



# The Dixmier problem, lamplighters and Burnside groups

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## Abstract

J. Dixmier asked in 1950 whether every non-amenable group admits uniformly bounded representations that cannot be unitarised. We provide such representations upon passing to extensions by abelian groups. This gives a new characterisation of amenability. Furthermore, we deduce that certain Burnside groups are non-unitarisable, answering a question raised by G. Pisier.

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## 1. Introduction

A group  $G$  is said to be *unitarisable* if every uniformly bounded representation  $\pi$  of  $G$  on a Hilbert space  $\mathcal{H}$  is unitarisable, i.e. there is an invertible operator  $S$  on  $\mathcal{H}$  such that  $S\pi(\cdot)S^{-1}$  is a unitary representation. Dixmier [3] proved that all amenable groups are unitarisable and asked whether unitarisability characterises amenability. Since unitarisability passes to subgroups and non-commutative free groups are not unitarisable, every group containing a non-commutative free group is non-unitarisable. For these facts and more background, we refer to Pisier [10,11].

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Recently, a criterion was discovered [4] that lead to examples without free subgroups (see [4,9]). We shall improve a strategy proposed in [7] in order to apply ergodic methods to the problem.

*Now are our brows bound with Victorious Wreathes*<sup>3</sup>

Let  $G$  and  $A$  be groups. Recall that the associated (restricted) *wreath product*, or *lamplighter group*, is the group

$$A \wr G = \bigoplus_G A \rtimes G,$$

wherein  $\bigoplus_G A$  is the restricted product indexed by  $G$  upon which  $G$  acts by permutation. We shall be interested in the case where  $A$  and hence also  $\bigoplus_G A$  is abelian.

**Theorem 1.** *For any group  $G$ , the following assertions are equivalent.*

- (i) *The group  $G$  is amenable.*
- (ii) *The wreath product  $A \wr G$  is unitarisable for all abelian groups  $A$ .*
- (iii) *The wreath product  $A \wr G$  is unitarisable for some infinite abelian group  $A$ .*

The above theorem leads to a partial answer to a question of G. Pisier, namely whether free Burnside groups are unitarisable (see e.g. [11]).

**Theorem 2.** *Let  $m, n, p$  be integers with  $m, n \geq 2$ ,  $p \geq 665$  and  $n, p$  odd. Then the free Burnside group  $B(m, np)$  of exponent  $np$  with  $m$  generators is non-unitarisable.*

## 2. Proofs

Let  $G$  be a group and  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ . We write  $\mathcal{L}(\mathcal{H})$  for the algebra of bounded operators of  $\mathcal{H}$ . A map  $D : G \rightarrow \mathcal{L}(\mathcal{H})$  is called a *derivation* if it satisfies the Leibniz rule  $D(gh) = D(g)\pi(h) + \pi(g)D(h)$ , or equivalently if the map  $\pi_D$  defined by

$$\pi_D(g) = \begin{pmatrix} \pi(g) & D(g) \\ 0 & \pi(g) \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$$

is a group homomorphism. In that case,  $\pi_D$  is a uniformly bounded representation if and only if  $D$  is a bounded derivation. Moreover,  $\pi_D$  is unitarisable if and only if  $D$  is inner, i.e. there is  $T \in \mathcal{L}(\mathcal{H})$  such that  $D(g) = \pi(g)T - T\pi(g)$ . (See Lemma 4.5 in [10] for a proof of this fact.) To set up a cohomological framework for studying this problem, we will view  $\mathcal{L}(\mathcal{H})$  as a coefficient  $G$ -module whose  $G$ -action is given by the conjugation  $g \cdot T = \pi(g)T\pi(g)^*$ . Then, the space of bounded derivations modulo inner derivations is canonically isomorphic to the first bounded cohomology group  $H_b^1(G, \mathcal{L}(\mathcal{H}))$ . Hence, to prove non-unitarisability of  $G$ , it suffices to produce a unitary  $G$ -representation  $(\pi, \mathcal{H})$  for which  $H_b^1(G, \mathcal{L}(\mathcal{H})) \neq 0$ .

<sup>3</sup> Shakespeare, *Richard III*, 1:1 (we quote from the 1623 *First Folio*).

We now undertake the proof of Theorem 1. It suffices to show that if  $A$  is infinite abelian and  $G$  is non-amenable, then the wreath product  $H = A \wr G$  is non-unitarisable.

We can and shall assume that  $A$  and  $G$  are countable. Indeed, since amenability is preserved under direct limits,  $G$  contains some countable non-amenable group  $G_0$ . Further, if  $A_0$  is a countable subgroup of  $A$ , then  $A_0 \wr G_0$  is a subgroup of  $A \wr G$ . Thus our claim follows since unitarisability passes to subgroups.

Let  $\mathbf{F}$  be a countable non-commutative free group. The proof relies on the following two facts. (1)  $H_b^1(\mathbf{F}, \mathcal{L}(\ell_2\mathbf{F})) \neq 0$ , see the proof of Theorem 2.7\* in [10], or [2]. (2) Every non-amenable countable group admits a free type  $\text{II}_1$  action whose orbits contain the orbits of a free  $\mathbf{F}$ -action [5], as described below. The strategy of the proof is to induce  $H_b^1(\mathbf{F}, \mathcal{L}(\ell_2\mathbf{F}))$  through this “randembedding” in the sense of [7].

We henceforth consider a non-amenable countable group  $G$  and the corresponding Bernoulli shift action on the compact metrisable product space  $X = [0, 1]^G$  endowed with the product of the Lebesgue measures. Gaboriau and Lyons prove in [5] that the resulting equivalence relation  $\mathcal{R} \subseteq X \times X$  contains the equivalence relation of some free measure-preserving  $\mathbf{F}$ -action upon  $X$ . In particular, we have commuting  $G$ - and  $\mathbf{F}$ -actions on  $\mathcal{R}$  given by the action on the first, respectively the second coordinate. These actions preserve the  $\sigma$ -finite measure on  $\mathcal{R}$  provided by integrating over  $X$  the counting measure on orbits. Each of these actions admits a fundamental domain; let  $Y \subseteq \mathcal{R}$  be a fundamental domain for  $\mathbf{F}$ . We may now forget the orbit equivalence relation and view  $\mathcal{R}$  just as a standard measure space with a measure-preserving  $G \times \mathbf{F}$ -action such that  $G$  admits a fundamental domain  $X$  of finite measure and  $\mathbf{F}$  admits a fundamental domain  $Y$ . We identify  $\mathcal{R}$  with  $Y \times \mathbf{F}$  in such a way that  $t^{-1}y \in \mathcal{R}$  corresponds to  $(y, t) \in Y \times \mathbf{F}$ . Then,  $s \in \mathbf{F}$  acts on  $Y \times \mathbf{F}$  by  $s(y, t) = (y, ts^{-1})$  and  $g \in G$  acts by  $g(y, t) = (g \cdot y, \alpha(g, y)t)$ , where  $g \cdot y \in Y$  is the (essentially) unique element in  $\mathbf{F}gy \cap Y \subset \mathcal{R}$  and  $\alpha(g, y) \in \mathbf{F}$  is the (essentially) unique element such that  $\alpha(g, y)gy = g \cdot y$ . It follows that  $\alpha$  satisfies the cocycle relation  $\alpha(gh, y) = \alpha(g, h \cdot y)\alpha(h, y)$ .

We now consider any countable infinite abelian group  $A$ . We claim that  $A$  has a representation into the unitaries of the von Neumann algebra  $L^\infty(Y)$  whose image generates  $L^\infty(Y)$  as a von Neumann algebra. By construction,  $Y$  is a standard Borel space with a  $\sigma$ -finite non-atomic measure. Furthermore, as far as the present claim is concerned, we may temporarily assume this measure finite since only its measure class is of relevance. Since  $A$  is countably infinite, its Pontryagin dual  $\widehat{A}$  (for  $A$  endowed with the discrete topology) is a non-discrete compact metrisable group. In other words, we have reduced to the case where we may assume that  $Y$  is  $\widehat{A}$  endowed with a Haar measure. Fourier transform establishes an isomorphism between  $L^\infty(\widehat{A})$  and the group von Neumann algebra  $L(A) \subseteq \mathcal{L}(\ell_2 A)$ , which is by definition generated by the unitary regular representation of  $A$ ; this proves the claim.

Returning to the main argument, we view  $A$  in the unitary group of  $L^\infty(Y) \cong L^\infty(Y) \otimes \mathbf{C}1_{\mathbf{F}} \subset L^\infty(\mathcal{R})$ . Since  $A$  and  $gAg^{-1} \subset L^\infty(Y)$  commute, this gives rise to a unitary representation of  $H = A \wr G$  on  $L^2(\mathcal{R})$ . We will prove that  $H_b^1(H, \mathcal{L}(L^2(\mathcal{R}))) \neq 0$ .

We write  $N = \bigoplus_G A$ . Since  $N$  is amenable and  $\mathcal{L}(L^2(\mathcal{R}))$  is a dual module, a weak-\* averaging argument shows that there is a canonical isomorphism

$$H_b^*(H, \mathcal{L}(L^2(\mathcal{R}))) \cong H_b^*(G, \mathcal{L}(L^2(\mathcal{R}))^N)$$

(see Corollary 7.5.10 in [6]). With the identification  $\mathcal{R} = Y \times \mathbf{F}$ , one has

$$\mathcal{L}(L^2(\mathcal{R}))^N = N' \cap \mathcal{L}(L^2(\mathcal{R})) = L^\infty(Y) \bar{\otimes} \mathcal{L}(\ell_2\mathbf{F}) \cong L^\infty(Y, \mathcal{L}(\ell_2\mathbf{F}))$$

(see Theorem IV.5.9 in [13]). Keeping track of the  $G$ -representation, one sees that  $g \in G$  acts on  $L^\infty(Y, \mathcal{L}(\ell_2\mathbf{F}))$  by  $(g \cdot f)(y) = \tau_{\alpha(g, g^{-1} \cdot y)}(f(g^{-1} \cdot y))$ , where  $\tau$  denotes the  $\mathbf{F}$ -action on  $\mathcal{L}(\ell_2\mathbf{F})$ . For ease of notation, we denote the coefficient  $\mathbf{F}$ -module  $\mathcal{L}(\ell_2\mathbf{F})$  by  $V$ . Then, one further has a  $G$ -isomorphism

$$L^\infty(Y, V) \cong L^\infty(\mathcal{R}, V)^{\mathbf{F}},$$

where  $f \in L^\infty(Y, V)$  corresponds to  $\tilde{f} \in L^\infty(\mathcal{R}, V)^{\mathbf{F}}$  defined by  $\tilde{f}(y, t) = \tau_t^{-1}(f(y))$ . Now,  $\mathbf{F}$  acts on  $L^\infty(\mathcal{R}, V)$  by  $(s \cdot F)(z) = \tau_s(F(s^{-1}z))$  and  $G$  acts by  $(g \cdot F)(z) = F(g^{-1}z)$ . Since both the  $\mathbf{F}$ -action and the  $G$ -action on  $\mathcal{R}$  admit a fundamental domain, Proposition 4.6 in [8] implies that

$$H_b^*(G, L^\infty(\mathcal{R}, V)^{\mathbf{F}}) \cong H_b^*(\mathbf{F}, L^\infty(\mathcal{R}, V)^G) \cong H_b^*(\mathbf{F}, L^\infty(X, V)).$$

(See also Proposition 5.8 in [7].) Since  $X = \mathcal{R}/G$  has a finite  $\mathbf{F}$ -invariant measure, the inclusion  $V \hookrightarrow L^\infty(X, V)$  has a  $G$ -equivariant left inverse. It follows that the corresponding morphism

$$H_b^*(\mathbf{F}, V) \longrightarrow H_b^*(\mathbf{F}, L^\infty(X, V))$$

is an injection. Therefore, putting all identifications together, we conclude that there are injections

$$H_b^*(\mathbf{F}, \mathcal{L}(\ell_2\mathbf{F})) \longrightarrow H_b^*(H, \mathcal{L}(L^2(\mathcal{R})))$$

in all degrees. Since  $H_b^1(\mathbf{F}, \mathcal{L}(\ell_2\mathbf{F})) \neq 0$ , this completes the proof.

Analysing the proof at the level of derivations, one observes that the above injection maps  $D: \mathbf{F} \rightarrow \mathcal{L}(\ell_2\mathbf{F})$  to  $\tilde{D}: H \rightarrow \mathcal{L}(L^2(Y, \ell_2\mathbf{F}))$  defined by

$$(\tilde{D}(ag)\xi)(y) = a(y)D(\alpha(g, g^{-1} \cdot y))\xi(g^{-1} \cdot y),$$

where  $a \in N$  is viewed as an element of  $L^\infty(Y)$ ,  $g \in G$  and  $\xi \in L^2(Y, \ell_2\mathbf{F})$ .

**Proof of Theorem 2.** By a theorem of Adyan [1], the free Burnside group  $G = B(2, p)$  is non-amenable. Therefore, Theorem 1 implies that  $(\bigoplus_{\mathbf{N}} \mathbf{Z}/n\mathbf{Z}) \wr G$  is non-unitarisable. Notice that this wreath product is a countably generated group of exponent  $np$ . Therefore, by the universal property of free Burnside groups, it is a quotient of  $B(\aleph_0, np)$ . In particular, the latter is non-unitarisable. It was shown by Širvanjan [12] that  $B(\aleph_0, np)$  embeds into  $B(2, np)$  which is therefore also non-unitarisable. Finally, each  $B(m, np)$  surjects onto  $B(2, np)$  as long as  $m \geq 2$ , concluding the proof.  $\square$

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