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The Bartle Bilinear Integration and Carleman Operators[1]

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Some characterizations of integrable functions in the bilinear sense of Bartle with respect to the injective tensor product are obtained. As a consequence it is shown that the kernels of Carleman compact operators coincide with these Bartle integrable functions. This result is applied to prove that every Carleman L-weak-compact operator is compact. An example showing the different behavior of the integrability with respect to the projective tensor product is given. A general Fubini theorem in this setting is shown. © 1999 Academic Press

INTRODUCTION

An operator u: $L_2([0,1]) \to L_2([0,1])$ is said to be integral or to be a kernel operator if there exists a measurable real function K, called the kernel of u, such that for every $\varphi \in L_2([0,1])$, the equality

$$u(\varphi)(s) = \int_0^1 K(s,t)\varphi(t)\,dt$$

holds for almost every s. A special case of these type of operators are the so-called Carleman operators. An integral operator is said to be Carleman if for almost every s, the sections $K(s,\cdot)$ of the kernel K are in $L^2([0,1])$. Thus a Carleman operator is given by a strongly measurable vector valued function $s \in [0,1] \to K(s,\cdot) \in L^2([0,1])$. The study of these operators was indicated by Carleman in the 1920's and has been the subject of attention of many other researchers.

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The concept of the Carleman operator can be extended in the following way: let L be an order continuous Köthe function space defined on the finite measure space (S, Σ, σ) [11, Section 1.b.17] and let X be a Banach space. An operator $u\colon X \to L$ is called a Carleman operator if there exists a strongly measurable function $f\colon S \to X^*$, called the kernel of u, such that for every $x \in X$ the equality $u(x)(s) = f(s)(x)$ holds for almost every $s \in S$ [6].

In 1956, Bartle [1] introduced the so-called Bartle bilinear integration which we describe now: let X, Y, and Z be Banach spaces and (S, Σ) be a measurable space. Let $\phi\colon X \times Y \to Z$ be a continuous bilinear map, $f\colon S \to X$ a strongly measurable function and $\nu\colon \Sigma \to Y$ a countably additive vector measure. The Bartle semivariation is the set function $\|\nu\|_\phi(A) = \sup\|\sum_{k=1}^m \phi(x_k, \nu(A_k))\|$, where $\|x_k\| \leq 1$ and (A_k) is a measurable partition of A. We assume that the Bartle semivariation of ν with respect to ϕ is dominated by the finite positive measure σ on Σ, that is to say, $\sigma(A) \to 0$ if and only if $\|\nu\|_\phi(A) \to 0$ (* property in the original terminology of Bartle). The function f is said to be integrable in the sense of Bartle with respect to ν and ϕ if

1. there exists a sequence (f_n) of simple functions that converges to f almost everywhere,

2. the sequence of indefinite integrals $\int_A \phi(f_n, d\nu)$ converges in the norm of Z for each measurable set A, where $\int_A \phi(f_n, d\nu) = \sum_{k=1}^m \phi(x_k, \nu(A_k \cap A)) \in Z$, if $f_n = \sum_{k=1}^m x_k \chi_{A_k}$.

The limit is usually denoted by $\int_A \phi(f, d\nu)$. It can be seen that the limit of the sequence $(\int_A \phi(f_n, d\nu))$ exists in the norm of Z uniformly for $A \in \Sigma$, and so the indefinite integral $\int \phi(f, d\nu)$ is countably additive.

In this paper we establish a connection between Carleman operators and integrable functions in the bilinear sense of Bartle by using the space $L^1(\nu)$ of scalar functions which are integrable with respect to ν [9] (see Section 2 for the definition).

In Section 2 we characterize the Bartle integrability of a function with respect to a certain vector measure ν and the injective tensor product in terms of the countably additivity of certain vector measure associated in a natural way to the function (cf. Theorem 2). We also obtain a characterization in terms of the L-weak compactness of the Carleman operator associated to the function (cf. Theorem 2). Some of these results show that there exist strong analogies between the Bartle bilinear integral with respect to the injective tensor product and the Pettis integral.

In Section 3 we identify the space of X-valued Bartle integrable functions with respect to the injective tensor product with the space of weak*-to-weak continuous compact Carleman operators from X^* into

$L^1(\nu)$ (cf. Theorem 4). We also prove that a Carleman operator from X into $L^1(\nu)$ is compact if and only if it is L-weak compact (cf. Proposition 7). This result should be compared with that obtained in [6, Cor. 16] and [5, Theorem 4.2]. We prove in Section 4 that the space of Bartle integrable functions with values in X is not complete when X is infinite dimensional and $L^1(\nu)$ is not purely atomic. Its completed space is $X \tilde{\otimes}_\epsilon L^1(\nu)$.

In Section 5 we prove that some of the characterizations of Bartle integrability with respect to the injective tensor product that we have obtained do no work for the projective tensor product. Finally, in Section 6 we use the results obtained in Section 2 to show a general Fubini theorem for the injective tensor product of two vector measures. This theorem includes those considered in [3] and [8].

The notation is standard as can be found for instance in [11] and [4]. If X is a Banach space, we shall denote by B_X its closed unit ball. If Y is another Banach space then $\mathscr{L}(X,Y)$ is the space of bounded linear operators from X into Y. By $\mathrm{Bil}(X \times Y)$ we denote the space of continuous bilinear forms defined on $X \times Y$; sometimes this space will be canonically identified with $\mathscr{L}(X, Y^*)$.

2. BARTLE BILINEAR INTEGRABILITY WITH RESPECT TO THE INJECTIVE TENSOR PRODUCT

If $\nu: \Sigma \to Y$ is a finitely additive vector measure defined on the σ-field Σ of subsets of the set S, the variation of ν is the measure $|\nu|$ defined by $|\nu|(A) = \sup\{\sum_{k=1}^n \|\nu(A_k)\|: (A_k) \text{ partition of } A\}$. The semivariation of ν is the set function $\|\nu\|(A) = \sup\{|y^*\nu|(A): \|y^*\| \leq 1\}$. If $\|\nu\|(S) < \infty$ we say that ν has bounded semivariation. Countably additive vector measures have bounded semivariation. A control measure for ν is a positive finite measure σ such that $\sigma(A) \to 0$ if and only if $\|\nu\|(A) \to 0$. A theorem by Bartle, Dunford, and Schwartz states that such a measure always exists if ν is countably additive which we suppose from now on. Moreover, this measure can be chosen as $\sigma = |y^*\nu|$, for certain $y^* \in Y^*$ [4], a result by Rybakov.

In this section we study the Bartle bilinear integration with respect to the injective tensor product, that is, the bilinear map is $\phi = \phi_\epsilon: X \times Y \to X \tilde{\otimes}_\epsilon Y$ given by $\phi_\epsilon(x, y) = x \otimes y$. We recall that the injective tensor product of X and Y is the completed space of $X \otimes Y$, the space of weak*-to-weak continuous finite rank operators from X^* into Y, endowed with the uniform norm of operators. In this case it is easy to see that the Bartle semivariation of ν with respect to ϕ_ϵ coincides with the semivariation of ν, so it is dominated by a control measure of ν.

Let us recall that a measurable function $\varphi\colon S \to \mathbb{R}$ is said to be integrable with respect to a countably additive vector measure $\nu\colon \Sigma \to Y$ if

1. for any $y^* \in Y^*$, φ is integrable with respect the scalar measure $|y^*\nu|$ and

2. for each measurable set A there exists an element in Y, which is denoted by $\int_A \varphi\, d\nu$, such that $y^*(\int_A \varphi\, d\nu) = \int_A \varphi\, dy^*\nu$ for every $y^* \in Y^*$.

The space of functions satisfying conditions (1) and (2) is denoted by $L^1(\nu)$ [9]. A norm is defined in $L^1(\nu)$ by setting $\|\varphi\| = \sup\{\int |\varphi(s)| d|y^*\nu|(s)\colon y^* \in B_{Y^*}\}$. With this norm $L^1(\nu)$ is an order continuous Köthe function space with the pointwise order. An equivalent norm is given by $\|\varphi\| = \sup\{\|\int_A \varphi\, d\nu\|\colon A \in \Sigma\}$.

If $f\colon S \to X$ is a function such that for any pair $x^* \in X^*$, $y^* \in Y^*$, x^*f is integrable with respect to $|y^*\nu|$, then we can define a finitely additive vector measure, which we will denote by $\int f\, d\nu$ with values in $\mathrm{Bil}(X^* \times Y^*)$ in the following way. For any $A \in \Sigma$, let us consider the bilinear map $\int_A f\, d\nu$ defined by

$$\left(\int_A f\, d\nu\right)(x^*, y^*) = \int_A x^* f\, dy^*\nu$$

on each pair $(x^*, y^*) \in X^* \times Y^*$. Using the closed graph theorem it can be seen that this bilinear map is separately continuous, therefore it is continuous. Since we have

$$\sup\left\{\left|\int_A x^* f\, dy^*\nu\right|\colon x^* \in B_{X^*}, y^* \in B_{Y^*}\right\} = \left\|\int_A f\, d\nu\right\|,$$

it follows from Nikodym–Grothendieck boundedness theorem [4] that this finitely additive measure has bounded semivariation.

In the next Lemma we show that if the function f is Bartle integrable with respect to ν and $\phi_\epsilon\colon X \times Y \to X \tilde{\otimes}_\epsilon Y$ then its indefinite integral is precisely the measure $\int f\, d\nu$. Note that $X \tilde{\otimes}_\epsilon Y$ is (isometrically) a subspace of $\mathrm{Bil}(X^* \times Y^*)$.

LEMMA 1. *Let $f\colon S \to X$ be a Bartle integrable function with respect to $\nu\colon \Sigma \to Y$ and $\phi_\epsilon\colon X \times Y \to X \tilde{\otimes}_\epsilon Y$. Then for every $x^* \in X^*$, $x^*f \in L^1(\nu)$ and for any $A \in \Sigma$ we have $\int_A \phi_\epsilon(f, d\nu) = \int_A f\, d\nu$ in $\mathrm{Bil}(X^* \times Y^*)$.*

Proof. If $f = x\chi_A$ then $\int_B \phi_\epsilon(f, d\nu) = x \otimes \nu(A \cap B) = \int_B f\, d\nu$, hence the lemma follows for simple functions.

Given an arbitrary Bartle integrable function f we can find a sequence (f_n) of simple functions converging to f almost everywhere and such that $\int_A \phi_\epsilon(f_n, d\nu)$ converges to $\int_A \phi_\epsilon(f, \nu)$ in $X \tilde{\otimes}_\epsilon Y$, uniformly for $A \in \Sigma$.

Therefore, for fixed $x^* \in X^*$, the sequence $(x^* f_n)$ is Cauchy in $L^1(\nu)$, and its limit must be $x^* f$. Thus f defines the vector measure $\int f\,d\nu$ as we said above.

For any measurable set A, we have that $(\int_A \phi_\epsilon(f_n, d\nu))(x^*, y^*) = \int_A x^* f_n\, dy^* \nu$ goes to $\int_A x^* f dy^* \nu$. It follows that $\int_A \phi_\epsilon(f, d\nu) = \int_A f\,d\nu$. ∎

Let us recall that if $u\colon X \to L$ is a bounded linear map and L is a Banach lattice then it is said that u is L-weak-compact if $u(B_X)$ is almost order bounded in L, that is, given $\epsilon > 0$ there exists $e \in L$, $e \geq 0$ such that $u(B_X) \subset [-e, e] + \epsilon B_L$, where $[-e, e] = \{z \in L\colon -e \leq z \leq e\}$. If L is an order continuous Köthe function space on (S, Σ, σ) it is known that this is equivalent to the condition $\lim_{\sigma(A) \to 0} \sup\{\|u(x)\chi_A\|\colon \|x\| \leq 1\} = 0$.

THEOREM 2. *Let $f\colon S \to X$ be a strongly measurable function. The following conditions are equivalent.*

1. *The function f induces an operator $x^* \to x^* f$ from X^* into $L^1(\nu)$ which is compact.*
2. *The function f induces an operator $x^* \to x^* f$ from X^* into $L^1(\nu)$ which is L-weak-compact.*
3. *For any pair $(x^*, y^*) \in X^* \times Y^*$, the function $x^* f$ is integrable with respect to $|y^* \nu|$ and the corresponding vector measure induced by f is countably additive.*
4. *For any pair $(x^*, y^*) \in X^* \times Y^*$, the function $x^* f$ is integrable with respect to $|y^* \nu|$ and the corresponding vector measure induced by f takes values into $X \tilde{\otimes}_\epsilon Y$.*
5. *The function f is Bartle integrable with respect to ν and the bilinear map $\phi_\epsilon\colon X \times Y \to X \tilde{\otimes}_\epsilon Y$.*

Proof. (1) ⇒ (2): It is clear since every compact operator is L-weak-compact.

(2) ⇒ (3): The first part of condition (3) follows from the definition of $L^1(\nu)$. Let σ be a control measure for ν. Observe that

$$\left\| \int_A f\,d\nu \right\| \leq \sup\{\|x^* f \chi_A\|_{L^1(\nu)}\colon x^* \in B_{X^*}\},$$

thus as the operator induced by f is L-weak-compact it follows that $\|\int_A f\,d\nu\| \to 0$ when $\sigma(A) \to 0$, so the measure $\int f\,d\nu$ is countably additive. This was proved in [2].

(3) ⇒ (5): Consider the measurable sets $A_n = \{s \in S\colon \|f(s)\| \leq n\}$. The functions $f\chi_{A_n}$ are strongly measurable and essentially bounded, hence by [1, Theorem 3], they are integrable in the sense of Bartle. It is clear that $(f\chi_{A_n})$ converges to f almost everywhere.

On the other hand, if A is any measurable set then by Lemma 1 we have that $\int_A \phi_\epsilon(f\chi_{A_n}, d\nu) = \int_A f\chi_{A_n}\, d\nu = \int_{A\cap A_n} f\, d\nu$, and the sequence $(\int_{A\cap A_n} f\, d\nu)$ converges to $\int_A f\, d\nu$, being $(A \cap A_n)$ non-decreasing and $\int f\, d\nu$ countably additive by hypothesis. As the measures $\int f\chi_{A_n}\, d\nu$ are countably additive and for every A they converge to $\int_A f\, d\nu$, it follows by [1] and Vitali–Hahn–Saks theorem that f is integrable in the sense of Bartle.

(5) ⇒ (1): As f is integrable Bartle we can get a sequence of X-valued simple functions (f_n) converging almost everywhere to f and such that the sequence of indefinite integrals $\int_A \phi_\epsilon(f_n\, d\nu)$ converges to $\int_A \phi_\epsilon(f, d\nu)$ in $X \tilde{\otimes}_\epsilon Y$ (even uniformly for $A \in \Sigma$). By Lemma 1 we have for $x^* \in X^*$ that $x^*f \in L^1(\nu)$. Let us denote by u_f the operator from X^* into $L^1(\nu)$ associated to f and u_{f_n} the operators associated to f_n in the same way. We have

$$\|u_f - u_{f_n}\| = \sup\left\{\|x^*(f - f_n)\|_{L^1(\nu)} : x^* \in B_{X^*}\right\}$$

$$\leq 4\sup\left\{\left|\int_A x^*(f - f_n)\, dy^*\nu\right| : x \in B_{X^*},\, y^* \in B_{Y^*},\, A \in \Sigma\right\}$$

$$\leq 4\sup\left\{\left\|\int_A f\, d\nu - \int_A f_n\, d\nu\right\| : A \in \Sigma\right\}.$$

As we know that $\int f_n\, d\nu$ converges in the semivariation norm to $\int f\, d\nu$ we get that u_f is limit of the finite rank operators u_{f_n}, in particular it is compact. Observe that (u_{f_n}) is a sequence of weak*-to-weak continuous operators.

(5) ⇒ (4): This follows by Lemma 1.

(4) ⇒ (3): On the basis of Orlicz–Pettis theorem [4] and according to [10, Lemma 1.1] it is enough to show that for every $x \in X^*$ and $y^* \in Y^*$ the scalar measure $(\int f\, d\nu)(x^*, y^*)$ is countably additive, which is plain because for any measurable set A, $(\int_A f\, d\nu)(x^*, y^*) = \int_A x^* f\, dy^*\nu$. ∎

Remark 1. Let us recall that a strongly measurable function $f: S \to X$ is said to be Dunford integrable with respect to a positive measure σ if $x^*f \in L^1(\sigma)$ for all $x^* \in X^*$. The Dunford indefinite integral $\int f\, d\sigma$ is the X^{**}-valued finitely additive vector measure given by $(\int_A f\, d\sigma)(x^*) = \int_A x^* f\, d\sigma$ for every measurable A and every $x^* \in X^*$.

Conditions 3 and 4 in Theorem 2 point out the strong analogy that exists between Pettis integrability and Bartle integrability with respect to the injective tensor product. Indeed, it is well known that the function f is Pettis integrable if and only if its Dunford indefinite integral is countably additive if and only if it takes values into X.

Remark 2. In the proof of Theorem 2 we have shown that if f is Bartle integrable with respect to the injective tensor product, then the operator u_f: $x^* \in X^* \to x^*f \in L^1(\nu)$ induced by f is in fact the limit of a sequence of weak*-to-weak continuous finite rank operators, therefore it belongs to $X \tilde{\otimes}_\epsilon L^1(\nu)$. We shall see in Section 5 that the corresponding condition fails for the projective tensor product.

Let us denote by $P_1(\nu, X)$ the space of (classes of) strongly measurable functions with values in X which are Bartle integrable with respect to ν and ϕ_e, endowed with the norm $\|f\| = \|u_f\|$. Observe that the operator determines the class of the function f, because it is strongly measurable. In view of Remark 2, $P_1(\nu, X)$ can be seen as a non-necessarily closed subspace of $X \tilde{\otimes}_\epsilon L^1(\nu)$. In fact, as the operators induced by X-valued simple functions are dense in $X \tilde{\otimes}_\epsilon L^1(\nu)$, this space can be seen as the completion of $P_1(\nu, X)$.

3. CARLEMAN OPERATORS

Theorem 2 shows that every $f \in P^1(\nu, X)$ is the kernel of the weak*-to-weak continuous compact Carleman operator u_f: $X^* \to L^1(\nu)$. Now we shall prove that, conversely, the kernel of such a Carleman operator is in $P^1(\nu, X)$.

Let us recall that if f: $S \to X^*$ is strongly measurable and for every $x \in X$, $xf \in L^1(\sigma)$, the weak*-integral of f over $A \in \Sigma$ is defined by $(w^* - \int_A f\,d\sigma)(x) = \int_A xf\,d\sigma$.

LEMMA 3. *Let σ be a positive finite measure. Let f: $S \to X^{**}$ be strongly measurable such that for every $x^* \in X^*$, $x^*f \in L^1(\sigma)$. Assume that for every $A \in \Sigma$, the weak*-integral $w^* - \int_A f\,d\sigma \in X$. Then $f(s) \in X$ almost everywhere.*

Proof. Let $A_n = \{s \in S: \|f(s)\| \le n\}$. Then $f\chi_{A_n}$ is Bochner integrable with values in X^{**}. As for every $A \in \Sigma$, $\int_A f\chi_{A_n}\,d\sigma = w^* - \int_A f\chi_{A_n}\,d\sigma = w^* - \int_{A \cap A_n} f\,d\sigma \in X$, it follows that $f(s) \in X$ almost everywhere in A_n. ∎

THEOREM 4. *$P_1(\nu, X)$ can be identified with the space of compact Carleman operators from X^* into $L^1(\nu)$ which are weak*-to-weak continuous.*

Proof. Let u: $X^* \to L^1(\nu)$ be a weak*-to-weak continuous compact Carleman operator. Let us denote by f: $S \to X^{**}$ its strongly measurable kernel. By Theorem 2 it suffices to show that f takes values essentially in X. Let σ be a Rybakov control measure for ν and i the inclusion from $L^1(\nu)$ into $L^1(\sigma)$. As $x^*f = u(x^*) \in L^1(\sigma)$ and $\int_A x^*f\,d\sigma =$

$(i \circ u)^*(\chi_A)(x^*)$, it follows from the weak*-to-weak continuity of $i \circ u$ that $w^* - \int_A f\, d\sigma = (i \circ u)^*(\chi_A) \in X$, and we can apply Lemma 3 to get $f(s) \in X$ almost everywhere. ∎

In order to establish the connection between Carleman operators from X into $L^1(\nu)$ we need the following version of Theorem 2 for Bartle integrable functions with values in a dual space, whose proof is omitted due to its similarity to that of Theorem 2. We just remark that the vector measure we need takes its values in $\mathrm{Bil}(X \times Y^*)$ and that we do not know that the corresponding condition (4) remains valid in this situation. We also observe that in this case, as in the proof of (5) ⇒ (1) in Theorem 2, it can be shown that the operator $x \to xf$ induced by the strongly measurable integrable function is the limit in the uniform norm of operators of a sequence of finite rank operators.

THEOREM 5. *Let $f: S \to X^*$ be a strongly measurable function. The following conditions are equivalent.*

1. *The function f induces an operator $x \to xf$ from X into $L^1(\nu)$ which is compact.*
2. *The function f induces an operator $x \to xf$ from X into $L^1(\nu)$ which is L-weak-compact.*
3. *The function f is Bartle integrable with respect to ν and the bilinear map $\phi_\epsilon: X^* \times Y \to X^* \tilde{\otimes}_\epsilon Y$.*

COROLLARY 6. *$P_1(\nu, X^*)$ can be identified with the space of compact Carleman operators from X into $L^1(\nu)$.*

As a consequence of Theorem 5, we obtain the following Proposition.

PROPOSITION 7. *Let L be an order continuous Köthe function space on a finite measure space (S, Σ, σ). Let $u: X \to L$ be a Carleman operator. Then u is compact if and only if u is L-weak-compact if and only if u is limit of finite rank operators.*

Proof. According to [2], there exists a countably additive vector measure $\nu: \Sigma \to L$ such that $L = L^1(\nu)$. The Proposition then follows from Theorem 5. ∎

Remark. The concept of Carleman operator can be defined for operators $u: X \to L$ when L is an abstract order continuous Banach lattice with weak unit, in such a way that for every representation of L as a Köthe function space, the definition is consistent with that given at the beginning of this section. Indeed, we say that u is Carleman if there exists a sequence (e_k) in L of positive and pairwise disjoint elements such that $\sum_k e_k$ is a weak unit in L and for each k we have $P_{e_k} \circ u: X \to I(e_k)$ is a compact operator, where P_{e_k} is the natural band projection and $I(e_k)$ is the ideal generated by e_k endowed with the norm $\|z\| = \inf\{c > 0: |z| \le c\|e_k\|^{-1}e_k\}$ (see [11]). In this context it is also true that every L-weak-compact Carleman operator is compact.

4. UNCOMPLETENESS OF $P_1(\nu, X)$

Let us recall that a countably additive vector measure ν is said to be non-purely atomic whenever any control measure for ν is not purely atomic. This is equivalent to the lattice $L^1(\nu)$ is not purely atomic.

THEOREM 8. *Let $\nu: \Sigma \to Y$ be a non-purely atomic vector measure and X an infinite dimensional Banach space. Then $P_1(\nu, X)$ is not complete.*

Proof. Let σ be a Rybakov control measure for ν. For each $n \in \mathbb{N}$ we use Dvoretsky–Rogers lemma to find vectors $x_1, \dots, x_m$ in X satisfying $\sum_{k=1}^m \|x_k\| = 1$ and such that $\sup\{\sum_{k=1}^m |x^*(x_k)|: x^* \in B_{X^*}\} \le 1/n$.

Without loss of generality we can suppose that S has measure 1 and such that σ has no atoms in it. Then we choose a partition $(A_1, A_2, \dots, A_m)$ of S such that $\sigma(A_k) = \|x_k\|$ for any $k = 1, \dots, m$. Consider the function

$$f_n(s) = \sum_{k=1}^{m} \frac{x_k}{\|x_k\|} \chi_{A_k}(s).$$

It is plain that $\|f_n(s)\| = 1$ for each $s \in S$. Now, we are going to prove that the sequence f_n tends to 0 in $P_1(\nu, X)$, that is, the sequence of associated operators u_{f_n} tends to 0 in $X \tilde{\otimes}_\epsilon L^1(\nu)$. Let $\epsilon > 0$. Then for any $x^* \in X^*$ we have

$$\|x^* f_n\| \le \|x^* f_n \chi_{[|x^* f_n| \ge \epsilon]}\| + \|x^* f_n \chi_{[|x^* f_n| < \epsilon]}\|,$$

where the norms are taken in $L^1(\nu)$ and $[|x^* f_n| \ge \epsilon]$ stands for the set $\{s \in S: |x^* f_n(s)| \ge \epsilon\}$.

The second summand is less than or equal to $\epsilon\|\chi_S\| = \epsilon\|\nu\|(S)$. To find a bound for the first summand just observe that

$$\sigma\big([|x^* f_n| \ge \epsilon]\big) \le \frac{1}{\epsilon}\|x^* f_n\|_{L^1(\sigma)} \le \frac{1}{\epsilon}\sum_{k=1}^{m} \big|x^*(x_k)\big| \le \frac{1}{n\epsilon}.$$

As ν is countably additive, given $\epsilon > 0$ we can find $\delta > 0$ such that if $\sigma(B) \le \delta$ then $\|\nu\|(B) \le \epsilon$. We choose no such that $1/n_0\epsilon \le \delta$; then for each $n \ge n_0$ as $|x^*(f_n(t))| \le \|f_n(t)\| = 1$, we have that

$$\|x^* f_n \chi_{[|x^* f_n| \ge \epsilon]}\| \le \|\chi_{[|x^* f_n| \ge \epsilon]}\| \le \|\nu\|\big([|x^* f_n| \ge \epsilon]\big) \le \epsilon.$$

This shows that the sequence (f_n) tends to 0 in $P_1(\nu, X)$, although it does not tend to 0 in measure.

Let $L^0(\sigma, X)$ be the F-space of classes of strongly measurable X-valued functions endowed with the topology of the convergence in measure. If $P_1(\nu, X)$ were complete, then the natural inclusion $P_1(\nu, X) \to L^0(\sigma, X)$ would be continuous because it has a closed graph. This would contradict the existence of the sequence f_n built above. ∎

5. THE PROJECTIVE TENSOR PRODUCT CASE

We recall that in Remark 2 following Theorem 2 we obtained that a necessary condition for a strongly measurable function $f\colon S \to X$ is Bartle integrable with respect to ν and $\phi_\epsilon\colon X \times Y \to X \tilde{\otimes}_\epsilon Y$ is that the operator $u_f\colon x^* \to x^* f$ induced by f belongs to $X \tilde{\otimes}_\epsilon L^1(\nu)$. In this section we shall see that the corresponding result for the projective tensor product fails.

Let us denote by $X \tilde{\otimes}_\pi Y$ the completed projective tensor product of X and Y, which is the completed space of $X \otimes Y$, this time with the norm

$$\|u\|_\pi = \inf\left\{ \sum_{k=1}^{n} \|x_k\| \, \|y_k\|\colon u = \sum_{k=1}^{n} x_k \otimes y_k \right\},$$

for $u \otimes X \in Y$. We denote by ϕ_π the bilinear map that associates a pair (x, y) to the tensor $x \otimes y$ in $X \tilde{\otimes}_\pi Y$.

If $u\colon X \to L$ is a bounded linear map and L is an order continuous Banach lattice then it is said that u is order bounded if $u(B_X)$ is an order bounded set in L, that is, there exists $g \in L$, $g \geq 0$, such that $|u(x)| \leq \|x\| g$ for every $x \in X$. The infimum of $\|g\|$ where $g \in L$ satisfies $|u(x)| \leq \|x\| g$, is known as the order bounded norm of u and is denoted by $\|u\|_m$. The space of order bounded operators will be denoted by $\mathscr{B}(X, L)$. The completion of $X \otimes L$ in $\mathscr{B}(X, L)$ under the norm $\|\cdot\|_m$ be denoted by $X \tilde{\otimes}_m L$. If L is an order continuous Köthe function space on a finite measure space (S, Σ, σ) then it is known that $X \tilde{\otimes}_m L$ can be isometrically identified with $L(X)$, the space of strongly measurable functions $f\colon S \to X$ such that $\varphi_f = \|f(\cdot)\| \in L$. See [7, Theorem 22]. We shall use the following lemmas.

LEMMA 9. *Let X be an infinite dimensional Banach space and L a Banach lattice whose dual is order continuous, one of them with the approximation property. Then there exists an operator $u \in X \tilde{\otimes}_m L$ that does not belong to $X \tilde{\otimes}_\pi L$.*

Proof. Suppose that every operator in $X \tilde{\otimes}_m L$ belongs to $X \tilde{\otimes}_\pi L$. As the projective norm is finer than the order bounded norm the open mapping theorem would imply that both norms are equivalent. Then their dual spaces would coincide. It is known [7] that the dual of $X \tilde{\otimes}_m L$ is the Banach space $\mathscr{B}(X, L^*)$. So we would obtain that $\mathscr{L}(X, L^*) = \mathscr{B}(X, L^*)$. This would contradict a result by Robert [13]. ∎

LEMMA 10. *Let L be a reflexive order continuous Banach lattice and let Y be an $\mathscr{L}^1$-space* ([12]). *The following conditions hold.*

1. *For every u: $\ell_2 \to L$ order bounded and every v: $L \to Y$, the composition $v \circ u$: $\ell_2 \to Y$ is nuclear.*

2. *For every v: $L \to Y$ there exists a constant C such that for every order bounded operator u: $\ell_2 \to L$ we have $\|v \circ u\|_\pi \leq C\|u\|_m$.*

Proof. To prove (1), let u: $\ell_2 \to L$ order bounded and let v: $L \to Y$. The operator u factorizes through a $C(K)$-space (the ideal generated by some bound) by means of two operators S_1: $\ell_2 \to C(K)$ and S_2: $C(K) \to L$ such that $u = S_2 \circ S_1$. Observe that the adjoint operator $(v \circ u)^*$ admits the factorization $(v \circ u)^* = S_1^* \circ S_2^* \circ v^*$, hence $(v \circ u)^*$ is nuclear being $v^* \circ S_2^*$ and S_1^* are absolutely 2-summing operators [4, p. 254]. Therefore $v \circ u = (v \circ u)^{**}$ is nuclear.

Let us observe that the space of nuclear operators from ℓ_2 into Y coincides with the projective tensor product $\ell_2^* \hat{\otimes}_\pi Y$. Statement 1 allows us to define the operator $u \in \mathscr{B}(\ell_2, L) \to v \circ u \in \ell_2^* \hat{\otimes}_\pi L$, which is bounded because of the closed graph theorem, and this proves statement 2.

THEOREM 11. *There exists a countably additive vector measure ν: $\Sigma \to L^1([0, \pi])$, and a strongly measurable function f: $S \to \ell_2$ such that f is integrable with respect to ν and ϕ_π but the operator u: $(\ell_2)^* \to L^1(\nu)$ associated to f does not belong to $\ell_2 \tilde{\otimes}_\pi L^1(\nu)$, i.e., is not nuclear.*

Proof. Let Σ be the Borel σ-field in $[0, \pi]$. Let us consider the vector measure ν defined on Σ with values in ℓ_2, given by

$$\nu(A) = \left(\sqrt{\frac{2}{\pi}} \int_A \sin(kt)\, dt \right).$$

This measure is well defined and countably additive because of the orthogonality of the sequence $\sin(kt)$.

The Banach space of integrable functions with respect to ν turns out to be $L^2([0, \pi])$: as ℓ_2 does not contain any copy of c_0 then $L^1(\nu)$ is the set of those measurable functions φ such that $\varphi \in L^1(|x^*\nu|)$ for every $x^* \in \ell_2^*$ [9]; moreover, since $(\sqrt{2/\pi}\, \sin(kt))$ is a complete orthonormal system in $L^2([0, \pi])$ the set $\{|x^*\nu|: x^* \in \ell_2^*\}$ is the positive cone in $L^2([0, \pi])$.

As $L^1([0,\pi])$ contains a subspace isomorphic to ℓ_2 we can consider ν as a measure with values in $L^1([0,\pi])$, obtaining this time that $L^1(\nu)$ is the lattice $L^2([0,\pi])$ with an equivalent norm.

Let us observe that the Lebesgue measure σ is a control measure for ν; in fact, there exists constants $c_1, c_2 > 0$ such that $c_1\sigma(A)^{1/2} \le \|\nu(A)\| \le \|\nu\|(A) \le c_2\sigma(A)^{1/2}$. As the dual of $L^1(\nu)$ is order continuous, by Lemma 9 there exists u in $\ell_2 \tilde{\otimes}_m L^1(\nu)$ which is not in $\ell_2 \tilde{\otimes}_\pi L^1(\nu)$. The operator u has associated a strongly measurable kernel, let us say $f\colon [0,\pi] \to \ell_2$, such that $u(x^*) = x^* f$ for every $x^* \in \ell_2^*$.

Next, we are going to see that f is Bartle integrable with respect to ν and $\phi_\pi\colon \ell_2 \times L^1([0,\pi]) \to \ell_2 \tilde{\otimes}_\pi L^1([0,\pi])$.

First we show that the Bartle semivariation of ν is dominated. Given a finite partition (A_k) of A, (x_k) in B_{ℓ_2}, and $w \in \mathscr{L}(L^1([0,\pi]), \ell_2^*)$, we have

$$\left|\sum_k w(\nu(A_k))(x_k)\right| \le \sum_k \|w \circ \nu(A_k)\| \le K_G\|w\|\,\|\nu\|(A)$$

since w is absolutely summing, where K_G is Grothendieck's constant [12]. It follows that $\|\sum_k \phi_\pi(x_k, \nu(A_k))\|_\pi \le K_G\|\nu\|(A)$ and therefore $\|\nu\|_{\phi_\pi}(A) \le K_G\|\nu\|(A) \le c_2 K_G\sigma(A)^{1/2}$. Conversely, given $x \in \ell_2$ with $\|x\| = 1$, we have $\|\nu\|_{\phi_\pi}(A) \ge \|x \otimes \nu(A)\|_\pi = \|\nu(A)\|_1 \ge c_1\sigma(A)^{1/2}$.

On the basis of the identification between $\ell_2 \tilde{\otimes}_m L^1(\nu)$ and the space of strongly measurable functions $L^1(\nu)(\ell_2)$, as the scalar simple functions are dense in L it is possible to find a sequence of simple functions (f_n) converging to f in the $\|\cdot\|_m$ norm. Then $\|f_n(\cdot) - f(\cdot)\|_{\ell_2} \to 0$ in $L^1(\nu)$, and so in $L^1([0,\pi])$. Therefore, by passing to a subsequence if necessary, we can assume as well that $f_n \to f$ almost everywhere f. Note that the sequence of associated operators u_{f_n} converges in the order bounded norm.

To finish the proof, we only need to show that for any measurable set A, the sequence of integrals $\int_A \phi_\pi(f_n, d\nu) \in \ell_2 \otimes L^1([0,\pi])$ converges in the projective norm.

Let $I_A\colon L^1(\nu) \to L^1([0,\pi])$ the integration operator given by $I_A(\varphi) = \int_A \varphi\, d\nu$. By Lemma 10 we have

$$\begin{aligned}\left\|\int_A \phi_\pi(f_n, d\nu) - \int_A \phi_\pi(f_m, d\nu)\right\|_\pi &= \left\|I_A \circ (u_{f_n} - u_{f_m})\right\|_\pi \\ &\le C\|u_{f_n} - u_{f_m}\|_m,\end{aligned}$$

which tends to zero. ∎

6. FUBINI'S THEOREM FOR THE INJECTIVE PRODUCT OF VECTOR MEASURES

Let Σ and Σ' two σ-algebras of subsets of the sets S and T, respectively. If $\mu: \Sigma \to X$, $\nu: \Sigma' \to Y$ are countably additive vector measures then we can define a finitely additive measure on the algebra $\Sigma \times \Sigma'$ generated by the measurable rectangles, taking $A \times B$ into $\mu \times \nu(A \times B) = \mu(A) \otimes \nu(B) \in X \tilde{\otimes}_{\epsilon} Y$. It is known [14] that this measure has a countably additive extension to the σ-algebra $\Sigma \otimes \Sigma'$ generated by $\Sigma \times \Sigma'$. We shall denote this extension by $\mu \otimes \nu$ and we call it the injective tensor product of μ and ν.

It has been seen in [15] that the classical Fubini theorem does not hold in this setting. Solving a problem posed in [15] we will see that the sections $f(s, \cdot)$ may not be even scalarly integrable with respect to ν. We denote Lebesgue measure on $[0,1]$ by λ.

THEOREM 12. *For every infinite dimensional Banach space X there exist a vector measure ν with values in X and a function f in $L^1(\lambda \otimes \nu)$ such that for every $s \in [0,1)$ the section $f(s, \cdot)$ is not scalarly integrable with respect to ν.*

Proof. We apply the Dvoretsky–Rogers theorem to find a sequence (x_n) in X such that $\sum_{n=1}^{\infty} x_n$ is an unconditionally convergent series and $\|x_n\| = (\sqrt{n} \log n)^{-1}$. We define the countably additive measure $\nu: \mathscr{P}(\mathbb{N}) \to X$ by setting $\nu(\{n\}) = x_n$ for all $n \in \mathbb{N}$, where $\mathscr{P}(\mathbb{N})$ denotes the power set of $\mathbb{N}$. Consider the injective tensor product measure $\lambda \otimes \nu$. Observe that $\mathbb{R} \otimes_{\epsilon} X$ and X are canonically isometric and that for every $x^* \in X^*$ the measure $x^* \circ (\lambda \otimes \nu)$ is in fact the product $\lambda \otimes (x^* \circ \nu)$ and $|\lambda \otimes (x^* \circ \nu)| = \lambda \otimes |x^* \circ \nu|$.

Let $\{A_n\}$ be the sequence of dyadic subintervals of $[0,1]$, $A_1 = [0,1]$, $A_2 = [0, 1/2)$, $A_3 = [1/2, 1)$, $A_4 = [0, 1/4), \dots$. We consider the function $f: [0,1] \times \mathbb{N} \to \mathbb{R}$ given by $f(s,n) = (1/\lambda(A_n))\chi_{A_n}(s)$.

First we see that f is scalarly integrable with respect to $\lambda \otimes \nu$. For, if $x^* \in X^*$ then

$$\int |f(s,n)| d|\lambda \otimes (x^* \circ \nu)|(s,n) = |x^* \circ \nu|(\mathbb{N}) < \infty.$$

In fact f is integrable with respect to $\lambda \otimes \nu$. For, if M is in the product σ-algebra and $M_n = \{s \in [0,1) : (s,n) \in M\}$, then

$$\int_M f \, d\lambda \otimes \nu = \sum_{n=1}^{\infty} \frac{\lambda(A_n \cap M_n)}{\lambda(A_n)} x_n \in X.$$

To finish the proof, we fix $s \in [0,1)$ and let (n_j) be the strictly increasing sequence in $\mathbb{N}$ such that $s \in A_{n_j}$. It is easy to see that $2^{j-1} \le n_j < 2^j$. We have

$$\left\| \frac{1}{\lambda(A_{n_j})} x_{n_j} \right\| \ge \frac{2^{j-1}}{2^{j/2} \log 2^j}$$

which tends to infinity. Thus the series

$$\sum_{j=1}^{\infty} \frac{1}{\lambda(A_{n_j})} x_{n_j}$$

is not weakly unconditionally Cauchy, hence

$$\sum_{j=1}^{\infty} \left| \frac{1}{\lambda(A_{n_j})} x^*(x_{n_j}) \right| = \infty$$

for some $x^* \in X^*$. This allows to conclude that the section $f(s, \cdot)$ is not scalarly integrable with respect to ν because

$$\int_{\mathbb{N}} f(s,n) d|x^* \circ \nu|(n) = \sum_{j=1}^{\infty} \left| \frac{1}{\lambda(A_{n_j})} x^*(x_{n_j}) \right|.$$

∎

Regarding Theorem 12, in order to state a general Fubini theorem for the injective product of two vector measures, we have to start with a function f such that almost all the sections $f(s, \cdot)$ are in $L^1(\nu)$, thus defining a function $S \to L^1(\nu)$. As a consequence of our Theorem 2, we obtain that it is enough to impose this condition.

THEOREM 13. *If $f\colon S \times T \to \mathbb{R}$ is in $L^1(\mu \otimes \nu)$ and $f(s, \cdot)$ belongs to $L^1(\nu)$ for almost every $s \in S$, then*

1. *The function $F\colon s \in S \to \int_T f(s,t)\, d\nu(t) \in Y$ is strongly measurable and Bartle integrable with respect to μ and ϕ_ϵ.*
2. *$\int_S \phi_\epsilon(F(s), d\mu(s)) = \int_{S \otimes T} f \, d\mu \otimes \nu$.*

Proof. Observe that, given a measurable $M \subset S \times T$, the function $\int_T \chi_M(\cdot, t)\, d\nu(t)$ is strongly measurable from S into $L^1(\nu)$, since [9, Theorem 2.2] and χ_M is the pointwise limit on $S \times T$ of a sequence of characteristic functions of unions of rectangles.

Let f_n be a sequence of measurable simple functions on $S \times T$, pointwise convergent to f and such that $|f_n(s,t)| \le 2|f(s,t)|$. By [9, Theorem 2.2] again, we have $f_n(s,\cdot) \to f(s,\cdot)$ in $L^1(\nu)$ for almost every s. The strong measurability of F follows from the fact that $\int_T f_n(s,t)\,d\nu(t)$ goes to $\int_T f(s,t)\,d\nu(t)$ and the previous observation.

Given $x^* \in X^*$ and $y^* \in Y^*$, we have

$$\int_S |y^* \circ F(s)|\,d|x^* \circ \mu|(s) \le \int_S \left(\int_T |f(s,t)|\,d|y^* \circ \nu|(t) \right) d|x^* \circ \mu|(t)$$

$$= \int_{S\times T} |f(s,t)|\,d|x^* \circ \mu \otimes y^* \circ \nu|(s,t) < +\infty.$$

Let $A \subset S$ be measurable. We also have

$$\int_A F\,d\nu = \int_{A\times T} f\,d\mu \otimes \nu \in X \tilde{\otimes}_\epsilon Y \subset \mathrm{Bil}(X^* \times Y^*),$$

because they coincide on $X^* \otimes Y^*$. As $f \in L^1(\mu \otimes \nu)$, statement 1 follows from Theorem 2. Statement 2 follows taking $A = S$ in the previous equality and Lemma 1. ∎

Remark. From Theorem 13, Fubini's theorems for bounded functions [3] or for functions in $L^1(|\mu| \otimes |\nu|)$ (in the case of vector measures with bounded variation) [8] follow.

Finally we shall see that in the case that one of the vector measures is purely atomic, the iterated integration can be done, if one integrates first with respect to the other measure. In the proof we make use of the characterization of Bartle bilinear integrability obtained in Section 2 for the case $\phi = \phi_\epsilon$.

PROPOSITION 14. *Let μ be a purely atomic countably additive vector measure with values in X and let ν be an arbitrary countably additive vector measure with values in Y. Let $f(s,t)$ be a real function, integrable with respect to the injective product $\mu \otimes \nu$. Then*

1. *For almost every s, the section $f(s,\cdot)$ is integrable with respect to ν;*
2. *The function $F\colon s \to \int_T f(s,t)\,d\nu(s) \in Y$ is strongly measurable and Bartle integrable with respect to μ and ϕ_ϵ;*
3. *$\int_S \phi_\epsilon(F, d\mu) = \int_{S\times T} f d(\mu \otimes \nu)$.*

Proof. We can assume that μ is defined on $\mathscr{P}(\mathbb{N})$ and that $\mu(\{n\}) \neq 0$ for every n.

As $f \in L^1(\mu \otimes \nu)$ we know that for any pair $x^* \in X^*$, $y^* \in Y^*$ we have

$$\int \left(\int |f(n,t)|\,d|y^* \circ \nu|(t) \right) d|x^* \circ \mu|(n) < \infty.$$

If we consider any Rybakov control measure for μ, then for all n the sections $f(n, \cdot)$ are scalarly integrable by the classical Fubini theorem.

Next, we show that for every n, $f(n, \cdot) \in L^1(\nu)$. This follows from [9, Theorem 2.2] and the fact that $\mu(\{n\}) \otimes \int_A f(n,t)\, d\nu(t) = \int_{\{n\}\times A} fd(\mu \otimes \nu)$ for every measurable A.

Now, given a sequence $f_j(n, s)$ of simple functions, converging pointwise to f with $|f_j| \leq 2|f|$, it follows from [9, Theorem 2.2] that the simple functions $\int_S f_j(\cdot, s)\, d\nu(s)$ converge to $\int_S f(\cdot, s)\, d\nu(s)$.

Statements 2 and 3 follow from Theorem 2 and the fact that $\int_A F\, d\mu = \int_{A \otimes T} fd(\mu \otimes \nu) \in X \tilde{\otimes}_\epsilon Y$ for every measurable A. ∎

REFERENCES

1. R. G. Bartle, A general bilinear vector integral, *Studia Math.* **XV** (1956), 337–352.
2. G. P. Curbera, Operators into L^1 of a vector measure and applications to Banach lattices, *Math. Ann.* **293** (1992), 317–330.
3. R. Rao Chivukula and A. S. Sastry, Product vector measures via Bartle integrals, *J. Math. Anal. Appl.* **96** (1983), 180–195.
4. J. Diestel and J. J. Uhl, "Vector Measures," Math. Surveys 15, Amer. Math. Soc., Providence, RI, 1977.
5. J. J. Grobler and P. van Eldik, Carleman operators in Riesz spaces, *Indag. Math.* **45** (1983), 421–433.
6. N. E. Gratsky and J. J. Uhl, Carleman and Korotkov operators on Banach spaces, *Acta Sci. Math.* **43** (1981), 207–218.
7. S. Heinrich, N. J. Nielsen, and G. H. Olsen, Order bounded operators and tensor products of Banach lattices, *Math. Scand.* **49** (1981), 99–127.
8. J. E. Huneycutt, Products and convolutions of vector valued set functions, *Studia. Math.* **XLI** (1972), 119–129.
9. D. R. Lewis, Integration with respect to vector measures, *Pac. J. Math.* **33** (1970), 157–165.
10. D. R. Lewis, Conditional weak compactness in certain inductive tensor products, *Math. Ann.* **201** (1973), 201–209.
11. J. Lindenstrauss and L. Tzafriri, "Classical Banach Spaces, vol. II," Ergebnisse der Mathematik und ihre Grenzgebiete, vol. 97, Springer Verlag, Berlin-Heidelberg-New York-Tokio, 1979.
12. G. Pisier, "Factorization of Linear Operators and Geometry of Banach Spaces," CBMS, vol. 60, Amer. Math. Soc., Providence, RI, 1984.
13. D. Robert, Sur les operateurs lineaires qui transforment la boule unité d'un espace de Banach en une partie latticiellement bornée d'un espace de Banach reticulé, *Israel J. Math.* **22** (1975), 354–360.
14. C. Swartz, A generalization of a Theorem of Duchon on products of vector measures, *J. Math. Anal. Appl.* **51** (1975), 621–628.
15. C. Swartz, Fubini's Theorem for tensor product measures, *Rev. Roum. Math. Pures et Appl.* **XXIX** (1984), 97–103.