( \Phi(j(a)) = \Psi'(a) \) for \( a \in A \). In other words, we have a measuring \( H \otimes A_T \to A_T \) with \( h(j(a)) = j(h(a)) \). The uniqueness of such a measuring is obvious.

It remains to prove that this measuring makes \( A_T \) an \( H \)-module. Using the coradical filtration \( H_0 \subset H_1 \subset \cdots \), it suffices to show that

\[
hk \left( \frac{a}{s} \right) = h \left( \frac{a}{s} \right) \quad \text{(\#)}
\]

for all \( l, m \geq 0, \ h \in H_l, \ k \in H_m, \) and \( a \in A, \ s \in T \). We proceed by induction on \( l + m \).

If \( l + m = 0 \), i.e., if \( h, k \in H_0 \), the equality (\#) holds, because \( H_0 = KG \). Assume that \( l + m \geq 1 \) and that the equality (\#) is true for \( h \in H_l, k \in H_m \) with \( l' + m' < l + m \).

Moreover, let \( h \in H_l, k \in H_m \). In view of Theorem 2.7, we can suppose that

\[
\Delta(h) = h \otimes g + \sum_{i=1}^r h_i \otimes h_i',
\]

for all \( l, m \geq 0, \ h \in H_l, \ k \in H_m, \) and \( a \in A, \ s \in T \). We proceed by induction on \( l + m \).

If \( l + m = 0 \), i.e., if \( h, k \in H_0 \), the equality (\#) holds, because \( H_0 = KG \). Assume that \( l + m \geq 1 \) and that the equality (\#) is true for \( h \in H_l, k \in H_m \) with \( l' + m' < l + m \).

Moreover, let \( h \in H_l, k \in H_m \). In view of Theorem 2.7, we can suppose that

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\Delta(h) = h \otimes g + \sum_{i=1}^r h_i \otimes h_i',
\]

for all \( l, m \geq 0, \ h \in H_l, \ k \in H_m, \) and \( a \in A, \ s \in T \). We proceed by induction on \( l + m \).
where \( g \in G, \ h_i \in H_{l-1}, \ h'_i \in H_l, \) and
\[
\Delta(k) = k \otimes f + \sum_{j=1}^r k_j \otimes k'_j,
\]

where \( f \in G, \ k_j \in H_{m-1}, \ k'_j \in H_m \) \((H_{-1} = 0)\). The latter implies that
\[
\Delta(hk) = hk \otimes gf + \sum_{i=1}^r h_i k \otimes h'_i f + \sum_{j=1}^r hk_j \otimes gk'_j + \sum_{i=1}^r \sum_{j=1}^r h_i k_j \otimes h'_i k'_j,
\]

whence
\[
h(k(a)) = h\left( k\left( \frac{a}{s} \right) \right) = h\left( k\left( \frac{a}{s} \right) f(s) \right) + \sum_{j=1}^r h\left( k_j\left( \frac{a}{s} \right) k'_j(s) \right)
\]
\[= h\left( k\left( \frac{a}{s} \right) \right) g(f(s)) + \sum_{j=1}^r h_i\left( k\left( \frac{a}{s} \right) h'_i f(s) \right)
\]
\[+ \sum_{j=1}^r h\left( k_j\left( \frac{a}{s} \right) \right) g(k'_j(s)) + \sum_{i=1}^r \sum_{j=1}^r h_i\left( k_j\left( \frac{a}{s} \right) \right) h'_i(k'_j(s)).
\]

On the other hand,
\[
h(k(a)) = hk(a) = hk\left( \frac{a}{s} \right)
\]
\[= hk\left( \frac{a}{s} \right) g(f(s)) + \sum_{i=1}^r h_i k\left( \frac{a}{s} \right) h'_i f(s)
\]
\[+ \sum_{j=1}^r hk_j\left( \frac{a}{s} \right) gk'_j(s) + \sum_{i=1}^r \sum_{j=1}^r h_i k_j\left( \frac{a}{s} \right) h'_i k'_j(s).
\]

By the inductive assumption,
\[
h_i\left( k\left( \frac{a}{s} \right) \right) = h_i k\left( \frac{a}{s} \right), \quad h\left( k_j\left( \frac{a}{s} \right) \right) = hk_j\left( \frac{a}{s} \right), \quad \text{and}
\]
\[
h_i\left( k_j\left( \frac{a}{s} \right) \right) = h_i k_j\left( \frac{a}{s} \right).
\]

The above equalities and the fact that \( A \) is an \( H \)-module imply that
\[
\sum_{i=1}^{r} h_i \left( k \left( \frac{a}{s} \right) \right) h_i' \left( f(s) \right) = \sum_{i=1}^{r} h_i k \left( \frac{a}{s} \right) h_i' \left( f(s) \right),
\]
\[
\sum_{j=1}^{t} h \left( k_j \left( \frac{a}{s} \right) \right) g \left( k_j'(s) \right) = \sum_{j=1}^{t} h k_j \left( \frac{a}{s} \right) g k_j'(s), \quad \text{and}
\]
\[
\sum_{i=1}^{r} \sum_{j=1}^{t} h_i \left( k_j \left( \frac{a}{s} \right) \right) h_i' \left( k_j'(s) \right) = \sum_{i=1}^{r} \sum_{j=1}^{t} h_i k_j \left( \frac{a}{s} \right) h_i' k_j'(s).
\]

Hence,
\[
h \left( k \left( \frac{a}{s} \right) \right) g \left( f(s) \right) = h k \left( \frac{a}{s} \right) g f(s),
\]
and consequently,
\[
h \left( k \left( \frac{a}{s} \right) \right) = h k \left( \frac{a}{s} \right).
\]

because \( g \left( f(s) \right) = g f(s) \in T \).

This proves Eq. (\( \ast \)), and thus the theorem follows. \( \square \)

**Corollary 3.5.** In the situation of the theorem, if \( H \) is connected, then the induced action of \( H \) on \( \Lambda_T \) exists for any multiplicative system \( T \subset A \). In particular, if \( A \) is noetherian, then we have the local, noetherian \( H \)-module algebras \( \Lambda_P \), where \( P \in \text{Spec} \, A \).

**Remark 3.6.** For the Hopf algebras from Example 2.3 the above theorem is well known and easy to prove.

### 4. Primary decomposition for \((H, A)\)-modules

Let \( A \) be a commutative, noetherian ring. Recall that an ideal \( Q \) in \( A \) is called primary if for any \( a, b \in A \) such that \( ab \in Q \) and \( a \notin Q \) there exists \( n \) with \( b^n \in Q \). If \( Q \) is primary, then its radical \( \sqrt{Q} \) is a prime ideal. The classical Lasker–Noether theorem [B,E] says that each ideal \( I \) in \( R \) admits an irredundant primary decomposition, that is, there are primary ideals \( Q_1, \ldots, Q_n \) such that \( I = Q_1 \cap \cdots \cap Q_n \), \( Q_j \not\subset \bigcap_{i \neq j} Q_i \), and the associated primes \( P_1 = \sqrt{Q_1}, \ldots, P_n = \sqrt{Q_n} \) are different. In general, such a decomposition is not unique, but the set \( \text{Ass}(I) = \{ P_1, \ldots, P_n \} \) is uniquely defined by \( I \).

Seidenberg, Brown, and Sato showed in [Se,Br,Sa], that if a derivation or a higher derivation of a rank \( n \in \mathbb{N} \cup \{ \infty \} \) of \( A \) is given, then every invariant ideal \( I \) admits an irredundant primary decomposition \( I = Q_1 \cap \cdots \cap Q_t \) such that all the ideals \( Q_i \) are invariant. These results give rise to the following question.
Question 1. Let $H$ be a Hopf algebra, and let $I$ be an invariant ideal in a commutative noetherian $H$-module algebra $A$. Is there an invariant irredundant primary decomposition of $I$, that is, is there an irredundant primary decomposition $I = Q_1 \cap \cdots \cap Q_t$, such that the ideals $Q_i$ are invariant?

Notice that if such a decomposition exists, then $g(P) \subseteq P$ for any group-like element $g \in H$ and $P \in \text{Ass}(I)$. This natural necessary condition is not satisfied for the ideal $(xy)$ in Example 1 of the introduction, and that is why no invariant irredundant primary decomposition of $(0)$ exists.

The main purpose of this section is to prove the following.

**Theorem 4.1.** Let $H$ be a pointed Hopf algebra and let $A$ be a commutative, noetherian $H$-module algebra. Moreover, let $I \subseteq A$ be an invariant ideal with $g(P) \subseteq P$ for all $g \in G(H)$ and $P \in \text{Ass}(I)$. Then there exists an irredundant primary decomposition $I = Q_1 \cap \cdots \cap Q_t$ such that all $Q_i$ are invariant.

The authors do not know how to generalize this theorem for Hopf algebras which are not pointed. In that case the natural necessary condition is probably that the set $\text{Ass}(I)$ is $H_0$-invariant, that is, given a $P \in \text{Ass}(I)$ and $h \in H_0$, $h(a) \in P$ for all $a \in P$. Unfortunately, we are able to prove neither that this condition holds if $I$ admits an invariant irredundant primary decomposition, nor that this condition is sufficient for the existence of an invariant primary decomposition of $I$.

Theorem 4.1 will be shown below not only for the invariant ideals but in a more general context. Namely, we prove it for all submodules of a given $(H,A)$-module $M$ which is finitely generated as an $A$-module. In order to recall the definition of an $(H,A)$-module, let us fix some notation. If $A$ is a module algebra over a Hopf algebra $H$, then for any $H$-module $M$ we write $h(m)$ instead of the traditional $hm$, and for all $A$-modules we use the traditional notation.

**Definition 4.2.** Let $H$ be a Hopf algebra and let $A$ be an $H$-module algebra. By an $(H,A)$-module we mean a vector space $M$ which is an $A$-module and an $H$-module such that $h(am) = \sum h_1(a)h_2(m)$ for $h \in H$, $a \in A$, $m \in M$. A submodule of an $(H,A)$-module $M$ is an $A$-submodule of $M$, which is also an $H$-submodule. If $N$ is a submodule of $M$, then the quotient space $M/N$ is an $(H,A)$-module in the natural way. A homomorphism of $(H,A)$-modules is a linear map which is a homomorphism both of $A$-modules and of $H$-modules.

Every invariant ideal of the $H$-module algebra $A$ is clearly an example of an $(H,A)$-module. In particular, the algebra $A$ itself is an $(H,A)$-module. Moreover, any invariant ideal is a submodule of this $(H,A)$-module.

Now let $n \in \mathbb{N} \cup \{\infty\}$ and $H(n) = \mathbb{K}(t_0 = 1, t_1, \ldots, t_n)$ be the Hopf algebra from Example 2.3. Moreover, let $A$ be an $(H(n),A)$-module algebra. Then we have the higher derivation $D = \{t_i : A \rightarrow A; \ 0 \leq i \leq n\}$ of the algebra $A$, and an $(H(n),A)$-module is an $A$-module $M$ together with a $D$-derivation, that is, together with a sequence of linear
maps \( \{ \delta_i : M \to M ; 0 \leq i \leq n \} \) such that \( \delta_0 = \text{Id} \) and \( \delta_s(am) = \sum_{i+j=s} \delta_i(a)\delta_j(m) \) for \( s = 0, \ldots, n \) and \( a \in A, m \in M \).

The following lemma gives more general examples of \((H, A)\)-modules.

**Lemma 4.3.** Assume that \( A \) is a module algebra over a Hopf algebra \( H \), and let \( V \) be an \( H \)-module.

1. The vector space \( A \otimes V \), together with \( a(b \otimes v) = ab \otimes v \) and \( h(a \otimes v) = \sum h_1a \otimes h_2v \) for \( a, b \in A, h \in H, v \in V \), is an \((H, A)\)-module.

2. The vector space \( \text{Hom}(A, V) \), together with \( (af)(b) = f(ba) \) and \( (hf)(b) = \sum h_2f(S(h_1(b))) \) for \( a, b \in A, h \in H, f \in \text{Hom}(A, V) \), is an \((H, A)\)-module.

**Proof.** Part (1) of the lemma is easy. As for part (2), \( \text{Hom}(A, V) \) is obviously an \( A \)-module, and, moreover, it is an \( H \)-module, because for \( h, k \in H, f \in \text{Hom}(A, V) \) and \( a \in A \) we have:

\[
(h(k(f)))(a) = \sum h_2(kf)(Sh_1(a)) = \sum h_2(k2f(Sk_1(Sh_1(a))))
\]

\[
= \sum (h_2k_2)(f(S(h_1k_1)(a))) = \sum ((hk)(f))\(a)\).
\]

It remains to prove the compatibility condition. Let \( h \in H, a, b \in A \). Then

\[
(h(af))(b) = \sum h_2((af)(Sh_1(b))) = \sum h_2(f((Sh_1(b))a))
\]

\[
= \sum h_3(f(Sh_2(b)h_1(a))) = \sum h_4(f(Sh_3(b)(Sh_2h_1(a))))
\]

\[
= \sum h_3(f(Sh_2(bh_1(a)))) = \sum (h_2f)(bh_1(a))
\]

\[
= \sum ((h_1a)(h_2f))(b),
\]

as was to be shown. \(\square\)

**Remark 4.4.** Let \( A \) be a module algebra over a Hopf algebra \( H \). Then we have the smash product algebra \( A \# H \) [M, 4.1.3], and the maps \( A \to A \# H, a \mapsto a \otimes 1 \), and \( H \to A \# H, h \mapsto 1 \otimes h \), are homomorphisms of algebras. If \( M \) is left \( A \# H \)-module then it is an \( A \)-module and an \( H \)-module, via these homomorphisms. A simple calculation shows that \( M \) is an \((H, A)\)-module. On the other hand, if \( M \) is an \((H, A)\)-module, then it is not difficult to show that \( M \) is a left \( A \# H \)-module, via \((a \otimes h)m = ah(m)\). Moreover, it is easy to check that a linear map \( f : M \to M' \) is a homomorphism of \((H, A)\)-modules if and only if it is a homomorphism of \( A \# H \)-modules.

The conclusion is that the category of \((H, A)\)-modules is equal to the category of \((H, A)\)-modules.
Recall that a submodule $N$ of $M$ is called primary if for all $a \in A$ and $y \in M - N$ such that $ay \in N$, there is an $n$ with $a^n \in (N : M)$. If $N$ is primary, then the ideal $(N : M)$ is primary, so that the ideal $P = \sqrt{(N : M)}$ is prime, and $N$ is called $P$-primary. As usual $\zeta(M)$ denotes the set of zero-divisors with respect to $M$, i.e., $\zeta(M) = \{a \in A; \exists_{y \in M} \text{ Ann}(m) = P\}$, and $\text{Ass}_A(M)$ denotes the set of all associated primes of $M$, that is, $\text{Ass}_A(M) = \{P \in \text{Spec}(A); \exists_{y \in M} \text{ Ann}(m) = P\}$. An ideal $P \in \text{Ass}(M)$ is called isolated when $P' \cap P$ imply $P' = P$.

We shall need the following information about primary decompositions.

**Theorem 4.5** [B,E]. Suppose that the ring $A$ is noetherian, and $M$ is a finitely generated $A$-module.

1. The set $\text{Ass}(M)$ is finite, nonempty if $M \neq 0$, $\bigcap_{P \in \text{Ass}(M)} P = \sqrt{(0 : M)}$, and $\bigcup_{P \in \text{Ass}(M)} P = \zeta(M) \cup \{0\}$.
2. Every submodule $N$ of $M$ admits an irredundant primary decomposition, i.e., there are primary submodules $N_1, \ldots, N_t$ of $M$ such that $N = N_1 \cap \cdots \cap N_t$, $N_j \not\subset \bigcap_{i \neq j} N_i$ for $j = 1, 2, \ldots, t$, and the prime ideals $P_1 = \sqrt{(N_1 : M)}, \ldots, P_t = \sqrt{(N_t : M)}$ are different. Moreover, $\{P_1, \ldots, P_n\} = \text{Ass}(M/N)$, and $N_i$ is uniquely determined by $N$, whenever $P_i$ is isolated.\
3. A submodule $N \subset M$ is primary if and only if $|\text{Ass}(M/N)| = 1$. If $N$ is primary, then $\zeta(M/N) \cup \{0\} = \sqrt{(N : M)}$ (this part follows from (1) and (2)).
4. If $N_1, N_2$ are submodules of $M$ and $N = N_1 \cap N_2$, then $\text{Ass}(M/N) \subset \text{Ass}(M/N_1) \cup \text{Ass}(M/N_2)$. Moreover, $N = N_1$, provided $\text{Ass}(M/N) \cap \text{Ass}(M/N_2) = \emptyset$.

In the rest of this section $H$ denotes a fixed pointed Hopf algebra with $G = G(H)$ and the coradical filtration $H_0 = KG \subset H_1 \subset \cdots$. Furthermore, $A$ denotes a fixed commutative, noetherian $H$-module algebra. It is easy to check that for any $(H, A)$-module $M$ the set $\text{Ass}(M) = \text{Ass}_A(M) \subset \text{Spec}(A)$ is $G$-invariant. By $\text{Ass}(M)^G$ we mean the set \{ $P \in \text{Ass}(M); \forall_{g \in G} g(P) = P$ \} = $\text{Ass}(M); \forall_{g \in G} g(P) \subset P$. Let us fix also an $(H, A)$-module $M$ which is supposed to be finitely generated as an $A$-module.

We are going to prove that if $N$ is a submodule of $M$ with $\text{Ass}(M/N)^G = \text{Ass}(M/N)$, then $N$, viewed as an $A$-submodule of $M$, admits an irredundant primary decomposition $N = N_1 \cap \cdots \cap N_t$ such that all $N_j$ are submodules of the $(H, A)$-module $M$. Observe that the condition $\text{Ass}(M/N)^G = \text{Ass}(M/N)$ is a necessary condition for the existence of such a decomposition.

The main idea of construction of the required decomposition of $N$ is patterned upon the construction of invariant primary decompositions of invariant ideals in a ring with a higher derivation [Sa], and the corresponding generalization of this construction for differential modules [N]. More precisely, first we define the class of $H$-irreducible submodules of $M$ and the class of $H$-primary submodules of $M$, and next we prove the following assertions:

(a) Each submodule of $M$ is an intersection of finite number of $H$-irreducible submodules of $M$.
(b) each $H$-irreducible submodule of $M$ is $H$-primary,
(c) each \( H \)-primary submodule \( N \) of \( M \) is primary as an \( A \)-submodule of \( M \), provided \( g(P) = P \) for all \( g \in G \) and \( P \in \text{Ass}(M/N) \).

Given a subset \( B \) of \( M \), we denote by \([B]\) the submodule of \( M \) generated by \( B \). In particular, each element \( m \in M \) determines a submodule \([m]\). Of course, each ascending chain of submodules stabilizes, because \( M \), being a finitely generated module over the (noetherian) ring \( A \), is noetherian.

**Lemma 4.6.** If \( N, L \subset M \) are submodules of the \((H, A)\)-module \( M \), then the ideal \((N : L)\) is invariant.

**Proof.** Let \( I = (N : L) \). In view of Theorem 2.7(1), it suffices to show that \( H_n(I) \subset I \) for all \( n \geq 0 \). Let \( a \in I \), \( m \in L \), and \( g \in G \). Then, clearly, \( g^{-1}(m) \in L \), and \( ag^{-1}(m) \in N \), whence \( g(a)m = g(a)g^{-1}(m) = g(\langle g^{-1}(m) \rangle) \in N \). This means that \( g(a) \in I \), and therefore \( H_0(I) \subset I \), because \( H_0 = KG \). Now assume that \( n > 0 \) and \( H_{n-1}(I) \subset I \). By Theorem 2.7(2), it is enough to prove that \( h'(I) \subset I \) for \( h' \in H_n \) such that \( \Delta(h') = h' \otimes e + f \otimes h' + w \) for some \( e, f \in G \) and \( w \in H_{n-1} \otimes H_{n-1} \). Let us fix such an \( h' \) and set \( h = e^{-1}h' \). As \( e(I) \subset I \), we need only to verify that \( h(I) \subset I \). Notice that

\[
\Delta(h) = h \otimes 1 + e^{-1}f \otimes h + \sum_{i=1}^{s} h_1 \otimes h'_i = h \otimes 1 + \sum_{i=1}^{s+1} h_1 \otimes h'_i,
\]

where \( h_{s+1} = e^{-1}f \), \( h'_{s+1} = h \), and \( h_i \in H_{n-1} \) for \( i = 1, \ldots, s + 1 \). Therefore, if \( a \in I \) and \( m \in L \), then

\[
N \ni h(am) = h(a)m + \sum_{i=1}^{s+1} h_1(a)h'_i(m).
\]

By the inductive assumption, \( h_1(a) \in I \) for all \( i = 1, \ldots, s + 1 \), whence \( h_1(a)h'_i(m) \in N \), because \( h'_i(m) \in L \). Hence

\[
h(a)m = h(am) - \sum_{i=1}^{s+1} h_1(a)h'_i(m) \in N,
\]

which implies that \( h'(a) = e(h(a)) \in e(I) \subset I \). Thus, \( H_n(I) \subset I \), and the lemma follows. \( \square \)

**Lemma 4.7.** If \( N \) is a submodule of \( M \) and \( J \) is an invariant ideal in \( A \), then \((N : J)\) is a submodule of \( M \).

**Proof.** Let \( L = (N : J) \). It is obvious that \( L \) is an \( A \)-submodule of \( M \). It remains to prove that \( L \) is an \( H \)-submodule of \( M \). As in the previous lemma, we show by induction that \( H_n(L) \subset L \) for \( n \geq 0 \). Let \( m \in L \), \( a \in J \), \( g \in G \). Then \( g^{-1}(a)m \in N \), whence
ag(m) = g(g^{-1}(a))g(m) = g(g^{-1}(a)m) \in N. This means that \(g(m) \in L\), and therefore \(H_0(L) \subset L\). Now suppose that \(n \geq 1\) and \(H_{n-1}(L) \subset L\). Again by Lemma 2.7(2), we need only to check that \(h'(L) \subset L\) for \(h' \in H_n\) with \(\Delta(h') = h \otimes e + f \otimes h' + w\) for some \(e, f \in G\) and \(w \in H_{n-1} \otimes H_{n-1}\). So, fix such an \(h'\) and set \(h = f^{-1}h'\). Then

\[
\Delta(h) = h \otimes f^{-1}e + 1 \otimes h + \sum_{i=1}^{s} h_i \otimes h'_i = 1 \otimes h + \sum_{i=0}^{s} h_i \otimes h'_i,
\]

where \(h_0 = h\), \(h'_0 = f^{-1}e\), and \(h'_i \in H_{n-1}\) for \(i = 0, \ldots, s\). Since \(f(L) \subset L\), it suffices to show that \(h(L) \subset L\). Let \(m \in L\). Then \(h(am) \in N\) for all \(a \in J\), because \(N\) is a submodule of \(M\). Moreover,

\[
h(am) = ah(m) + \sum_{i=0}^{s} h_i(a)h'_i(m).
\]

But \(h'_i(m) \in L\) by the inductive assumption, and \(h_i(a) \in J\), because the ideal \(J\) is invariant. Consequently, \(ah(m) = h(am) = \sum_{i=0}^{s} h_i(a)h'_i(m) \in N\), which implies that \(h(m) \in L\). The lemma is proved. \(\square\)

To formulate the next lemma, for a given \(a \in A\), we define a special \(A\)-submodule of \(M\) denoted by \(\Theta_a\). For \(r \in A\), let \(M_r = \bigcup_{m=0}^{r}(0 : r^m)\), where \((0 : r^m) = \{m \in M; r^m = 0\} \subset M\). Observe that \(g(M_r) = M_{g(r)}\) for all \(g \in G\). Now, by definition,

\[
\Theta_a = \bigcap_{g \in G} M_{g(a)} = \bigcap_{g \in G} g(M_a).
\]

**Lemma 4.8.** \(\Theta_a\) is a submodule of \(M\) for arbitrary \(a \in A\).

**Proof.** Again, it is enough to prove that \(H_n(\Theta_a) \subset \Theta_a\) for all \(n \geq 0\). Certainly, \(H_0(\Theta_a) \subset \Theta_a\). Suppose that \(n > 0\) and \(H_{n-1}(\Theta_a) \subset \Theta_a\). As in the proof of Lemma 4.7, we need only to prove that \(h(\Theta_a) \subset \Theta_a\) for \(h \in H_n\) with \(\Delta(h) = 1 \otimes h + \sum_{i=0}^{s} h_i \otimes h'_i\), where \(h'_i \in H_{n-1}\) for \(i = 0, \ldots, s\). So, let us fix such an \(h\) and an \(m \in \Theta_a\) = \(\bigcap_{g \in G} M_{g(a)}\). Moreover, let \(g \in G\). We have to show that \(h(m) \in M_{g(a)}\). Since \(M\) is noetherian as an \(A\)-module, there exists \(k = k_{a, x} \in \mathbb{N}\) such that \(M_a = (0 : a^k)\) and \(M_{g(a)} = (0 : g(a)^k)\). Hence, \(g(a)^k m = 0\), which implies that

\[
0 = h(g(a)^k m) = g(a)^k h(m) + \sum_{i=0}^{r} h_i(g(a)^k)h'_i(m).
\]

Furthermore, by the induction assumption, \(h'_i(m) \in (0 : g(a)^k)\) for all \(i = 0, \ldots, s\). It follows that
Theorem 4.5(2), there exists a primary decomposition that are disjoint. Let $O \subseteq G$. It suffices to show that any $\Theta a \subseteq \Theta a'' \subseteq \Theta a' \subseteq \Theta a$. Therefore, $\Theta a'' \subset \Theta a'$ by Theorem 4.5(1), the prime ideal $P$ contains $\sqrt{(0 : M)}$, it follows that $P_i \subset P$ for

$$g(a)^{2k}h(m) = g(a)^k g(a)^k h(m) = - \sum_{i=0}^{s} g(a)^k h_i(g(a)^k)h_i'(m)$$

$$= - \sum_{i=0}^{s} h_i(g(a)^k)g(a)^k h_i'(m) = 0.$$ Consequently, $h(m) \in (0 : g(a)^{2k}) = (0 : g(a)^k) = M_{g(a)}$, and the lemma is proved. \hfill $\Box$

**Definition 4.9.** A submodule $N$ of the $(H, A)$-module $M$ is called $H$-primary if for any invariant ideal $I \subset A$ and any submodule $L$ of $M$ such that $IL \subset N$ and $L \not\subset N$ there exists an $n$ with $I^n \subset (N : M)$.

Observe that a submodule $N \subset M$ is $H$-primary if and only if $(0)$ is an $H$-primary submodule of $M/N$.

**Lemma 4.10.** Let $N$ be a submodule of $M$. Then the following conditions are equivalent:

(a) $N$ is $H$-primary.
(b) The action of the group $G$ on the set $\text{Ass}(M/N)$ is transitive, that is, $\text{Ass}(M/N) = \{g(P) : g \in G\}$ for some $P \in \text{Ass}(M/N)$.
(c) There exists a $P \in \text{Ass}(M/N)$ and a $P$-primary $A$-submodule $Q$ of $M$ such that $N = \bigcap_{g \in G} g(Q)$.

**Proof.** We can assume that $N = (0)$.

Let $a \Rightarrow b$. Suppose $(0) \subset M$ is $H$-primary, and let $P$ be an isolated prime in $\text{Ass}(M)$. It suffices to show that any $G$-orbit in $\text{Ass}(M)$ contains $P$, because different orbits are disjoint. Let $O = \{P_1, \ldots, P_s\}$ be such an orbit, and let $\text{Ass}(M) = \{P_1, \ldots, P_t, P_{t+1}, \ldots, P_s\}$. If $t = s$, then trivially $P \in O$. So, we can assume that $t > s$. By Theorem 4.5(2), there exists a primary decomposition $(0) = \bigcap_{i=1}^{t} L_i$ of $A$-submodules of $M$ such that each $L_i$ is $P_i$-primary. Set

$$N' = \bigcap_{i=1}^{s} L_i, \quad N'' = \bigcap_{i=t+1}^{s} L_i, \quad I = \bigcap_{i=1}^{s} P_i.$$ It is obvious that $N'' \neq 0$, the ideal $I$ is $G$-invariant, and that $I^n M \subset N'$ for some $n$. Let $0 \neq m \in N''$. We show that $I \subset \sqrt{(0 : M)}$. Let $a \in I$. Then $a^m \in N' \cap N'' = (0)$, whence $g(a)^m = 0$ for all $g \in G$, because $I$ is $G$-invariant. This means that $m \in \Theta a$. By Lemma 4.8, it follows that the submodule $[m] \subset M$ (generated by $m$) is contained in $\Theta a$. In particular, there exists a $k$ such that $[m] \subset (0 : a^k)$. Hence, $a^k[m] = 0$, i.e., $a^k \in (0 : [m])$. As, by Lemma 4.6, the ideal $(0 : [m])$ is invariant, this implies that $[a^k] \subset (0 : [m])$. Therefore, $[a^k][m] = 0$. But $m \neq 0$, so that there exists an $l$ with $[a^l][l] M = 0$, because $0$ is $H$-primary. In particular, $a^l M = 0$, which makes clear that $\bigcap_{i=1}^{t} P_i = I \subset \sqrt{(0 : M)}$. As, by Theorem 4.5(1), the prime ideal $P$ contains $\sqrt{(0 : M)}$, it follows that $P_i \subset P$ for
some $i \leq s$. Consequently, $P = P_i \in O$, because $P$ is isolated. The implication $a \Rightarrow b$ is proved.

$b \Rightarrow c$. By the assumption, $\text{Ass}(M) = \{P, g_1(P), \ldots, g_s(P)\}$ for some $P \in \text{Ass}(M)$ and $g_1 \in G$. In particular, all prime ideals in $\text{Ass}(M)$ are isolated. Let $P_i = g_i(P)$, and let $(0) = N_1 \cap \cdots \cap N_s$ be an irredundant primary decomposition of $(0)$ such that each $N_i$ is $P_i$-primary. Then for every $i = 1, \ldots, s$, $0 = g_i(N_1) \cap \cdots \cap g_i(N_s)$ is also an irredundant primary decomposition and $g_i(N_1)$ is $P_i$-primary. In view of Theorem 4.5(2), $N_i = g_i(N_1)$ for all $i$, and thus the implication $b \Rightarrow c$ follows.

c $\Rightarrow a$. By the assumption, there exist a primary $A$-submodule $Q$ of $M$ and $g_1 = 1, g_2, \ldots, g_s \in G$ such that $(0) = g_1(Q) \cap \cdots \cap g_s(Q)$ is a primary decomposition of $(0)$. Let $P = \sqrt{Q}$. Suppose that $IL = (0)$ for some nonzero submodule $L$ of $M$ and an invariant ideal $I$ in $A$. Then clearly $L \not\subset g_j(L)$ for some $j$, whence $L = g_j^{-1}(L) \not\subset Q$. As $IL = (0) \subset Q$ and $Q$ is $P$-primary, this implies that $I \subset P$. Hence $I = g_i(L) \subset g_i(P)$ for $i = 1, \ldots, s$, because $I$ is $G$-invariant. It follows that $I \subset \bigcap_{i=1}^s g_i(P) = \bigcap_{p \in \text{Ass}(M)} P'$. By Theorem 4.5(1), the latter ideal equals $\sqrt{(0 : M)}$, which implies that $I \subset \sqrt{(0 : M)}$. Consequently, the submodule $(0) \subset M$ is $H$-primary, as was to be shown. The proof of the lemma is complete. □

**Corollary 4.11.** If $N_1$, $N_2$ are $H$-primary submodules of $M$ such that $\text{Ass}(M/N_1) = \text{Ass}(M/N_2)$, then $N = N_1 \cap N_2$ is an $H$-primary submodule of $M$ and $\text{Ass}(M/N) = \text{Ass}(M/N_1) = \text{Ass}(M/N_2)$.

**Proof.** In view of Theorem 4.5(4), $\text{Ass}(M/N) \subset \text{Ass}(M/N_1) \cup \text{Ass}(M/N_2) = \text{Ass}(M/N_1)$, and from the above lemma we know that $\text{Ass}(M/N_1)$ is an orbit of the action of $G$ on $\text{Ass}(M/N_1)$. This implies that $\text{Ass}(M/N) = \text{Ass}(M/N_1)$. Again using the above lemma, we conclude that $N$ is $H$-primary. □

The next corollary is an immediate consequence of the lemma.

**Corollary 4.12.** If $N$ is an $H$-primary submodule of $M$, then all ideals in $\text{Ass}(M/N)$ are isolated.

**Definition 4.13.** A submodule $N \subset M$ is said to be $H$-irreducible if it is not the intersection of two strictly larger submodules of $M$.

**Lemma 4.14.** If $N$ is an $H$-irreducible submodule of $M$, then $N$ is $H$-primary.

**Proof.** Let $I$ be an invariant ideal of $A$, and let $L$ be a submodule of $M$ such that $IL \subset N$ and $I \not\subset \sqrt{(N : M)}$. We have to show that $L \subset N$. Let $N = \bigcap_{i=1}^k N_i$ be an irredundant primary decomposition of $N$ as an $A$-submodule of $M$. Since $I \not\subset \sqrt{(N : M)} = \bigcap_{i=1}^k \sqrt{(N_i : M)}$, there exists an $s \leq k$ with $I \not\subset \sqrt{(N_i : M)}$ for $i = 1, \ldots, s$ and $I \subset \sqrt{(N_i : M)}$ for $i = s + 1, \ldots, k$. This implies that $L \subset N_i$ for $i \leq s$, because $IL \subset N \subset N_i$ and $N_i$ is a primary. Hence, $L \subset \bigcap_{i=1}^s N_i = N$, whenever $s = k$. So, let $s < k$. As $A$ is noetherian, there exists an $n$ such that $I^n \subset (N_j : M)$ for $j = s + 1, \ldots, k$. Hence, $(N_j : I^n) = M$ for $j > s$. Further, $I \not\subset \zeta(M/N_i)$ for $i \leq s$, because $\zeta(M/N_i) \cup \{0\}$
Theorem 4.5(3). It follows that \((N_i : I) = N_i\) for \(i \leq s\), whence \((N_i : I^n) = N_i\) for \(i \leq s\). This in turn implies that

\[
N \subset (N : I^n) \cap (N + I^n M) = \bigcap_{i=1}^{k} (N_i : I^n) \cap (N + I^n M)
\]

\[
= \bigcap_{i=1}^{s} N_i \cap (N + I^n M) \subset \bigcap_{i=1}^{s} N_i \cap \bigcap_{j=s+1}^{k} N_j = N,
\]

because \(N + I^n M \subset N + N_j = N_j\) for \(j = s+1, \ldots, k\).

Thus, we see that \(N = (N : I^n) \cap (N + I^n M)\). Now observe that \(N + I^n M\) is a submodule of \(M\), because the ideal \(I\) is invariant, and \((N : I^n)\) is also a submodule of \(M\), by Lemma 4.7. Furthermore, \(N \neq N + I^n M\), because it is assumed that \(I \not\subset \sqrt{(N : M)}\). Since \(N\) is \(H\)-irreducible, this implies that \(N = (N : I^n)\). On the other hand, we know that \(I^n L \subset N\), as \(I L \subset N\). The result is that \(L \subset (N : I^n) = N\), as was to be shown. \(\square\)

The above lemmas allow one to prove the following.

**Theorem 4.15.** Let \(N\) be a submodule of \(M\). Then there exist submodules \(N_1, \ldots, N_t\) of \(M\) satisfying the conditions:

(a) \(N = \bigcap_{i=1}^{r} N_i\).

(b) Each \(N_i\) is \(H\)-primary.

(c) \(\text{Ass}(M/N) = \bigcup_{i=1}^{r} \text{Ass}(M/N_i)\) and this union is the decomposition of the set \(\text{Ass}(M/N)\) on orbits of the action of \(G\) on \(\text{Ass}(M/N)\).

**Proof.** Since the \((H, A)\)-module \(M\) is noetherian, the standard application of the Kuratowski–Zorn lemma shows that \(N = \bigcap_{i=1}^{r} N_i\) for some \(H\)-irreducible submodules \(N_i\). By Lemma 4.14, each submodule \(N_i\) is \(H\)-primary, and by Lemma 4.10, \(\text{Ass}(M/N_i)\) is an orbit of the action of \(G\) on \(\text{Ass}(M/N)\). In view of Corollary 4.11, replacing \(\{N_1, \ldots, N_t\}\) by a smaller set \(\{N_1, \ldots, N_s\}\) of \(H\)-primary submodules if necessary, we can assume that \(\text{Ass}(M/N_i) \cap \text{Ass}(M/N_j) = \emptyset\) for \(i \neq j\). Moreover, from Theorem 4.5(4) we derive that \(\text{Ass}(M/N) \subset \bigcup_{i=1}^{t} \text{Ass}(M/N_i)\). It follows that there exists \(s \leq t\) such that \(\text{Ass}(M/N) = \bigcup_{i=1}^{s} \text{Ass}(M/N_i)\) and \(\text{Ass}(M/N_j) \cap \text{Ass}(M/N) = \emptyset\) for \(j > s\) (after possible re-ordering). Again by Theorem 4.5(4), \(N = N_1 \cap \cdots \cap N_s\), and the theorem is proved. \(\square\)

Now we are in position to prove the main result of this section. Recall that we assume \(H\) is a fixed pointed Hopf algebra and \(A\) is a fixed commutative, noetherian \(H\)-module algebra. Moreover, \(M\) is a fixed \((H, A)\)-module which is supposed to be finitely generated as an \(A\)-module.

**Theorem 4.16.** Let \(N\) be a submodule of the \((H, A)\)-module \(M\) with \(\text{Ass}(M/N)^G = \text{Ass}(M/N)\). Then there exists an irredundant primary decomposition \(N = \bigcap_{i=1}^{t} N_i\) such
that all $N_i$ are submodules of $M$. In particular, if $I$ is an invariant ideal in $A$, then there exists an irredundant primary decomposition $I = \bigcap_{j=1}^r I_j$ such that each ideal $I_j$ is invariant.

**Proof.** From the above theorem we know that there exists a primary decomposition $N = \bigcap_{i=1}^r N_i$ such that all $N_i$ are $H$-primary submodules of $M$, $\text{Ass}(M/N) = \bigcup_{i=1}^r \text{Ass}(M/N_i)$, and $\text{Ass}(M/N_i)$ are different orbits of the action of $G$ on $\text{Ass}(M/N)$. But $|\text{Ass}(M/N_i)| = 1$, because $\text{Ass}(M/N)^G = \text{Ass}(M/N)$. Therefore, by Theorem 4.5(3), each $N_i$ is a primary $A$-submodule of $M$. The conclusion is that $N = \bigcap_{i=1}^r N_i$ is the required irredundant primary decomposition of $N$. The theorem follows. \[\square\]

An immediate consequence of the theorem is the following.

**Theorem 4.17.** If the Hopf algebra $H$ is connected and $N$ is a submodule of $(H, A)$-module $M$, then there exists a irredundant primary decomposition $N = \bigcap_{i=1}^r N_i$ of $A$-submodules of $M$ such that all $N_i$ are submodules of $M$.

The following elementary example shows, in the situation of the above theorem, that a submodule $N$ of $M$ can admit an irredundant primary decomposition $N = Q_1 \cap \cdots \cap Q_t$ such that not all $Q_i$ are submodules of $M$.

**Example 4.18.** Let $H = K[t]$ with $t$ primitive, and let $A = K[x, y]$. Then $H$ is a connected Hopf algebra, and the derivation $t : A \to A$ with $t(x) = x = t(y)$ makes $A$ an $H$-module algebra. Notice that the ideal $(x^2, xy)$ is invariant, and that $(x^2, xy) \cap (x) = (x^2, x - y) \cap (x)$ are irredundant primary decompositions of $I$ [B, II, §3, Example 4]). The ideals $(x^2, xy), (x), (x^2, x - y)$ are clearly invariant, while the ideal $(x^2, y)$ is not, because $t(y) = x \notin (x^2, y)$.

Let $H = \bigoplus_{i \geq 0} H_i$ be a graded Hopf algebra, and let $A = \bigoplus_{j \geq 0} A_j$ be a graded $H$-module algebra ($H_i(A_j) \subset A_{i+j}$ for all $i, j$). By a graded $(H, A)$-module we mean an $(H, A)$-module $M$ together with a vector space grading $M = \bigoplus_{j \geq 0} M_j$ such that $H_i(M_j) \subset M_{i+j}$ and $A_j M_j \subset M_{i+j}$. In view of [E, Proposition 3.12], if $A$ is commutative and $M$ is a graded $(H, A)$-module, then $\text{Ass}(M)$ consists of graded (prime) ideals. A graded submodule of a graded $(H, A)$-module $M$ is a graded subspace of $M$ which is also a submodule of $M$. Proceeding exactly as above, one can prove the following.

**Theorem 4.19.** Assume that $H$ is connected ($H_0 = K$), and that the graded $H$-module algebra $A$ is commutative and noetherian. Moreover, let $M$ be a graded $(H, A)$-module which is finitely generated as an $A$-module. Then each graded submodule $N \subset M$ possesses an irredundant primary decomposition consisting of graded submodules of $N$. 
5. The convolutionally reduced Hopf algebras and invariance of the associative primes

In this section $H$ is a fixed Hopf algebra (not necessarily pointed) and $G = G(H)$ is the group of its group-like elements. All $H$-module algebras $A$ under consideration are supposed to be commutative and noetherian, and all $(H, A)$-modules are supposed to be finitely generated as $A$-modules. Given an $(H, A)$-module $M$, $Ass(M)$ stands for the set of all prime ideals associated with $M$ as an $A$-module. In the last section we proved that every submodule $N$ of $M$ with $Ass(M/N)^G = Ass(M/N)$ admits an irredundant primary decomposition $N = N_1 \cap \cdots \cap N_t$ such that all $N_i$ are submodules of $M$. In this section our main objective is to investigate the following.

Question 2. Let $A$ be an $H$-module algebra and let $M$ be an $(H, A)$-module. Is every ideal in $Ass(M)$ invariant? In particular, is every prime ideal associated to an invariant ideal in $A$ invariant?

In view of Example 2 from the introduction, the answer to this question is, in general, "no." Moreover, a natural necessary condition is that $Ass(M) = Ass(M)^G$.

Let us start with the following simple observation.

Lemma 5.1. Suppose $H$ is pointed and $A$ is an $H$-module algebra such that for any primary invariant ideal $I$ in $A$ the nilradical of the quotient $H$-module algebra $A/I$ is invariant. Then for each $(H, A)$-module $M$ with $Ass(M)^G = Ass(M)$ all ideals in $Ass(M)$ are invariant.

Proof. From Theorem 4.16 we know that the submodule $(0) \subset M$ admits a primary decomposition

$$0 = \bigcap_{i=1}^{t} N_i$$

such that every $N_i$ is a $P_i$-primary submodule in $M$, where $\{P_1, \ldots, P_t\} = Ass(M)$. In particular, the ideal $Q_i = (N_i : M)$ is primary for $i = 1, \ldots, t$. Moreover, by Lemma 4.6, the ideals $Q_i$ are invariant. Hence, by the assumption, the ideal $P_i / Q_i$, being the nilradical of the quotient $H$-module algebra $A/Q_i$, is invariant for every $i = 1, \ldots, t$. Consequently, all the ideals $P_i$ are invariant, as required. \qed

Recall that an algebra is called reduced, if it has no nonzero nilpotent elements.

Definition 5.2. We say that the Hopf algebra $H$ is convolutionally reduced, if there exists a subcoalgebra $C$ of $H$ satisfying the following conditions.

(a) $C$ (as a set) generates $H$ as an algebra.
(b) For any commutative reduced algebra $B$ the convolution algebra Hom($C, B$) is reduced.
It is easy to see that condition (b) is equivalent to the condition (b′)

For every field extension $K \subset L$ the convolution algebra $\text{Hom}(C, L)$ is reduced.

Also notice that if $f : H \to H'$ is a surjective homomorphism of Hopf algebras and $H$ is convolutionally reduced, then $H'$ is convolutionally reduced, too.

**Example 5.3.** If $T$ is an arbitrary group, then the group algebra $KT$ is convolutionally reduced, because the convolution algebra $\text{Hom}(C, B)$ is isomorphic to the product $\prod_T B$ for any algebra $B$.

**Example 5.4.** Let $H = K[t]$ with $t$ primitive. Then for any algebra $B$ the convolution algebra $\text{Hom}(H, B)$ is the algebra $B_\delta[[t]]$ of formal power series with divided powers $(B_\delta[[t]]) = \prod_{i \geq 0} Bt_i$ with $t_it_j = \binom{i+j}{j} t_{i+j}$. It follows that in characteristic 0 the Hopf algebra $H$ is convolutionally reduced, whereas it is not the case in characteristic $p > 0$.

**Example 5.5.** The Hopf algebra $H(\infty) = K(t_0 = 1, t_1, \ldots)$ from Example 2.3 is convolutionally reduced. The reason is that the subcoalgebra $C = \sum_{i \geq 0} Kt_i$ (with $\Delta(t_m) = \sum_{i+j=m} t_i \otimes t_j$ and $\varepsilon(t_i) = \delta_{i0}$) generates $H$ as an algebra, and the convolution algebra $\text{Hom}(C, B)$ is isomorphic to the formal power series algebra $B[[t]]$ for any algebra $B$. Later on, we show that for any $n$ the Hopf algebra $H(n)$ is convolutionally reduced, whenever $\text{char } K = 0$.

**Example 5.6.** An important role in algebraic topology and in invariant theory of finite groups is played by the Steenrod algebra $P^*$ (without the Bockstein operator) defined over the simple field $F_p$ of characteristic $p > 0$. As an $F_p$-algebra $P^*$ is generated by the Steenrod (squaring) operations $P^i, i \geq 0$, satisfying certain relations (called the Adem–Wu relations) [Mi]. The Hopf algebra structure in $P^*$ is determined by $\Delta(P^m) = \sum_{i+j=m} P^i \otimes P^j$. It follows that the Hopf algebra $P^*$ is a homomorphic image of the Hopf algebra $H(\infty)$ from the previous example for $K = F_p$. Consequently, $P^*$ is a convolutionally reduced Hopf algebra.

**Remark 5.7.** In [Sm] L. Smith gave a very nice and simple construction of the Steenrod Hopf algebra $P^*$ defined over an arbitrary finite field $F_q$ (for $F_q = F_p$ the construction gives the classical Steenrod algebra). In the same manner one proves that $P^*$ is a convolutionally reduced Hopf algebra.

All Hopf algebras in the above examples were pointed. It is not so in the next example.

**Example 5.8.** Let $G$ be a finite group and let $H$ be the Hopf algebra dual to the group Hopf algebra $KG$. Then for any algebra $B$ the convolution algebra $\text{Hom}(H, B)$ is isomorphic to $BG$. Therefore, when $G$ is Abelian and $(\text{char } K, |G|) = 1$, then $H$ is convolutionally reduced. To see that $H$ need not be pointed, one simply verifies that if $K$ is the field of rationals and $G$ is Abelian, then $H$ is pointed if and only if $|G| \leq 2$. If $G$ is not Abelian
and the field $K$ is algebraically closed, then it is not difficult to prove (using the Maschke theorem and the Wedderburn theorem) that the Hopf algebra $H$ is not convolutionally reduced.

The significance of the convolutionally reduced Hopf algebras is illustrated by the following.

**Theorem 5.9.** Suppose that the Hopf algebra $H$ is convolutionally reduced. Then for each $H$-module algebra $A$ the following conditions hold:

1. The nilradical $N(A)$ of the algebra $A$ is invariant.
2. If $H$ is pointed, then for any $(H,A)$-module $M$ with $\text{Ass}(M)^G = \text{Ass}(M)$ all ideals in $\text{Ass}(M)$ are invariant. In particular, if the Hopf algebra $H$ is connected, then the ideals in $\text{Ass}(M)$ are invariant for any $(H,A)$-module $M$.

**Proof.** In view of Lemma 5.1, it suffices to prove part (1) of the theorem.

Let $C$ be a subcoalgebra of $H$ from Definition 5.2, and let $A$ be a fixed $H$-module algebra. The inclusion $C \hookrightarrow H$ induces a surjective homomorphism of algebras $\pi: \text{Hom}(H,A) \rightarrow \text{Hom}(C,A)$, and the action of $H$ on $A$ gives us the algebra homomorphism $\Phi: A \rightarrow \text{Hom}(H,A)$, $\Phi(a)(h) = h(a)$. In order to prove that the nilradical $N(A)$ of $A$ is invariant, it is clearly enough to show that $c(a) \in N(A)$ for all $c \in C$, $a \in N(A)$. This in turn reduces to proving that $c(a) \in P$ for any prime ideal $P$, because $N(A) = \bigcap_{P \in \text{Spec}(A)} P$. So, let us fix $a \in N(A)$, $c \in C$, $P \in \text{Spec}(A)$, and denote by $L$ the quotient field of the domain $A/P$. Then $L$ is a field extension of $K$, and the natural inclusion $A/P \hookrightarrow L$ induces an injective algebra homomorphism $\text{Hom}(C, A/P) \rightarrow \text{Hom}(C, L)$. By the assumption we know that the convolution algebra $\text{Hom}(C, L)$ is reduced. Hence the algebra $\text{Hom}(C, A/P)$ is also reduced. As the homomorphism $\Phi: A \rightarrow \text{Hom}(C, A)$ induces the algebra homomorphism $\Psi: A \rightarrow \text{Hom}(C, A/P)$, $\Psi(a)(c) = \Phi(a)(c) + P = c(a) + P$, and $a \in A$ is a nilpotent element, this implies that $c(a) \in P$. The theorem follows.

Now we show how to get the known results on primary decomposition and the associated primes of differential modules.

Recall that if $D = \{D_i: A \rightarrow A; \ 0 \leq i \leq n\}$ is a higher derivation of an algebra $A$, and $M$ is an $A$-module, then a $D$-derivation of $M$ is a sequence of linear maps $\mathcal{D} = \{D_i: M \rightarrow M; \ 0 \leq i \leq n\}$ such that $D_0 = \text{Id}$ and

$$D_i(am) = \sum_{s=0}^{i} D_s(a)D_{i-s}(m).$$

**Corollary 5.10.** Let $A$ be a commutative, noetherian algebra with a higher derivation $D$ of degree $n \in \mathbb{N} \cup \{\infty\}$, and let $M$ be a finitely generated $A$-module with a $D$-derivation $\mathcal{D} = \{D_i: M \rightarrow M\}$. 

(1) For any $D$-invariant submodule $N \subset M$ (i.e., $D_i(N) \subset N$ for all $i$) there is a primary decomposition $N = N_1 \cap \cdots \cap N_s$ such that all $N_i$ are $D$-invariant. In particular, every $D$-invariant ideal in $A$ possesses a primary decomposition consisting of $D$-invariant ideals.

(2) Every $P \in \text{Ass}(M)$ is invariant, whenever $n = \infty$ or $\text{char } K = 0$.

Proof. Let $H(n) = K\langle t_0 = 1, t_1, \ldots, t_n \rangle$ be the (connected) Hopf algebra from Example 2.3. Then $t_i(a) = D_i(a)$, $t_i(m) = D_i(m)$ for $a \in A$, $m \in M$ make $A$ an $H(n)$-module algebra and $M$ an $(H(n), A)$-module. Now part (1) of the corollary is a consequence of Theorem 4.17. In view of Example 5.5, part (2) follows from the above theorem.

Remark 5.11. If $n = 1$, the corollary was proved in [Se, Theorem 1] (for ideals) and in [N, Theorem 6] (for modules). The case $n = \infty$ was considered in [Br, Theorem 1] and [W, Theorem 11.3]. For arbitrary $n$ the corollary was proved in [Sa, Theorem] (for ideals).

The next theorem is an extension of the main results of [NS] and [Ne].

Theorem 5.12. Let $P^*$ be the Steenrod Hopf algebra defined over a finite field $F_q$, and let $A$ be a $P^*$-module algebra. Moreover, let $M$ be an $(H, A)$-module.

(1) Every submodule $N$ of $M$ admits an irredundant primary decomposition consisting of submodules of $M$. If $A$ and $M$ are graded, then any graded submodule of $M$ admits an irredundant primary decomposition consisting of graded submodules of $M$.

(2) All ideals in $\text{Ass}(M)$ are invariant (as we mentioned in the previous section, if $A$ and $M$ are graded, then the ideals from $\text{Ass}(M)$ are also graded).

In particular, each invariant ideal $I$ in $A$ admits an irredundant primary decomposition $I = Q_1 \cap \cdots \cap Q_n$ such that the ideals $Q_i$ are invariant and all ideals in $\text{Ass}(I)$ are invariant.

Proof. Since the Hopf algebra $P^*$ is connected, the first part of the theorem is a consequence of Theorems 4.17 and 4.19. The second one is a consequence of Theorem 5.9, because $P^*$ is convolutionally reduced.

Theorem 5.13. Suppose the Hopf algebra $H$ is connected and $\text{char } K = 0$. Then for any field extension $K \subset L$ the convolution algebra $\text{Hom}(H, L)$ is reduced. In particular, $H$ is convolutionally reduced.

For the proof of the theorem we need the following.

Lemma 5.14. Let $C$ be a finite dimensional connected coalgebra with the unique grouplike element $c_0$, and let $H(C) = T(C)/(c_0 - 1)$ (see Section 2). If $\text{char } K = 0$, then for any field extension $K \subset L$ the convolution algebra $\text{Hom}(H(C), L)$ is reduced.
Proof. By Lemma 2.8, there exists a linear basis \( \{c_0, c_1, \ldots, c_n\} \) of \( C \) such that

\[
\Delta(c_k) = c_k \otimes c_0 + c_0 \otimes c_k + \sum_{i,j=0}^{k-1} \alpha_{ijk} c_i \otimes c_j
\]

for \( k = 1, \ldots, n \) and some \( \alpha_{ijk} \in K \). In particular, \( H(C) = K \langle c_1, \ldots, c_n \rangle \). Denote by \( B \) the natural linear basis in \( H(C) \) composed from the products of the form \( c_i c_j \ldots c_k \), where \( i, j, \ldots, k \in \{1, \ldots, n\} \), and fix an \( f \in \text{Hom}(H(C), L) \) with \( f \ast f = 0 \). For the proof of the lemma it suffices to show \( f(a) = 0 \) for arbitrary \( a \in B \).

Given an \( s \geq 1 \), let us denote by \( \leq \) the lexicographical order in the set \( \mathbb{N}^s \). Observe that this (good) order preserves the natural addition in \( \mathbb{N}^s \), that is, if \( (\alpha_1, \ldots, \alpha_s) \leq (\beta_1, \ldots, \beta_s) \) and \( (\gamma_1, \ldots, \gamma_s) \leq (\delta_1, \ldots, \delta_s) \), then \( (\alpha_1 + \gamma_1, \ldots, \alpha_s + \gamma_s) \leq (\beta_1 + \delta_1, \ldots, \beta_s + \delta_s) \).

Now let us define the function \( w : B \to \mathbb{N}^n \) by

\[
w(a = c_{i_1} c_{i_2} \ldots c_{i_k}) = (w_n(a), \ldots, w_1(a)),
\]

where \( w_j(a) \) is the number of \( c_j \) which appear in the product \( c_{i_1} c_{i_2} \ldots c_{i_k} \).

It is clear that \( w(ba) = w(ab) = w(a) + w(b) \), and that

\[
\text{if } w(a) \leq w(b) \text{ and } w(c) \leq w(d), \text{ then } w(ac) \leq w(bd)
\]

for all \( a, b, c, d \in B \).

The function \( w \) will be frequently used in the rest of the proof.

Notice that each element of the basis \( B \) can be uniquely written in the form (called canonical) \( d_1^{i_1} \ldots d_t^{i_t} \), where \( d_j \in \{c_1, \ldots, c_n\} \), \( d_i \neq d_{i+1} \) for \( i = 1, \ldots, t-1 \), and \( i,t \geq 1 \).

Now to each \( a \in B \) with the canonical form \( a = d_1^{i_1} \ldots d_t^{i_t} \) we associate the sequence \( P(a) = (w_n(a), \ldots, w_1(a), t, i_1, \ldots, i_t) \) and the element \( a^{(2)} = d_1^{2i_1} \ldots d_t^{2i_t} \).

Furthermore, in the set \( B \) we define the relation \( < \) by

\[
a < b \iff P(a) < P(b),
\]

and for \( a \in B \) we set \( H_n(C) = \sum_{b < a} Kb \subset H(C) \). The key role in the proof is played by the equality

\[
\Delta(a^{(2)}) = n(a)a \otimes a + v,
\]

where \( n(a) \) is a nonzero natural number and \( v \in H_n(C) \otimes H(C) + H(C) \otimes H_n(C) \).

First, we show how to get the lemma, using (3). It is obvious that every nonempty subset \( B' \) of \( B \) contains a minimal element \( b \) with respect to the relation \( < \) (i.e., if \( b' \in B \) and \( b' < b \), then \( b' \notin B' \)). Therefore, if the set \( B_f = \{ b \in B ; f(b) \neq 0 \} \) is nonempty, then it contains a minimal element \( a \). This means that \( f(H_n(C)) = 0 \), which together with (3)
implies that \( 0 = (f \circ f)(a^{(2)}) = n(a)f(a)^2 \), where \( n(a) \in \mathbb{N} \). This is a contradiction, because \( f(a) \neq 0 \) and \( \text{char } K = 0 \). Thus, we see that the set \( B_f \) is empty and the lemma follows.

So, it remains to prove (3). To this end, we need another two relations in the set \( B \):

\[
\begin{align*}
  a \approx b, & \quad \text{if } w(a) = w(b), \\
  a \ll b, & \quad \text{if } w(a) < w(b).
\end{align*}
\]

(4) \( a \approx b \), if \( w(a) = w(b) \).

(5) \( a \ll b \), if \( w(a) < w(b) \).

The relation \( \approx \) is clearly an equivalence relation. Notice that \( c_1 \ll c_2 \ll \cdots \ll c_k \). Also notice that if \( a \not\approx b \), then \( a \ll b \) or \( b \ll a \), and if \( a \ll b \), then \( a < b \). From the properties of the function \( w \) it results that the relation \( \ll \) preserves products, that is, if \( a, b, c, d \in B \) and \( a \ll b \) or \( a \approx b \), and \( c \ll d \), then \( ac \ll bd \), \( ca \ll bd \), \( ac \ll db \), \( ca \ll db \). Further, for \( k = 1, \ldots, n \) and arbitrary \( a_1, \ldots, a_k \in B \), if \( a_1, \ldots, a_k \ll c_k \), then \( a_1 \cdots a_k \ll c_k \).

From the fact that \( c_1 \ll c_2 \ll \cdots \ll c_k \) we immediately derive the equalities:

\[
\Delta(c_k) = c_k \otimes 1 + 1 \otimes c_k + \sum_{a, a' \ll c_k} \alpha_{a,a'} a \otimes a',
\]

(6) where \( \alpha_{a,a'} \in K, k = 1, \ldots, n \).

Now we are going to show that for any \( q \in \mathbb{N} \) and any \( k \leq n \) one gets the following equality:

\[
\Delta(c_k^q) = \sum_{r=0}^q \binom{q}{r} c_k^r \otimes c_k^{q-r} + \sum_{d, d' \ll c_k^q} \beta_{d,d'} d \otimes d'
\]

for some \( \beta_{d,d'} \in K \).

(7)

Let us apply induction on \( q \). If \( q = 1 \), then (7) is true, by (6. Assume (7) holds for some \( q \). Then we have

\[
\begin{align*}
\Delta(c_k^{q+1}) &= \Delta(c_k)\Delta(c_k^q) \\
&= \left( c_k \otimes 1 + 1 \otimes c_k + \sum_{a, a' \ll c_k} \alpha_{a,a'} a \otimes a' \right) \\
&\quad \times \left( \sum_{r=0}^q \binom{q}{r} c_k^r \otimes c_k^{q-r} + \sum_{d, d' \ll c_k^q} \beta_{d,d'} d \otimes d' \right) \\
&= (c_k \otimes 1 + 1 \otimes c_k) \left( \sum_{r=0}^q \binom{q}{r} c_k^r \otimes c_k^{q-r} \right) \\
&\quad + (c_k \otimes 1 + 1 \otimes c_k) \left( \sum_{d, d' \ll c_k^q} \beta_{d,d'} d \otimes d' \right)
\end{align*}
\]
\begin{align*}
&+ \left( \sum_{a,a' \ll c_k} \alpha_{a,a'} a \otimes a' \right) \left( \sum_{r=0}^{q} \left( \frac{q}{r} \right) c_k^r \otimes c_k^{q-r} \right) \\
&+ \left( \sum_{a,a' \ll c_k} \alpha_{a,a'} a \otimes a' \right) \left( \sum_{d,d' \ll c_k^q} \beta_{d,d'} d \otimes d' \right) \\
&= \sum_{r=0}^{q+1} \left( \frac{q+1}{r} \right) c_k^r \otimes c_k^{q+1-r} + \sum_{d,d' \ll c_k^q} \beta_{d,d'} (c_k d \otimes d' + d \otimes c_k d') \\
&+ \sum_{a,a' \ll c_k} \sum_{r=0}^{q} \alpha_{a,a'} \left( \frac{q}{r} \right) a c_k^r \otimes a' c_k^{q-r} \\
&+ \sum_{a,a' \ll c_k} \sum_{d,d' \ll c_k^q} \alpha_{a,a'} \beta_{d,d'} d a d' \otimes a' d'.
\end{align*}

Observe that the inequalities \( a a' \ll c_k \) and \( d d' \ll c_k^q \) imply that \( c_k d d' \ll d c_k d' \ll c_k^{q+1} \), \( a c_k^{q-r} a' \ll c_k^{q+1} \), \( a d a' \ll a d' \ll c_k^{q+1} \).

Hence

\begin{align*}
&\sum_{d,d' \ll c_k^q} \beta_{d,d'} (c_k d \otimes d' + d \otimes c_k d') + \sum_{a,a' \ll c_k} \sum_{r=0}^{q} \alpha_{a,a'} \left( \frac{q}{r} \right) a c_k^r \otimes a' c_k^{q-r} \\
&+ \sum_{a,a' \ll c_k} \sum_{d,d' \ll c_k^q} \alpha_{a,a'} \beta_{d,d'} d a d' \otimes a' d' = \sum_{e,e' \ll c_k^{q+1}} \gamma_{e,e'} e \otimes e',
\end{align*}

for some \( \gamma_{e,e'} \in K \). Consequently,

\begin{align*}
\Delta(c_k^{q+1}) = \sum_{r=0}^{q+1} \left( \frac{q+1}{r} \right) c_k^r \otimes c_k^{q+1-r} + \sum_{e,e' \ll c_k^{q+1}} \gamma_{e,e'} e \otimes e',
\end{align*}

and thus (7) is proved.

Now for any \( a \in B \) with the canonical form \( a = d_1^{j_1} \ldots d_t^{j_t} \) we consider the set \( A_a \) of all sequences \( (j_1, \ldots, j_t) \) with \( 0 \leq j_k \leq 2i_k \) for \( k = 1, \ldots, t \). Furthermore, for \( \lambda = (j_1, \ldots, j_t) \in A_a \) we set: \( a_\lambda = d_1^{j_1} \ldots d_t^{j_t}, a_\lambda' = d_1^{2i_1-j_1} \ldots d_t^{2i_t-j_t} \).

For the proof of (3) we also need the following two equalities

\begin{align*}
\Delta(a^{(2)}) = \sum_{\lambda \in A_a} m_\lambda a_\lambda \otimes a_\lambda' + \sum_{e,e' \ll a^{(2)}} \beta_{e,e'} e \otimes e', \tag{8}
\end{align*}
where \( m_\lambda \in \mathbb{N} \setminus \{0\} \) and \( \beta_{e,e'} \in K \), and
\[
a_\lambda = a = a' \lor a_\lambda < a \lor a'_\lambda < a \quad \text{for all } \lambda \in \Lambda_a.
\]

(9)

It is clear that these formulas imply (3).

In order to prove (8), we show that for a given \( b \in B \) with the canonical form \( b = d_1^{i_1} \ldots d_t^{i_t} \) one has
\[
\Delta(b^{(2)}) = \prod_{k=1}^{t} \left( \sum_{r_k=0}^{2i_k} \left( \frac{2i_k}{r_k} d_k^{r_k} \otimes d_k^{2i_k-r_k} \right) \right) + \sum_{ee' \ll b^{(2)}} \beta_{e,e'} e \otimes e'.
\]

(10)

where \( \beta_{e,e'} \in K \). Again we use induction, this time on \( t \). If \( t = 1 \), then \( b = d_1^{i_1} \). Set \( q = 2i_1 \), so that, from (7)
\[
\Delta(b^{(2)}) = \Delta(d_1^{i_1}) = \sum_{r=0}^{q} \left( \frac{q}{r} \right) d_1^{r} \otimes d_1^{q-r} + \sum_{aa' \ll d_1^{i_1}} \alpha_{a,a'} a \otimes a'.
\]

Now making use of the induction assumption and (7), we obtain that
\[
\Delta(b^{(2)}) = \left( \prod_{k=1}^{t} \left( \sum_{r_k=0}^{2i_k} \left( \frac{2i_k}{r_k} d_k^{r_k} \otimes d_k^{2i_k-r_k} \right) \right) + \sum_{ee' \ll b^{(2)}} \beta_{e,e'} e \otimes e' \right) \\
\times \left( \sum_{r=0}^{2i_t} \left( \frac{2i_t}{r} d_t^{r} \otimes d_t^{2i_t-r} \right) + \sum_{aa' \ll d_t^{i_t}} \alpha_{a,a'} a \otimes a' \right)
\]
\[
= \prod_{k=1}^{t} \left( \sum_{r_k=0}^{2i_k} \left( \frac{2i_k}{r_k} d_k^{r_k} \otimes d_k^{2i_k-r_k} \right) \right) \left( \sum_{aa' \ll d_t^{i_t}} \alpha_{a,a'} a \otimes a' \right) \\
+ \sum_{ee' \ll b^{(2)}} \sum_{r=0}^{2i_t} \left( \frac{2i_t}{r} e' \otimes e'^{2i_t-r} \right) \beta_{e,e'} e' \otimes e' a^{2i_t-r} \\
+ \sum_{ee' \ll b^{(2)}} \sum_{aa' \ll d_t^{i_t}} \beta_{e,e'} a a' a \otimes a' e' a'.
\]
Further, observe that the product
\[
\prod_{k=1}^{t-1} \left( \sum_{r_k=0}^{2i_k} \binom{2i_k}{r_k} d_k^{r_k} \otimes d_k^{2i_k-r_k} \right) \left( \sum_{\alpha, a, a' \in d_k^{2i_k}} \alpha_{a, a'} a \otimes a' \right)
\]
is a sum of elements of the form
\[
a d_1^{r_1} \ldots d_{t-1}^{r_{t-1}} \otimes d_1^{2i_1-1} \ldots d_{t-1}^{2i_{t-1}-1} a',
\]
where \(a \in K\). Since clearly \(d_1^{r_1} \ldots d_{t-1}^{r_{t-1}} \otimes d_1^{2i_1-1} \ldots d_{t-1}^{2i_{t-1}-1} \approx b^{(2)}\), then
\[
d_1^{r_1} \ldots d_{t-1}^{r_{t-1}} a d_1^{2i_1-1} \ldots d_{t-1}^{2i_{t-1}-1} a' \ll b^{(2)}, \quad \text{because } a a' \ll d_1^{2i_1}.
\]
It follows that
\[
\prod_{k=1}^{t-1} \left( \sum_{r_k=0}^{2i_k} \binom{2i_k}{r_k} d_k^{r_k} \otimes d_k^{2i_k-r_k} \right) \left( \sum_{\alpha, a, a' \in d_k^{2i_k}} \alpha_{a, a'} a \otimes a' \right) = \sum_{\alpha, a, a' \in b^{(2)}} \alpha_{a, a'} a \otimes a'.
\]
Besides, \(ed_t' e' d_t^{2i_t-r} \approx ee' d_t^{2i_t} \ll b^{(2)}\) and \(eae'a' \approx ee'a a' \ll b^{(2)}\) for \(ee' \ll b^{(2)}\) and \(aa' \ll d_t^{2i_t}\). Hence
\[
\Delta(b^{(2)}) = \prod_{k=1}^{t} \left( \sum_{r_k=0}^{2i_k} \binom{2i_k}{r_k} d_k^{r_k} \otimes d_k^{2i_k-r_k} \right) + \sum_{\alpha, a, a' \in b^{(2)}} \gamma_{\alpha, a, a'} \otimes a'
\]
for some \(\gamma_{\alpha, a, a'} \in K\). This proves (10). Now (8) follows, because
\[
\prod_{k=1}^{t} \left( \sum_{r_k=0}^{2i_k} \binom{2i_k}{r_k} d_k^{r_k} \otimes d_k^{2i_k-r_k} \right) = \sum_{\lambda \in A_k} m_\lambda b_\lambda \otimes b_\lambda'
\]
for some \(m_\lambda \in \mathbb{N} \setminus 0\).

It still remains to prove (9).

Let us fix an \(a \in B\) with the canonical form \(a = d_1^{i_1} \ldots d_t^{i_t}\) and a \(\lambda \in A_a\). Moreover, set \(a_1 = a_\lambda\), \(a_2 = a_\lambda'\), and assume that neither \(a_1 \triangleleft a\) nor \(a_2 \triangleleft a\). This means that \(P(a_1) \geq P(a)\) and \(P(a_2) \geq P(a)\). In particular, \(w(a_1) \geq w(a)\) and \(w(a_2) \geq w(a)\), whence \(w(a_1) = w(a) = w(a_2)\), because \(w(a_1) + w(a_2) = w(a_1 a_2) = w(a^2) = 2w(a)\). Hence the inequalities \(P(a_i) \geq P(a), i = 1, 2\), imply that \(0 < j_k < 2i_k\) for \(k = 1, \ldots, t\), which in turn implies that \(a_1 = d_1^{i_1} \ldots d_t^{i_t}\) and \(a_2 = d_1^{2i_1-1} \ldots d_t^{2i_t-1}\) are the canonical forms of the elements \(a_1\) and \(a_2\). It follows that \(P(a_1) = (w(a), t, j_1, \ldots, j_t)\) and \(P(a_2) = (w(a), t, 2i_1 - j_1, \ldots, 2i_t - j_t)\). But \(P(a_1) \geq P(a) = (w(a), t, i_1, \ldots, i_t), i = 1, 2,\) which is possible only when \(j_k = i_k = 2i_k - j_k\) for \(k = 1, \ldots, t\). The conclusion is that \(a_1 = a = a_2\), and thus (9) has been proved. This completes the proof of the lemma. \(\square\)
Once the lemma is proved, we can give the proof of the theorem.

**Proof of Theorem 5.13.** Let \( K \subset L \) be a field extension, and let \( f \in \text{Hom}(H, L) \) be a nilpotent. Further, let \( h \) be a fixed element in \( H \). By [M, Theorem 5.1.1], there exits a finite dimensional subcoalgebra \( C \subset H \) containing \( h \). From the construction of the Hopf algebra \( H(C) \) it follows that the inclusion \( C \hookrightarrow H \) can be uniquely extended to a homomorphism of bialgebras \( \varphi: H(C) \to H \). This homomorphism induces the homomorphism of the convolution algebras \( \varphi^*: \text{Hom}(H, L) \to \text{Hom}(H(C), L) \). In view of Lemma 5.14, the algebra \( \text{Hom}(H(C), L) \) is reduced. Therefore, \( \varphi^*(f) = 0 \). On the other hand, \( \varphi^*(f)(h) = f(\varphi(h)) = f(h) \). Hence \( f(h) = 0 \), which means that \( f = 0 \).  

\[ \ast \]

**Theorem 5.15.** If the Hopf algebra \( H \) is pointed and cocommutative, and \( \text{char} K = 0 \), then the convolution algebra \( \text{Hom}(H, L) \) is reduced for any field extension \( K \subset L \). In particular, \( H \) is convolutionally reduced.

**Proof.** By [M, 5.6.4, 5.6.5], \( H = U(P(H)) \otimes KG \) as coalgebras, where \( P(H) \) is the Lie algebra of all primitive elements in \( H \). Moreover, \( U(P(H)) \) is a connected Hopf algebra. Suppose \( f \in \text{Hom}(H, L) \setminus \{0\} \). Then \( f(t \otimes g) \neq 0 \) for some \( t \in U(P(H)) \) and \( g \in G \). Consider the linear map

\[ \phi: U(P(H)) \to H, \quad \phi(h) = h \otimes g. \]

It is obvious that \( \phi \) is a homomorphism of coalgebras, so that the dual map \( \phi^*: \text{Hom}(H, L) \to \text{Hom}(U(P(H)), L) \) is a homomorphism of algebras. Moreover, \( \phi^*(f) \neq 0 \), because \( \phi^*(f)(t) = f(t \otimes g) \neq 0 \). From Lemma 5.14 we know that the convolution algebra \( \text{Hom}(U(P(H)), L) \) is reduced, and therefore \( \phi^*(f)^2 \neq 0 \). Hence \( f^2 \neq 0 \), because \( \phi^*(f^2) = \phi^*(f)^2 \).  

From Theorems 5.9 and 5.13 we immediately get the following.

**Theorem 5.16.** Let \( \text{char} K = 0 \) and let the Hopf algebra \( H \) be connected. Then for any \( H \)-module algebra \( A \) and any \( (H, A) \)-module \( M \) all prime ideals associated with \( M \) are invariant.

**Corollary 5.17.** Assume the Hopf algebra \( H \) is connected and convolutionally reduced, and \( A \) is an \( H \)-module algebra. Moreover, assume \( A \) is an UFD (as a ring) and \( a \in A \). If the ideal \( (a) \) is invariant and \( a = p_1^{\alpha_1} \cdots p_s^{\alpha_s} \), where \( p_i \) are indecomposable and \( p_i \not\sim p_j \) for \( i \neq j \), then the ideals \( (p_1), \ldots, (p_s) \) are invariant.

**Proof.** Since \( A \) is UFD, \( \text{Ass}(\langle a \rangle) = \{ (p_1), \ldots, (p_s) \} \), and now the corollary results from Theorem 5.9.  

\[ \square \]

To show another interesting consequence of Theorem 5.13 we need the following simple lemma.
Lemma 5.18. If $C$ is a connected coalgebra and $1 < \dim_K C < \infty$, then the convolution algebra $C^* = \text{Hom}(C, K)$ is not reduced.

Proof. Let $1, c_1, \ldots, c_n$ be a linear basis of $C$ with $\Delta(c_i) = c_i \otimes 1 + 1 \otimes c_i + \sum_{j,k=1}^{n-1} \alpha_{jk}c_j \otimes c_k$ for some $\alpha_{jk} \in K$ (see Lemma 2.8). Then $c_n^2 = 0$ in $C^*$, where $c_n^*(c_i) = \delta_{n,i}$. □

The next theorem seems to be “folklore.”

Theorem 5.19. If $\text{char} K = 0$ and the Hopf algebra $H$ is connected and finite dimensional, then $H = K$.

Proof. By Theorem 5.13, the convolution algebra $\text{Hom}(H, K)$ is reduced. Now the above lemma implies that $H = K$. □

Theorem 5.13 is not true for all pointed Hopf algebras.

Example 5.20. Consider the Taft Hopf algebra $T(2) = K[g, t]/(g^2 - 1, t^2, gt + tg)$ described in Example 2.5. This is a four-dimensional algebra with basis $1, g, t, gt$, and $\Delta(g) = g \otimes g$, $\Delta(t) = t \otimes 1 + g \otimes t$, $\Delta(gt) = gt \otimes g + 1 \otimes gt$. Let $f \in \text{Hom}(T(2), K)$ be defined as follows:

$$f(1) = f(g) = f(gt) = 0, \quad f(t) = 1.$$  

One easily checks that $f \ast f = 0$, whence $f$ is a nonzero nilpotent. This makes clear that Theorem 5.13 is not true for $T(2)$. Notice also that if we set $A = K[X]/(X^2)$ and define the action of $T(2)$ on the algebra $A$ by $g(x) = -x$, $t(x) = 1$, where $x = X + (X^2)$, then the nilradical of $A$ is not invariant. This in turn implies that the conclusion of Theorem 5.9 fails for $T(2)$.

Now for a given $H$-module algebra $A$ and an $(H, A)$-module $M$ with $\text{Ass}(M)^G = \text{Ass}(M)$ we find a condition which implies that all ideals in $\text{Ass}(M)$ are invariant. However, we are able to do it only in characteristic 0. According to Lemma 5.1, of importance will be the invariability of the nilradicals of some quotient algebras of $A$.

Let us start with the following elementary and well-known fact.

Lemma 5.21. Let $B$ be a commutative algebra over an infinite field $L$ and let $0 \neq f \in B[x_1, \ldots, x_n]$. Then there exists a sequence $(\alpha_1, \ldots, \alpha_n) \in L^n$ such that $f(\alpha_1, \ldots, \alpha_n) \neq 0$.

From now on, we assume that $\text{char} K = 0$, and that all algebras under consideration are commutative and noetherian (it does not mean that the Hopf algebra $H$ is supposed to be commutative and noetherian as an algebra!). For any algebra $B$ we denote by $N(B)$
the nilradical of $B$. Obviously, $N^s \neq 0$ and $N^{s+1} = 0$ for some $s = s(B)$. Given an endomorphism $\sigma$ of $B$, we define the map $\gamma_\sigma : N \to N$ by formula

$$\gamma_\sigma(b) = \sigma(b)^s + \sigma(b)^{s-1}b + \ldots + \sigma(b)b^{s-1} + b^s.$$ 

In particular, if $B$ is an $H$-module algebra, then for each group-like element $g \in G$ we have the map $\gamma_g : N \to N$ induced by the automorphism $g : B \to B$. Notice that $\gamma_1(b) = (s+1)b^s$.

**Lemma 5.22.** If $\sigma$ is an automorphism of an algebra $B$ such that $\gamma_\sigma(y) \neq 0$ for some $y \in N = N(B)$, then there exist generators $y_0, \ldots, y_n$ of the ideal $N$ such that $\gamma_\sigma(y_i) \neq 0$ for $i = 0, \ldots, n$.

**Proof.** Let $t_1, \ldots, t_n$ be any generators of the ideal $N$. For a fixed $i \leq s$ we define the polynomial $f_i \in B[X]$ by $f_i(X) = \sum_{j=0}^s (\sigma(y)X + \sigma(t_i))^j(yX + t_i)^{s-j}$. Since $\gamma_\sigma(y) \neq 0$, the polynomial $f_i$ is not equal 0. Hence there exists an $\alpha_i \in K$ such that $f_i(\alpha_i) \neq 0$, because the field $K$ is infinite (char $K = 0$). On the other hand, for $x \in K$ we have $f_i(x) = \gamma_\sigma(yx + t_i)$, because the map $\sigma : B \to B$ is an algebra automorphism. It follows that $\gamma_\sigma(y_i) \neq 0$ for $y_i = y\alpha_i + t_i$, $i = 1, \ldots, n$. Besides, it is obvious that $y_0 = y, y_1, \ldots, y_n$ generate the ideal $N$. □

**Definition 5.23.** Let $T$ be an arbitrary subset of the group $G$. By induction we define a sequence $\{T_0, T_1, \ldots\}$ of subspaces of $H$ as follows:

(a) $T_0 = H_0 = KG$.
(b) if $n > 0$, then $T_n$ is the subspace of $H$ spanned by the set

$$\{h \in H; \Delta(h) = h \otimes 1 + g \otimes h + w, \text{ for some } g \in T \text{ and } w \in T_{n-1} \otimes T_{n-1}\}.$$

It is easy to see that $\{T_n\}_{n \geq 0}$ is an increasing sequence of subcoalgebras of $H$ and $T_n \subset H_n$.

**Lemma 5.24.** Assume $B$ is an $H$-module algebra with $N = N(B)$, and $T$ is a subset of $G$ such that $T_{n-1}(N) \subset N$ for some $n > 0$. Moreover, suppose that $h \in T_n$ is such that $\Delta(h) = h \otimes 1 + g \otimes h + \sum_i h_i \otimes h_i'^{1}$ for some $g \in G$ and $h_i, h_i' \in T_{n-1}$. Then for all $a \in N$ we have the equality

$$0 = h(a^{t+1}) = \gamma_g(a)h(a).$$

**Proof.** Notice that $h_i(a^k) \in N^k$ and $h_i'(a^k) \in N^k$, because $T_{n-1}(N) \subset N$ and $T_{n-1}$ is a coalgebra. By induction on $k$, we show that $h(a^{k+1}) \equiv (g(a)^k + g(a)^{k-1}a + \ldots + a^k)h(a) \mod N^{k+1}$. (11)

For $k = 0$ the equality is trivial. Assume $k > 0$ and (11) is true for $k - 1$. Then
\[ h(a^{k+1}) = h(a) a^k + g(a) h(a^k) + \sum_{i} h_i(a) h_i^\prime(a^k) \]
\[ \equiv h(a) a^k + g(a) h(a^k) \mod N^{k+1} \]
\[ \equiv h(a) a^k + g(a)(g(a)^{k-1} + g(a)^{k-2}a + \cdots + a^{k-1})h(a) \mod N^{k+1} \]
\[ \equiv (g(a)^k + g(a)^{k-1}a + \cdots + g(a)a^{k-1} + a^k)h(a) \mod N^{k+1}. \]

which proves the equality (11). As \( N^{s+1} = 0 \), the lemma follows from (11) for \( k = s \). \( \square \)

**Lemma 5.25.** Suppose \((0)\) is a primary ideal in an \( H \)-module algebra \( B \) with \( N = N(B) \), and \( T \) is a subset of \( G \) is such that the function \( \gamma_g : N \rightarrow N \) is nonzero for each \( g \in T \). Then \( h(N) \subset N \) for \( h \in \bigcup_n T_n \).

**Proof.** Using induction on \( n \), we show that \( T_n(N) \subset N \) for all \( n \). Since \( T_0 = KG \), \( T_0(N) \subset N \). Let \( n > 0 \) and let \( T_{n-1}(N) \subset N \). Moreover, let \( h \in T_n \). In order to show that \( h(N) \subset N \), we can obviously assume that \( \Delta(h) = h \otimes 1 + g \otimes h + \sum_i h_i \otimes h_i^\prime \), where \( g \in T \) and \( h_i, h_i^\prime \in T_{n-1} \). From the assumption we know that there is an \( r_{g} \in N \) such that \( \gamma_{g}(r_{g}) \neq 0 \). Therefore, by Lemma 5.22, there exist generators \( x_1, \ldots, x_t \) of the ideal \( N \) with \( \gamma_{g}(x_i) \neq 0 \) for \( i = 1, \ldots, t \). In view of Lemma 5.24, it follows that \( 0 = h(x_i^{k+1}) = \gamma_{g}(x_i) h(x_i) \). In particular, this means that for each \( i \) \( h(x_i) \) is a zerodivisor in \( A \). Hence \( h(x_i) \in N \) for \( i = 1, \ldots, t \), because the ideal \((0)\) is primary. Consequently, \( h(N) \subset N \), and we are done. \( \square \)

The above considerations suggest the following.

**Definition 5.26.** An \( H \)-module algebra \( B \) is called \( \gamma \)-nonzero, if there exists a subset \( T \subset G \) satisfying the conditions:

(a) The function \( \gamma_g : N \rightarrow N \) is nonzero for each \( g \in T \),
(b) The set \( \bigcup_{n \geq 0} T_n \) generates \( H \) as an algebra.

**Theorem 5.27.** Let \( A \) be an \( H \)-module algebra and let \( M \) be an \((H, A)\)-module with \( \text{Ass}(M)^G = \text{Ass}(M) \). Moreover, let \((0) = \bigcap_{i=1}^n M_i \) be an irredundant primary decomposition of \((0) \subset M \) such that all the \( M_i \) are submodules of \( M \). If the \( H \)-module algebra \( A/(M_i : M) \) is \( \gamma \)-nonzero for each \( i = 1, \ldots, n \), then all ideals in \( \text{Ass}(M) \) are invariant.

**Proof.** Let \( Q_i = (M_i : M) \). By Lemma 4.6, every primary ideal \( Q_i \) is invariant. Moreover, \( \text{Ass}(M) = \{ P_1, \ldots, P_n \} \), where \( P_i = \sqrt{Q_i} \). For each \( i = 1, \ldots, n \) the \( H \)-module algebra \( A/Q_i \) is \( \gamma \)-nonzero, so that there exists a subset \( T \subset G \) such that the function \( \gamma_{g_i} : P_i/Q_i \rightarrow P_i/Q_i \) is nonzero for each \( g \in T \), and the set \( \bigcup_{n \geq 0} T_n \) generates \( H \) as an algebra. In view of Lemma 5.25, \( T_n(P_i/Q_i) \subset P_i/Q_i \) for all \( n \geq 0 \), which implies that the ideal \( P_i/Q_i \) is invariant. The result is that the ideals \( P_i \) are invariant. \( \square \)
Now we want to prove that Theorem 5.16 is a consequence of the above theorem. To this end, we need the following.

**Lemma 5.28.** Let $B$ be an algebra with $N = N(B) \neq 0$, and let $s$ be, as above, the natural number such that $N^s \neq 0$ and $N^{s+1} = 0$. Then there are generators $y_0, \ldots, y_n$ of the ideal $N$ with $y_i^s \neq 0$ for all $i = 0, \ldots, n$.

**Proof.** Choose any generators $w_1, \ldots, w_n$ of the ideal $N$. Since $N^s \neq 0$, we can find a sequence $(i_1, \ldots, i_n) \in \mathbb{N}^n$ such that $i_1 + \cdots + i_n = s$ and $w_1^{i_1} \cdots w_n^{i_n} \neq 0$.

Now consider the polynomial $f = (x_1 w_1 + \cdots + x_n w_n)^s \in B[x_1, \ldots, x_n]$. Since char $K = 0$, the coefficient of $f$ at the monomial $x_1^{i_1} \cdots x_n^{i_n}$ is nonzero. Hence there are elements $a_1, \ldots, a_n \in K$ such that $f(a_1, \ldots, a_n) \neq 0$, by Lemma 5.21.

Let $y = a_1 w_1 + \cdots + a_n w_n$. Then $y^s = f(a_1, \ldots, a_n) \neq 0$, which implies that the polynomial $f_i(t) = (ty + w_i)^s \in B[t]$ is nonzero for $i = 1, \ldots, n$. By Lemma 5.21, $f_i(a_i) \neq 0$ for some $a_i \in K$. Now let $y_i = a_i y + w_i$ and let $y_0 = y$. Then $\{y_0, y_1, \ldots, y_n\}$ are the required generators of the ideal $N$, and so the lemma is proved. \(\square\)

**A second proof of Theorem 5.16.** In view of Theorem 5.27, it suffices to show that every $H$-module algebra $B$ is $\gamma$-nonzero. So, let $B$ be an $H$-module algebra with $N(B)^s \neq 0$ and $N(B)^{s+1} = 0$. From Lemma 5.28 we know that there is an element $y \in N(B)$ such that $y^s \neq 0$. Hence $\gamma_\sigma(y) = (s + 1)y^s \neq 0$, because char $K = 0$. This proves that $B$ is $\gamma$-nonzero, because $H$ is connected. \(\square\)

To indicate another application of Theorem 5.27 let us introduce the following.

**Definition 5.29.** An endomorphism $f : V \to V$ of a vector space $V$ is called $E$-independent if it is diagonalizable and the eigenvalues of $f$ are linearly independent over $\mathbb{N}$. An endomorphism $\sigma : B \to B$ of an algebra $B$ is called $E$-independent if it is $E$-independent as an endomorphism of the underlying vector space.

It is easy to see that every $E$-independent endomorphism $f : V \to V$ is an automorphism. Moreover, if $f$ is $E$-independent and $W$ is an $f$-invariant subspace of $V$, then the restriction $f|W : W \to W$ is also $E$-independent.

**Lemma 5.30.** Let $B$ be an algebra with $N = N(B) \neq 0$ and $N^s \neq 0$, $N^{s+1} = 0$. Moreover, let $\sigma$ be an $E$-independent automorphism of $B$. Then the function $\gamma_\sigma : N \to N$ ($\gamma_\sigma(a) = \sum_{i=0}^{s} a^{i}\sigma(a)^{s-i}$) is nonzero.

**Proof.** By Lemma 5.28, $y^s \neq 0$ for some $y \in N$. Furthermore, as $N$ is a $\sigma$-invariant subspace of $A$, $N = \bigoplus_{t \in W} N_t$, where $N_t = \{a \in N; \ \sigma(a) = ta\}$ and $W$ is the set of eigenvalues of $\sigma$. Hence the element $y$ can be written in the form $y = y_1 + \cdots + y_k$, where $y_i \in N_{t_i}$ for some $t_1, \ldots, t_k \in W$. 
Now consider the polynomial \( f(x_1, \ldots, x_k) \in A[x_1, \ldots, x_k] \) defined by
\[
f(x_1, \ldots, x_k) = \sum_{l=0}^{s} (t_1 y_1 x_1 + \cdots + t_k y_k x_k)^l (y_1 x_1 + \cdots + y_k x_k)^{s-l}.
\]

If \( \alpha_1, \ldots, \alpha_k \in K \), then clearly \( f(\alpha_1, \ldots, \alpha_k) = \gamma \sigma (y_1 \alpha_1 + \cdots + y_k \alpha_k) \). On the other hand,
\[
f(x_1, \ldots, x_k) = \sum_{l=0}^{s} (t_1 y_1 x_1 + \cdots + t_k y_k x_k)^l (y_1 x_1 + \cdots + y_k x_k)^{s-l}
\]
\[
= \sum_{l=0}^{s} \left( \sum_{j_1 + \cdots + j_k = l} n(j_1, \ldots, j_k) t_1^{j_1} \cdots t_k^{j_k} y_1^{j_1} \cdots y_k^{j_k} \right)
\]
\[
\times \left( \sum_{r_1 + \cdots + r_2 = s-l} n(r_1, \ldots, r_2) t_1^{r_1} \cdots t_k^{r_k} y_1^{r_1} \cdots y_k^{r_k} \right)
\]
for some \( n(j_1, \ldots, j_k), n(r_1, \ldots, r_2) \in \mathbb{N} \setminus 0 \). Hence
\[
f(x_1, \ldots, x_k) = \sum_{m_1 + \cdots + m_k = s} \left( \sum_{j_1 + \cdots + j_k = l} n(m_1, \ldots, m_k, j_1, \ldots, j_k) t_1^{j_1} \cdots t_k^{j_k} \right)
\]
\[
\times y_1^{m_1} \cdots y_k^{m_k} x_1^{m_1} \cdots x_k^{m_k}
\]
for some \( n(m_1, \ldots, m_k, j_1, \ldots, j_k) \in \mathbb{N} \setminus 0 \). Since \( \text{char } K = 0 \), from the condition \( y^s = (y_1 + \cdots + y_k)^s \neq 0 \) we derive that there exist \( i_1, \ldots, i_k \) with \( i_1 + \cdots + i_k = s \) and \( y_1^{i_1} \cdots y_k^{i_k} \neq 0 \). Note that if \( j_l \leq i_l \) for \( l = 1, \ldots, k \), then \( y_1^{j_1} \cdots y_k^{j_k} \neq 0 \) and \( \sigma(y_1^{j_1} \cdots y_k^{j_k}) = t_1^{j_1} \cdots t_k^{j_k} x_1^{j_1} \cdots x_k^{j_k} \). Therefore, \( t_1^{j_1} \cdots t_k^{j_k} \) is an eigenvalue of the automorphism \( \sigma \). Now observe that the coefficient of \( f \) at the monomial \( x_1^{i_1} \cdots x_k^{i_k} \), equal to
\[
\sum_{l=0}^{s} \left( \sum_{j_1 + \cdots + j_k = l} n(i_1, \ldots, i_k, j_1, \ldots, j_k) t_1^{j_1} \cdots t_k^{j_k} \right) y_1^{i_1} \cdots y_k^{i_k},
\]
is nonzero, because the set of the eigenvalues of the automorphism \( \sigma \) is linearly independent over \( \mathbb{N} \). This means that the polynomial \( f \) is nonzero, so that, by Lemma 5.21, there are \( \alpha_1, \ldots, \alpha_k \in K \) such that \( f(\alpha_1, \ldots, \alpha_k) \neq 0 \). The conclusion is that for \( x = \alpha_1 y_1 + \cdots + \alpha_k y_k \) we have \( \gamma \sigma(x) \neq 0 \), as was to be shown. \( \square \)

From Lemma 5.30 and Theorem 5.27 one gets the following.

**Theorem 5.31.** Let \( A \) be an \( H \)-module algebra and let \( T \) be a subset of \( G \) satisfying the conditions:
(a) For each \( g \in T \) the automorphism \( g : A \rightarrow A \) is \( E \)-independent.
(b) The set \( \bigcup_n T_n \) generates \( H \) as an algebra.

Then for any \((H, A)\)-module \( M \) with \( \text{Ass}(M)^G = \text{Ass}(M) \) all ideals in \( \text{Ass}(M) \) are invariant.

**Proof.** Let \( M \) be an \((H, A)\)-module with \( \text{Ass}(M)^G = \text{Ass}(M) \), and let \( P \in \text{Ass}(M) \).

By Theorem 4.16, there exist a \( P \)-primary submodule \( M' \subset M \). By Theorem 4.6, it follows that the ideal \( (M' : M) \) in \( A \) is \( P \)-primary and invariant. So we can consider the induced action of \( H \) on the algebra \( A/(M' : M) \). Let \( g \in T \). Since the automorphism \( g : A \rightarrow A \) is \( E \)-independent, the induced automorphism \( g : A/(M' : M) \rightarrow A/(M' : M) \) is also \( E \)-independent. Hence, by Lemma 5.30, there exists an \( x \in P + (M' : M) \) such that \( \gamma_g(x) \neq 0 \). In view of Theorem 5.27, this implies that the ideal \( P \) is invariant. The theorem follows.

The next result is an application of the above theorem to skew derivations.

**Theorem 5.32.** Let \( \sigma \) be an \( E \)-independent automorphism of an algebra \( A \), and let \( d : A \rightarrow A \) be a \( \sigma \)-derivation of \( A \). Moreover, let \( I \) be an ideal in \( A \) such that \( \sigma(I) = I \), \( \delta(I) \subset I \), and \( \sigma(P) = P \) for \( P \in \text{Ass}(I) \). Then all the ideals in \( \text{Ass}(I) \) are \( d \)-invariant.

**Proof.** Consider the Hopf algebra \( H = K\langle g, g^{-1}, t \rangle \) from Example 2.4. If we put \( g(a) = \sigma(a) \), \( t(a) = d(a) \) for \( a \in A \), then \( A \) becomes an \( H \)-module algebra. Moreover, it is obvious that \( I \) is an invariant ideal in \( A \).

Let \( T = \{g\} \). Then \( T_1 \supset Kt \), which implies that the set \( T_0 \cup T_1 \) generates \( H \) as an algebra. The conclusion now follows from Theorem 5.31.

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**References**


