Approximate Solutions and Error Bounds for Quasilinear Elliptic Boundary Value Problems*

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An error bound for a quasilinear elliptic boundary value problem (including the case of nonlinear differential boundary conditions) is obtained as a positively weighted sum of the absolute defects of the operator equations. Once an approximate solution is computed, using linear programming, by minimizing this error bound over a discrete grid, a corresponding realistic error bound over the whole domain of definition can also be obtained by solving an associated linear program.

1. INTRODUCTION

This paper is concerned with the numerical determination of approximate solutions and error bounds by linear programming for the following quasilinear elliptic boundary value problem:

\[ \begin{align*}
    L_2[u] + g_2(x, y, u) &= r_2(x, y) \quad \text{in } R_2, \\
    Q(g_j): L_1[u] + g_1(x, y, u) &= r_1(x, y) \quad \text{in } R_1, \\
    u &= r_0(x, y) \quad \text{in } R_0,
\end{align*} \]

where \( L_2 = -(a(x, y)(\partial^2/\partial x^2) + b(x, y)(\partial^2/\partial x \partial y) + c(x, y)(\partial^2/\partial y^2)) \) and \( a(\partial^2/\partial x^2) + b(\partial^2/\partial x \partial y) + c(\partial^2/\partial y^2) \) is a linear uniform elliptic differential operator such that \( L_2[c] \geq 0 \) for any constant \( c \). \( L_1 \equiv (\partial/\partial y) \) is an outward non-tangential directional derivative, and \( g_j, j = 1, 2, \) may be nonlinear in \( x, y \) and \( u \). \( R_2 \) is a bounded, simply-connected and open domain in \( E^2 \). Its boundary is composed of two mutually disjoint parts \( R_0 \) and \( R_1 \), each (possibly empty) consisting of a finite number of smooth arcs. Each point of \( R_1 \) lies on the boundary of an open ball which lies entirely in \( R_2 \). (This is the inside sphere property. See Friedman [7, p. 55].) \( r_j \in C^0(R_j), j = 0, 1, 2. \)

In Section 2 some preliminary material is given. In Section 3, an error bound for a given approximate solution to the problem \( Q(g_j) \) is derived, making use of the conditionally inverse-positive property of the operators \( L_1 \) and \( L_2 \) and the local

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Lipschitz conditions of the functions $g_1$ and $g_2$. This error bound is a positively weighted sum of the absolute defects of the operator equations. In Section 4 by means of this error bound, a theoretical constrained minimization problem is formulated by which we can determine an approximate solution and a corresponding error bound. Section 5 discusses the various advantages of such approach. In Section 6, a computational scheme, making use of linear programming, is suggested to solve this constrained minimization problem. Section 7 presents some numerical results and extensions.

Rosen [11] considered a particular case of the problem $Q(g_j)$, in which the quasilinear boundary condition (1.2) is not included. Though our approach is slightly different, his results motivated this research work.

2. PRELIMINARY

**Definition 1.** Let $f$ be a function defined on a set $X \subset \mathbb{R}^2$. Set

$$
\| f \|_X \equiv \inf_{(x, y) \in X} |f(x, y)|.
$$

If there is no ambiguity, the subscript $X$ may be omitted.

**Definition 2.** $R = R_0 \cup R_1 \cup R_2$.

**Definition 3.** $V(\bar{R}) = C^q(R_0) \cap C^4(R_1) \cap C^2(R_2)$.

**Definition 4.** $g_j(u) = g_j(x, y, u); g_j'(u) = (\partial g_j(x, y, u)/\partial u)$.

**Definition 5.** For $j = 1, 2$, suppose $g_j'$ exists and is bounded. For a fixed $v \in V(\bar{R})$ and a constant $\xi > 0$, define

$$
p_j \equiv p_j(\xi, v, x, y) = \min_\eta \{ |g_j'(\eta)| : |\eta - v| \leq \xi \}.
$$
for each \((x, y) \in R_j\), and
\[
\hat{g}_j'(u) = \hat{g}_j(x, y, u)
\]
\[
= \begin{cases} 
  g_j(u) & v + \xi \geq u \geq v - \xi \quad \text{in } R_j, \\
  g_j(v + \xi) + (u - v - \xi)g_j'(v + \xi) & u > v + \xi \quad \text{in } R_j, \\
  g_j(v - \xi) - (v - \xi - u)g_j'(v - \xi) & v - \xi > u \quad \text{in } R_j.
\end{cases}
\] (2.1)

The following lemma is obvious.

**Lemma 1.** Let \(p_j\) and \(\hat{g}_j\) be defined as in Definition 5. Then \(\hat{g}_j\) is differentiable with respect to \(u\), and, for a fixed \((x, y) \in R_j\) and arbitrary \(u \in V(\mathbb{R})\), we have
\[
\hat{g}_j'(u) \equiv \frac{\partial \hat{g}_j(x, y, u)}{\partial u} \geq p_j(\xi, v, x, y) \quad (x, y) \in R_j, \quad j = 1, 2
\] (2.2)
or, equivalently,

\[
\hat{g}_j(x, y, v_1) - \hat{g}_j(x, y, v_2) \geq p_j(\xi, v, x, y)(v_1 - v_2) \quad (x, y) \in R_j, \quad j = 1, 2
\] (2.3)

where \(v_1 \geq v_2\), \(v_1, v_2 \in V(\mathbb{R})\).

**In particular,**

\[
g_j(x, y, v_1) - g_j(x, y, v_2) \geq p_j(\xi, v, x, y)(v_1 - v_2) \quad (x, y) \in R_j, \quad j = 1, 2
\] (2.4)

where \(v_1 \geq v_2\), \(|v_i - v| \leq \xi\), \(v_i \in V(\mathbb{R})\), \(i = 1, 2\).

**Definition 6.** (2.4) is called a one-sided local Lipschitz condition.

**Definition 7.** The problem \(Q(g_j)\) is said to be inverse-positive if, for every \(v_1, v_2 \in V(\mathbb{R})\), we have
\[
L_j[v_1] + g_j(x, y, v_1) \geq L_j[v_2] + g_j(x, y, v_2) \quad \text{in } R_j, \quad j = 1, 2
\]
\[
v_1 \geq v_2 \quad \text{in } R_0 \quad \Rightarrow v_1 \geq v_2 \quad \text{on } \mathbb{R}.
\]

**Notation.** We say \([(L_j + g_j, R_j), j = 1, 2; (I, R_0)]\) is inverse-positive, where \(I\) is an identity operator.

An important feature of an inverse-positive problem is that it can have at most one solution.

The elliptic problem \(Q(g_j)\) is not always inverse-positive. For the linear case, where \(g_j(x, y, u) = uk_j(x, y), j = 1, 2\), the following conditionally inverse-positive property, as a consequence of the maximum principle of Hopf [8] and a theorem by Oleinik [9], is stated in Protter and Weinberger [10, Theorem 13, p. 78].
LEMMA 2. Consider the linear problem $Q(\mathbf{u}_j)$, where $\mathbf{k}_j$ are bounded in $\mathbb{R}_j$, $j = 1, 2$. Let there exist a positive function $\mu \in V(\bar{\mathbb{R}})$ such that

$$(L_j + k_j)[\mu] \geq 0 \quad \text{in} \quad \mathbb{R}_j, \quad j = 1, 2$$

and that the three conditions (i) $(L_1 + k_1)[\mu] \equiv 0$ in $\mathbb{R}_1$; (ii) $(L_2 + k_2)[\mu] \equiv 0$ in $\mathbb{R}_2$; and (iii) $\mathbb{R}_0$ is empty, do not hold simultaneously. Then $Q(\mathbf{u}_j)$ is inverse-positive.

3. DERIVATION OF ERROR BOUND

By means of the local Lipschitz condition (2.4) and the above conditionally inverse-positive property, we can derive an error bound for any approximate solution to $Q(\mathbf{g}_j)$. In fact, we have the following theorem.

THEOREM 3. For the problem $Q(\mathbf{g}_j)$, the following assumptions are made: (1) For any $u \in V(\bar{\mathbb{R}})$, $g_j(u)$ exists and is bounded on $\mathbb{R}_j$ (the closure of $\mathbb{R}_j$), $j = 1, 2$.

(2) Let $\mathbf{g}_j$ be defined in Definition 5 and $Q(\mathbf{g}_j)$ have a solution in $V(\bar{\mathbb{R}})$.

Let $v \in V(\bar{\mathbb{R}})$ and $\xi$, a scalar, be given satisfying the following conditions:

(3) $v \in V(\bar{\mathbb{R}})$ is an approximate solution to $Q(\mathbf{g}_j)$. $\lambda_j$, $j = 0, 1, 2$, are scalars such that

$$\begin{align*}
|\lambda_0| &\geq |L_0[v] - g_j(v) - r_j| \quad \text{in} \quad \mathbb{R}_0, \\
|\lambda_j| &\geq |v - r_j| \quad \text{in} \quad \mathbb{R}_j, \quad j = 1, 2.
\end{align*}$$

(3.1)

(4) For $p_j(\xi, v, x, y)$ as defined in Definition 5 and

$$\bar{p}_j(\xi, v) = \begin{cases} 
\min\{\inf_{(x,y)\in \mathbb{R}_j} p_j(\xi, v, x, y), \lambda_j/\lambda_0\} & \text{if } \lambda_0 \neq 0, \\
\min\{\inf_{(x,y)\in \mathbb{R}_j} p_j(\xi, v, x, y), 0\} & \text{if } \lambda_0 = 0.
\end{cases}$$

there exists a solution $\mu$ to the following system of differential inequalities.

$$\begin{cases}
(L_j + p_j(\xi, v, x, y))[\mu] \geq 1 \quad \text{in} \quad \mathbb{R}_j, \quad j = 1, 2 \\
\mu > 0 \quad \text{on} \quad \bar{\mathbb{R}} \\
\xi \geq \lambda_0 + \mu \sum_{i=1}^{2} (\lambda_i - \lambda_0 \bar{p}_i(\xi, v)) \quad \text{on} \quad \bar{\mathbb{R}}.
\end{cases}$$

(3.3)

Then, there exists exactly one solution $u \in V(\bar{\mathbb{R}})$ of $Q(\mathbf{g}_j)$ such that

$$|u(x, y) - v(x, y)| \leq \rho(x, y) = \lambda_0 + \mu(x, y) \sum_{i=1}^{2} (\lambda_i - \lambda_0 \bar{p}_i) \quad \text{on} \quad \bar{\mathbb{R}}.$$ 

(3.5)
Proof. Consider two fixed but arbitrary functions \( v_j \in V(R), \ j = 1, 2 \). Define \( \bar{g}_j = \bar{g}_j(v_j) \) as in (2.1) and (2.2), \( j = 1, 2 \). We shall first prove that the linear problem

\[
[(L_j + \bar{g}_j, R_j), j = 1, 2; (I, R_0)]
\]

is inverse-positive. In fact, by (2.2) and (3.3), there exists a positive \( \mu \in V(R) \) such that

\[
(L_j + \bar{g}_j)[\mu] \geq (L_j + p_j)[\mu] \geq 1 \quad \text{in} \quad R_j, \quad j = 1, 2.
\]

Hence, follows from Lemma 2, with \( k_j \) replaced by \( \bar{g}_j \), that (3.6) is inverse-positive. Now, let \( u \) be a solution of \( Q(g_j) \). For \( j = 1, 2 \), let \( \bar{g}_j(v) - \bar{g}_j(u) = \bar{g}_j(\bar{v}_j)(v - u) \) where \( \bar{v}_j = \theta_j v + (1 - \theta_j)u, 0 < \theta_j < 1 \). Since \( \bar{g}_j(v) = g_j(v) \), we have

\[
(L_j + \bar{g}_j(\bar{v}_j))[v + p - u] = (L_j + \bar{g}_j)[v] - (L_j + \bar{g}_j)[u] + (L_j + \bar{g}_j(\bar{v}_j))[p]
\]

\[
= (L_j + g_j)[v] - r_j
\]

\[
+ \sum_{i=1}^{2} \left[ (L_j + \bar{g}_j(\bar{v}_j)) \right] \left[ \lambda_i + \mu \sum_{i=1}^{2} (\lambda_i - \lambda_0 \bar{p}_i) \right].
\]

By (3.1), (3.2), (3.7) and (2.2),

\[
(L_j + \bar{g}_j(\bar{v}_j))[v + p - u] \geq -\lambda_j + \left\{ \lambda_0 \bar{g}_j(\bar{v}_j) + \sum_{i=1}^{2} (\lambda_i - \lambda_0 \bar{p}_i) \right\}
\]

\[
\geq -\lambda_j + (\lambda_j + (\lambda_j - \lambda_0 \bar{p}_j))
\]

\[
\geq 0.
\]

On the boundary segment \( R_0 \), (3.1) implies

\[
v + p - u = v + [\lambda_0 + \mu \sum (\lambda_i - \lambda_0 \bar{p}_i)] - u
\]

\[
\geq v + \lambda_0 + 0 - r_0 \geq 0.
\]

Hence, by (3.6), (3.8) and (3.9) we get

\[
v + p \geq u \quad \text{on} \quad R.
\]

Similarly, we have

\[
v - p \leq u \quad \text{on} \quad R.
\]

Next, by (3.4)

\[
|v - u| \leq p \leq \xi \quad \text{on} \quad R.
\]

But then (2.1) implies that \( u \) is also a solution of \( Q(g_j) \).
Lastly, suppose that \( Q(g) \) has two solutions \( u_1 \) and \( u_2 \in V(R) \) satisfying (3.10). For \( j = 1, 2 \), set
\[
\bar{g}_j'(\bar{u}) = g_j(u_1) - g_j(u_2),
\]
where \( \bar{u} = \theta u_1 + (1 - \theta) u_2 \), \( 0 < \theta < 1 \). Then, \( |\bar{u} - v| \leq \xi \),
\[
\hat{g}_j'(\bar{u}) = g_j'(\bar{u}), \quad j = 1, 2,
\]
and
\[
(L_j + \hat{g}_j'(\bar{u}))[u_1 - u_2] = (L_j + g_j)[u_1] - (L_j + g_j)[u_2] = 0 \text{ in } R_j, \quad j = 1, 2 \text{ (3.11)}
\]
\[
\begin{align*}
&u_1 - u_2 = 0 \\
&u_1 - u_2 = 0 \\
&\text{in } R_0.
\end{align*}
\]
Since \( [(L_j + \hat{g}_j, R_j), j = 1, 2; (I, R_0)] \) is inverse-positive, (3.11) has the only solution \( u_1 - u_2 = 0 \) on \( R \).

The basic idea of Theorem 3 is as follows: For \( g_j' \equiv g_j'(\bar{u}) \), if \( [(L_j + g_j', R_j), j = 1, 2; (I, R_0)] \) is not inverse-positive in the whole domain of \( u \), we want to find a set \( Z = \{ u \mid v - \rho \leq u \leq v + \rho, (x, y) \in R \} \) in which it has this property (i.e., locally inverse-positive), and, hence, \( Q(g_j) \) has at most one solution in \( Z \). This is done by first taking an approximate solution \( v \) as the 'center' of \( Z \). If \( \rho \) is determined in Theorem 3, then it can be a possible 'radius' of \( Z \). The constrained minimization problem formulated in the next section is devised for finding the 'smallest possible' \( \rho \) such that \( Z \) may 'trap' a solution of \( Q(g_j) \).

4. CONSTRAINED MINIMIZATION PROBLEM

In this section, we derive from the error bound formula (3.5) of the last section a constrained minimization problem. In solving this problem by some numerical techniques and linear programming method, we can obtain an approximate solution and the corresponding error bound for the problem \( Q(g_j) \).

Notations

For a given function \( v^k \), define
\[
g_j(v^k) = g_j(x, y, v^k);
q_j^k := g_j'(v^k) \quad \text{and} \quad \frac{\partial g_j(x, y, v^k)}{\partial u};
\]
and
\[
G_j^k := v^k q_j^k - g_j(v^k) - r_j.
\]

Constrained minimization problem

Suppose \( Q(g_j) \) satisfies the Assumptions (1) and (2) of Theorem 3. Given suitable initial approximation \( v^0 \) and constants \( \xi^0, \hat{\rho}^0 \) and \( \bar{\rho}^0 \), \( j = 1, 2 \), the \( k \)th cycle of the following iterative process starts with known \( v^{k-1}, \xi^{k-1}, \hat{\rho}^{k-1} \) and \( \bar{\rho}^{k-1} \):
Step 1. Let \( \psi^k(x, y) \in V(R) \) and \( \delta^k_j, j = 0, 1, 2 \), solve
\[
\min_{\psi, \delta_j} \left\{ \left[ 1 - \tilde{\mu}^{k-1} \sum_{j=1}^2 \tilde{\delta}^{j-1} \right] \delta_0 + \tilde{\mu}^{k-1} \sum_{j=1}^2 \delta_j \right\},
\]
subject to
\[
\begin{align*}
\delta_j &\geq (L_j + q_j^{k-1})[\psi] - G_j^{k-1} \geq -\delta_j, \quad \text{in } R_j, \quad j = 1, 2, \\
\delta_0 &\geq v - r_0 \geq -\delta_0, \quad \text{in } R_0.
\end{align*}
\]

Step 2. For \( j = 1, 2 \), evaluate
\[
\lambda_0^k = \| v^k - r_0 \|_{R_0}, \quad \lambda_j^k = \| L_j[v^k] + g_j(v^k) - r_j \|_{R_j},
\]
\[
\rho_j^k(x, y) = \rho_j(\xi^{k-1}, v^k, x, y) = \min \{ g_j(x) \mid g_j \leq \xi^{k-1} \} \text{ for fixed } (x, y)
\]
and
\[
\tilde{\rho}_j^k = \begin{cases} 
\min \{ \inf_{(x, y) \in R_j} \rho_j^k(x, y), \lambda_j^k/\lambda_0^k \} & \text{if } \lambda_j^k \neq 0, \\
\inf_{(x, y) \in R_j} \rho_j^k(x, y) & \text{if } \lambda_j^k = 0.
\end{cases}
\]

Step 3. Let the scalar \( \tilde{\mu}^k \) and the function \( \mu^k \in V(R) \) solve
\[
\min_{\mu, \delta} \left\{ \tilde{\mu} \mid (L_j + \rho_j^k(x, y)) \mu \geq 1 \text{ in } R_j, \quad j = 1, 2 \right\}.
\]

Set \( \xi^k = \lambda_0^k + \tilde{\mu}^k \sum_{j=1}^2 (\lambda_j^k - \lambda_0^k \tilde{\rho}_j^k) \).

Theorem 4. In the above iterative process, if at the \( k \)-th cycle we have \( \xi^k \leq \xi^{k-1} \),
then the error bound (3.5) holds with
\[
\rho(x, y) = \lambda_0^k + \mu^k(x, y) \sum (\lambda_j^k - \lambda_0^k \tilde{\rho}_j^k(\xi^k, v^k)),
\]
where \( \tilde{\rho}_j(\xi, v) = \inf_{(x, y) \in R_j} \rho_j(\xi, v, x, y) \).

Proof. Clearly we have only to show that (3.3) and (3.4) hold for \( \xi = \xi^k, v = v^k \)
and \( \mu = \mu^k \).

Since \( \tilde{\mu}^k \geq \mu^k \geq 0 \), and, for fixed \( v^k \) and \( (x, y) \), \( \rho_j \) and \( \tilde{\rho}_j \) are both monotone
nonincreasing functions of \( \xi \), it follows from Step 3 that
\[
(L_j + \rho_j(\xi^k, v^k, x, y)) \mu \geq 1 \text{ in } R_j, \quad j = 1, 2,
\]
\[
\xi^k = \lambda_0^k + \tilde{\mu}^k \sum (\lambda_j^k - \lambda_0^k \tilde{\rho}_j(\xi^{k-1}, v^k))
\geq \lambda_0^k + \mu^k(x, y) \sum (\lambda_j^k - \lambda_0^k \tilde{\rho}_j(\xi^k, v^k)) \text{ on } R.
5. Discussion

(1) By (4.1) we see that the approximate solutions $v^k$ are obtained by minimizing a positively weighted sum of the absolute defects of the linearized differential equations and the identity boundary equation. Also, by its structure, the error bound $p$ is a monotonic decreasing function of each of the quantities $\lambda_j$ and $\mu(x, y)$. $\lambda_j$ (see (4.2)) are absolute defects of the operator equations. $\mu(x, y)$ (see (4.5)) is obtained by minimizing an upper bound. Hence, we may say that the approximate solution is obtained by minimizing its error bound in certain sense.

A similar error bound, which depends on the maximum of the absolute defects instead of a sum of them, can also be derived. But it can be shown that the error bound discussed in this paper is more realistic. See Cheung [4].

(2) Suppose, instead of the local Lipschitz condition (2.4), $g_j$ satisfies a global Lipschitz condition, i.e., there exists a bounded function $k_j$, independent of $u$, such that

$$g_j(v_1) - g_j(v_2) \geq (v_1 - v_2)k_j \text{ in } R_j, \text{ for all } v_1 \geq v_2, v_i \in V(R), \quad i = 1, 2.$$  

(5.1)

Theorem 3 and the constrained minimization problem of Section 4 still hold, with $p_j = k_j$. However, a local Lipschitz condition has at least two advantages:

(i) The global Lipschitz condition (5.1) may not exist; whereas, we can always construct a local Lipschitz condition provided $g_j'(u)$ is bounded for every $u$. For example, consider $g(x, y, u) = e^{-u}$. Then no global Lipschitz condition exists, but

$$g(x, y, v_1) - g(x, y, v_2) \geq -e^{-(x-y)}(v_1 - v_2),$$

where $v_i \in V(R), | v_i - v | \leq \xi, i = 1, 2$ and $v_1 \geq v_2$.

(ii) A local Lipschitz condition gives a 'better' error bound than a global Lipschitz condition. In order to show this, we first prove the following lemma.

**Lemma 5.** Let $\hat{\mu}^i$ be the minimum value of the minimization problem:

$$Q^i: \min_{\mu, \hat{\mu}} \{ \mu \mid (L_j + k_j)[\mu] \geq 1 \text{ in } R_j, \quad j = 1, 2 \}$$

$$\hat{\mu} \geq \mu \geq 0 \text{ on } R,$$

(Notice that the superscript $i$ does not imply iteration here.) If $k_j^1 \geq k_j^2, j = 1, 2$, then $\hat{\mu}^1 \leq \hat{\mu}^2$.

**Proof.** Let $\mu^2$ be a minimizing function of $Q^2$. Then

$$\{(L_j + k_j)[\mu^2] \geq (L_j + k_j^2)[\mu^2] \geq 1 \text{ in } R_j, \quad j = 1, 2, \}$$

$$\hat{\mu}^2 \geq \mu^2 \geq 0 \text{ on } R,$$

i.e., $\hat{\mu}^2$ and $\mu^2$ also satisfy the constraints of $Q^1$. It follows that $\hat{\mu}^1 \leq \hat{\mu}^2$. \[\square\]
Now, for $j = 1, 2$, let $k_j^1 = p_j(\xi, v, x, y)$ be the local Lipschitz function as defined in (2.4) and $k_j^2 = k_j$ be the global Lipschitz function as defined in (5.1). Obviously, $p_j \geq k_j$. It follows from Lemma 5 that $\mu^1 \leq \mu^2$. Generally, we may hope that $\mu^1 \leq \mu^2$ on $R$ also. This implies a better error bound for local Lipschitz condition (see (4.6)).

6. Computational Method

In this section, we consider a computational method of solving the constrained minimization problem of Section 4. Let $\{\phi_i(x, y)\}_{i=1}^m \subset V(R)$ be a set of suitably chosen functions. Assume the function $\mu$ and the approximate solution $v$ to be of the form

$$\mu(\beta; x, y) = \sum_{i=1}^m \beta_i \phi_i(x, y) \quad \text{and} \quad v(x, y) = \sum_{i=1}^m \alpha_i \phi_i(x, y).$$

Let $D_j$ and $D_j^*$ be two discretizations of the region $R_j$, where $D_j^*$ has finer grid sizes than $D_j$ (Fig. 2).

**Figure 2**

**Computational Method**

Given suitable initial approximation $v^0$ and constants $\xi^0$, $\mu^0$ and $p_j^0$, the $k$th cycle of the following iterative process starts with known $\xi^{k-1}$, $\mu^{k-1}$, $p_j^{k-1}$ and $v^{k-1}(x, y) = \sum \alpha_i^{k-1} \phi_i(x, y)$.

**Step 1.** Let $\alpha_i^k$, $i = 1, \ldots, m$, and $\delta_j^k$, $j = 0, 1, 2$, solve

$$\min_{\alpha_i, \delta_j} \left[ \left| 1 - \mu^{k-1} \sum_{j=1}^2 \frac{\delta_j^{k-1}}{\delta_j} \right| \delta_0 + \mu^{k-1} \sum_{j=1}^2 \delta_j \right]$$

subject to

$$\delta_j \geq \sum \alpha_i^k (L_j + q_j^{k-1}) \phi_i - G_j^{k-1} \geq -\delta_j \text{ in } D_j, \quad j = 1, 2,$$

$$\delta_0 \geq \sum \alpha_i^k \phi_i - r_0 \geq -\delta_0 \text{ in } D_0.$$
Step 2. For $j = 1, 2$, evaluate

$$\lambda_0^k = \|v^k - r_0\|_{D_0^*}, \quad \lambda_j^k = \|L_j[v^k] + g_j(v^k) - r_j\|_{D_j^*}$$

$$p_j^k(x, y) \equiv p_j(\xi^{k-1}, v^k, x, y) \approx \min_{\eta} \{g_j'(\eta) \mid |\eta - v^k| \leq \xi^{k-1}\} \text{ for fixed } (x, y)$$

and

$$p_j^k(x, y) = \begin{cases} 
\min_{(x,y) \in D_j^*} \{p_j^k(x, y), \lambda_j^k/\lambda_0^k\} & \text{if } \lambda_0^k \neq 0, \\
\min_{(x,y) \in D_j^*} p_j^k(x, y) & \text{if } \lambda_0^k = 0.
\end{cases}$$

Step 3. Choose suitable small positive constants $\epsilon_j, j = 0, 1, 2$. Let $\beta_i^k$, $i = 0, 1, \ldots, m$, solve

$$\begin{align*}
\min_{\beta_0} & \left\{ \sum_{i=1}^m \beta_i(L_j + p_j^k(x, y))[\phi_i] \geq 1 + \epsilon_j \text{ in } D_j, \quad j = 1, 2 \right. \\
& \left. \beta_0 \geq \sum_{i=1}^m \beta_i \phi_i \geq \epsilon_0 \quad \text{on } D = D_0 \cup D_1 \cup D_2 \right\} \quad (6.2)
\end{align*}$$

Let

$$\tilde{\beta}^k = \max_{(x,y) \in D^*} \sum_{i=1}^m \beta_i^k \phi_i$$

and

$$\xi^k = \lambda_0^k + \tilde{\beta}^k \sum_{j=1}^2 (\lambda_j^k - \lambda_0^k \tilde{p}_j^k),$$

where $D^* = D_0^* \cup D_1^* \cup D_2^*$.

Choice of Initial Values for the Parameters $\xi^0, \tilde{\beta}^0, \tilde{p}_j^0$ and Approximation $v^0$

By Theorem 4, we see that the initial $\xi^0$ should overestimate the error of the initial approximation $v^0$. Usually if $\xi^0$ is large enough we should have $\xi^0 \geq \xi^*$, and, hence, an error bound is obtained at a single cycle. However, difficulty may arise if $\xi^0$ is too large, (6.2) may have no feasible solution.

Without better values, we may set $\beta_0 = 1, \beta_0 = 0, j = 1, 2$ and $v^0(x, y) \equiv 0$ (or 1) and $\xi^0 = 1$.

Choice of the Parameters $\epsilon_j, j = 0, 1, 2$

In (6.2), we add the positive quantities $\epsilon_j$ to the right sides. If the density of discretization is fine enough and the differential operators satisfy some Lipschitz conditions, it can be shown (for detail, see Cheung [4]) that a solution $\mu = \sum \beta_i \phi_i$ of the discretized problem (6.2) also satisfies the inequalities (4.5) over the whole region. Therefore, the error bound is valid over $\tilde{R}$ instead of over $D^*$ only.
Criterion for Terminating the Iterative Process

The iterative process may stop whenever $\xi^k \leq \xi^{k-1}$. However, in practice this is usually satisfied at the first cycle. To obtain better accuracy, we may use the following criterion:

Let

$$\delta^k = \| L_2[v^k] + q^{k-1}v^k - G^{k-1} \|_D $$

and

$$\lambda^k = \| L_2[v^k] + s_2(v^k) - r_2 \|_{D^*} $$

For a preassigned quantity $\epsilon_N$ (convergence tolerance of Newton's method), the iterative process is stopped at the $k$th cycle when

$$\frac{|\lambda^k - \delta^k|}{\lambda^k} \leq \epsilon_N.$$

Since $\lambda^k$ and $\delta^k$ are quantities obtained during the iterative process, only little additional computation is required.

Linear Programming Formulation

It is easy to show that both (6.1) and (6.2) are linear programs of the form (for detail, see Rosen [11]):

$$\min_{\pi} \{d'\pi \mid A'\pi \geq -c\}, \quad (6.3)$$

where $d$, $\pi$ and $c$ are vectors and $A$ is a matrix. denotes the transpose. With $w = -\pi$, (6.3) is equivalent to the dual problem of a standard linear program (see Dantzig [6]):

$$\max_{w} \{d'w \mid A'w \leq c\}. \quad (6.4)$$

Instead of solving (6.4) directly, most available computer linear programming code (e.g. SIMPLEX [13]) are derived so as to solve its primal problem

$$\min_{z} \{c'z \mid Az = d, z \geq 0\}. \quad (6.5)$$

(6.3) and (6.5) have the following relations which are well known in the duality theory of linear programming:

(i) If (6.5) has an optimal solution $z^*$ with optimal basis $B$, then $\pi^* = -w^* - -(B')^{-1}z^*$ is an optimal solution to (6.3). In some computer linear programming codes, $\pi^*$ is one of the output data. Hence, we can directly obtain an optimal solution to (6.3) by solving (6.5).
(ii) If \((6.5)\) has an infinite (negative) solution, \((6.3)\) has no feasible solution. This fact can be used to test the inverse-positive property of the given problem \(Q(g_j)\).

**Sizes of the Linear Programs**

For the linear program \((6.1)\), the dimension of the matrix \(A\) is \((m + 3) \times 2n\), where \(m\) is the number of basis functions \(\{\phi_i\}_{i=1}^{m}\) and \(n\) is the total number of grid points over the three meshes \(D_j\), \(j = 0, 1, 2\).

For the linear program \((6.2)\), the dimension of \(A\) is \((m + 1) \times (3n - n_0)\), where \(n_0\) is the number of grid points on \(D_0\).

7. **Extension and Numerical Results**

**Extension**

Throughout our previous discussions, it was assumed that there is only one boundary differential operator. In fact we can consider the more general case where there are \((J - 1)\) of them. Let the interior elliptic operator be defined over a bounded simply-connected open domain \(R_j\). Then, by similar argument, we have the error bound

\[
\rho(x, y) = \lambda_0 \cdot \mu(x, y) \sum_{j=1}^{J} (\lambda_j - \lambda_0 \bar{\phi}_j).
\]

Obviously, all previous results can be generalized.

Extensions to parabolic and hyperbolic problems had been considered by Cheung [4].

**Example.** (mixed BVP on square domain).

\[
\begin{cases}
-Au + g_2(u) = r_3 & 0 < x < 0.6, \quad 0 < y < 0.6, \quad R_3, \\
- (\partial u/\partial x) + g_4(u) = r_2 & x = 0, \quad 0 < y < 0.6, \quad R_2, \\
(\partial u/\partial y) + g_3(u) = r_1 & 0 < x < 0.6, \quad y = 0, \quad R_1, \\
u = e^{0.1x} \cos x & 0 \leq x \leq 0.6, \quad y = 0, \quad R_0, \\
u = e^{0.06} \cos(0.6 + y) & x = 0.6, \quad 0 \leq y \leq 0.6, \quad R_0,
\end{cases}
\]

where \(g_1 = u, g_2 = 0.002 \cos u, g_3 = -0.002yu^2\);

\[
r_1 = e^{0.1x}(\cos(x + 0.6) - \sin(x + 0.6));
\]

\[
r_2 = \sin y - 0.1 \cos y + 0.002 \cos(\cos y);
\]

and

\[
r_3 = (1.99 - 0.002 ye^{0.1x} \cos(x + y)) e^{0.1x} \cos(x + y) + 0.2 e^{0.1x} \sin(x + y).
\]
**Exact solution** \( u = e^{0.1x} \cos(x + y) \)

**Algorithm**

It is not known whether this problem is inverse-positive. The computational method discussed in Section 6 is applied, starting with \( v = 1, \xi_0 = 0.4, \rho_0 = 1 \) and \( \hat{p}_2^0 = \hat{p}_3^0 = \hat{p}_4^0 = 0. \) \( \| \cdot \|_{D_j} \) is approximated by \( \| \cdot \|_{D_j^*} \), where \( D_j^*, j = 0, 1, 2, \) are uniform meshes on the boundary all with the same grid size 0.0125, and \( D_3^* \) is an interior uniform square mesh with grid size 0.025. The termination criterion is applied with \( \epsilon_n = 0.0016 \).

**Discretization method**

The iterative linear programs (6.1) and (6.2) are solved (see Section 6) with \( \epsilon_j = 0.0001, j = 0, 1, 2, 3 \). \( D_j^* \) is the same as \( D_j^*, j = 0, 1, 2, \) but has grid size 0.025. \( D_3^* \) is the same as \( D_3^* \) but has grid size 0.075.

**Function space**

\( S^{(3+4, 3+4)} \) (bicubic elementary splines with 5 knots in both the \( x \)- and \( y \)-directions. See Appendix.)

**Computer and LP code**

CDC 3600, RS MSUB (Clasen [5]).

**Time** (including compilation time)

3 minutes 50 seconds.

### TABLE I

Coefficients of Approximate Solution

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>( x )</th>
<th>( x^2 )</th>
<th>( (x)^2_+ )</th>
<th>( (x-0.15)^2_+ )</th>
<th>( (x-0.30)^2_+ )</th>
<th>( (x-0.45)^2_+ )</th>
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<td>0.09999</td>
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<td>-0.03230</td>
<td>0.00940</td>
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<td>-9.26311</td>
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<td>$u - v$</td>
<td>$\rho$</td>
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</tr>
</tbody>
</table>

* grid points;  $^o$ max error;  $^{*}$ max error bound;  $^{*}$ min error bound. 
Numerical results

After a single iteration, the termination criterion is satisfied and the following results are obtained (omitting superscripts):

\[ \lambda_0 = 9.00 \times 10^{-5}, \quad \lambda_1 = 8.29 \times 10^{-4}, \quad \lambda_2 = 1.03 \times 10^{-5}, \quad \lambda_3 = 3.98 \times 10^{-3}, \]
\[ \bar{p}_1 = 1.14 \times 10^{-1}, \quad \bar{p}_2 = -2.00 \times 10^{-3}, \quad \bar{p}_3 = -3.37 \times 10^{-3}, \quad \xi = 4.89 \times 10^{-3}. \]

Each entry in Table I is the coefficient of a basis function which is the product of the corresponding functions at the top row and on the leftmost column.

Table II shows the values of the approximate solution, the actual error and error bound at some points distributed fairly uniformly over the whole region. Figure 3 shows the errors along 4 horizontal lines at equal distance. The error curve for \( y = 0 \) oscillates as expected for uniform approximation to boundary data. This is not true for the other 3 curves since we only minimize the absolute defects of the differential equations there.

Appendix

Monovariate Splines

In the following we consider the space of spline functions of degree \( m \) with \( n + 1 \) knots in terms of the basic splines defined by

\[ (x)_+^m = \begin{cases} x^m & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases} \]
Elementary Splines

Let \( x_0, x_1, \ldots, x_n \) be a set of knots over \([x_0, x_n]\). An arbitrary spline of degree \( m \) is given by

\[
\sum_{i=0}^{m-1} a_i \phi_i(x) = \sum_{i=0}^{n-1} \alpha_{m+i}(x - x_i).^m.
\]

This space is of dimension \( m + n \).

Given a function \( \theta(x) \in C^m([0, a]) \), it was proved in Cheung [3, Theorem 1.2, pp. 8–10] that there exists a sequence of splines of fixed degree \( m \) and with uniform knots which converges (as the number of knots tends to infinity) to \( \theta(x) \) together with its derivatives up to the order \( m \). This is the denseness property of splines. On the other hand, the evaluation, differentiation, and integration of splines are simple. Also, when \( m \) is small, evaluation of the values of the functions and their derivatives usually will not lead to large relative round-off errors. (For references, see Ahlberg, Nilson, and Walsh [1] and Schoenberg [12].)

Bivariate Splines

A convenient way of obtaining splines of two variables is to form products of one-dimensional splines. Let \( \{\phi_i(x)\}_{i=1}^{m+n} \) be a basis for a space of dimension \( m + n \) and \( \{\varphi_j(y)\}_{j=1}^{M+N} \) be a basis for another space of dimension \( M + N \), then \( \{\phi_i(x) \varphi_j(y)\}_{i=1}^{m+n} \{j=1}^{M+N} \) would form a basis for the product space of dimension \((m + n) \cdot (M + N)\).

Notations. If \( \phi_i \) and \( \varphi_j \) are elementary splines the product space is denoted by \( S_{(m+n, M+N)}(\mathbb{R}) \).

Bivariate splines had been used to approximate the solutions of partial differential equations by Birkhoff, Schultz and Varga [2].

References

3. T. Y. Cheung, Spline approximation of the Cauchy problem \((\partial^m u/\partial x^m \partial y^n) = f(x, y, u, \ldots)\), Computer Sci. Tech. Report #29, Univ. of Wisconsin, Madison, 1968.