Sets of Even Type in $\text{PG}(3, 4)$, alias the Binary (85, 24) Projective Geometry Code

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To characterize Hermitian varieties in projective space $\text{PG}(d, q)$ of $d$ dimensions over the Galois field $\text{GF}(q)$, it is necessary to find those subsets $K$ for which there exists a fixed integer $n$ satisfying (i) $3 < n < q - 1$, (ii) every line meets $K$ in $1$, $n$ or $q + 1$ points. $K$ is called singular or non-singular as there does or does not exist a point $P$ for which every line through $P$ meets $K$ in $1$ or $q + 1$ points. For $q$ odd, a non-singular $K$ is a non-singular Hermitian variety (M. Tallini Scafati “Caratterizzazione grafica delle forme hermitiane di un $S_{n,q}$” Rend. Mat. Appl. 26 (1967), 273–303). For $q$ even, $q > 4$ and $d = 3$, a non-singular $K$ is a Hermitian surface or “looks like” the projection of a non-singular quadric in $\text{PG}(4, q)$ (J. W. P. Hirschfeld and J. A. Thas “Sets of type $(1, n, q + 1)$ in $\text{PG}(d, q)$” to appear). The case $q = 4$ is quite exceptional, since the complements of these sets $K$ form a projective geometry code, a $(21, 11)$ code for $d = 2$ and an $(85, 24)$ code for $d = 3$. The full list of these sets is given.

1. INTRODUCTION

In $\text{PG}(d, q)$, the projective space of $d$ dimensions over the Galois field $\text{GF}(q)$, a subset $\mathcal{H}$ is a $k_{n,d,q}$ if $n$ is a fixed integer satisfying $1 \leq n \leq q$ such that

(i) $|\mathcal{H}| = k$;
(ii) $|\mathcal{H} \cap l| = 1, n$ or $q + 1$ for every line $l$;
(iii) $|\mathcal{H} \cap l| = n$ for some line $l$.
A line \( l \) is an \( r \)-secant of \( \mathcal{K} \) if \( |l \cap \mathcal{K}| = r \). A point \( P \) of \( \mathcal{K} \) is singular if the only lines through it are 1-secants and \((q + 1)\)-secants. \( \mathcal{K} \) is singular or non-singular as it has singular points or not. An \( m \)-dimensional subspace of \( PG(d, q) \) is denoted \( \Pi_m \).

Tallini Scafati [7] showed, in a proof valid only when \( q \) is odd (see [3]), that for \( 3 < n < q - 1 \) and \( d > 3 \) a non-singular \( k_{n,d,a} \) is a non-singular Hermitian variety \( \mathcal{U}_{d,a} \); here \( n = q^{1/2} + 1 \).

When \( q \) is even there is another example apart from \( \mathcal{U}_{d,a} \), namely the projection of a non-singular quadric in \( PG(d + 1, q) \), [3]: here \( n = \frac{1}{2}q + 1 \).

This gives one example for \( d \) odd and two for \( d \) even. For \( q > 4 \), it is shown in [3] that the only non-singular \( k_{n,3,a} \) with \( 3 \leq n \leq q - 1 \) are \( \mathcal{U}_{d,a} \) and \( \mathcal{K}_1 \), which is characterised by the property of containing a triangle. In fact, \( \mathcal{K}_1 \) comprises a plane \( \pi \) plus \( \frac{1}{2}q \) cones with common base in \( \pi \) a set of \( q + 1 \) points no three collinear. One way to construct \( \mathcal{K}_1 \) is as the projection \( \mathcal{R}_3 \) of a non-singular quadric in \( PG(4, q) \).

It seemed, paradoxically, that the smallest case \( q = 4 \) was the most complicated: this arose partly from the coincidence for \( q = 4 \) of the numbers \( q^{1/2} + 1 \) and \( \frac{1}{2}q + 1 \). However, the problem turns out to be quite a different type for \( q = 4 \), and its complexity increases with \( d \). In this paper we give a complete list of sets \( k_{3,d,4} \) for \( d = 2 \) and \( d = 3 \).

If \( \mathcal{K} \) is a \( k_{3,d,4} \), then \( |\mathcal{K} \cap l| = 1, 3 \) or 5 for every line \( l \). Hence \( |\mathcal{K} \cap l| \) is odd, and \( \mathcal{K} \) is accordingly called a set of odd type if this condition is satisfied: this includes some sets which do not have 3-secants, namely \( PG(d, 4) \) and its primes (hyperplanes). If \( \mathcal{K} \) is of odd type, its complement \( \mathcal{K}^c \) is of even type, since \( |\mathcal{K}^c \cap l| = 0, 2 \) or 4 for every line \( l \).

If \( \mathcal{K} \) and \( \mathcal{K}' \) are of odd type, then \( \mathcal{K} \vee \mathcal{K}' \), the complement of the symmetric difference of \( \mathcal{K} \) and \( \mathcal{K}' \), is also of odd type. Thus the sets of odd type in \( PG(d, 4) \) form a vector space over \( GF(2) \). We are most grateful to M. Dehon for pointing out this closure property: this was the essential point which enabled the list to be completed for \( d = 3 \).

In fact, it is perhaps more natural to consider the sets of even type. Let \( M_d \) be the \( r_d \) by \( c_d \) incidence matrix of points and lines of \( PG(d, 4) \). Here \( r_d = (4^d - 1)/3 \) and \( c_d = r_d a_{d-1}/r_1 = (4^d - 1)(4^d - 1)/45 \). The columns generate a vector subspace \( S \) of the \( r_d \)-dimensional vector space \( V \) over \( GF(2) \). Let the points of \( PG(d, 4) \) be \( P_i \), \( i = 1, 2, \ldots, r_d \), numbered in the same order as the corresponding rows of \( M_d \). With \( \mathcal{K} \) a set of odd type and so \( \mathcal{K}^c \) a set of even type, write \( \mathcal{K}^c = (a_1, a_2, \ldots, a_{r_d})^T \), where \( a_i = 1 \) if \( P_i \in \mathcal{K}^c \) and \( a_i = 0 \) if \( P_i \notin \mathcal{K}^c \). Then \( \mathcal{K}^c \) is orthogonal to each vector in \( S \). Conversely, each element of the orthogonal complement \( E \) of \( S \) corresponds to a set of even type. The vector subspace \( E \) is known as a projective geometry code. For \( d = 2 \) and 3 respectively, \( E \) has dimension 11 and 24; \( E \) is correspondingly a binary \((21, 11)\) code and a binary \((85, 24)\) code, Goethals and Delsarte [1]. We are most grateful to J.-M. Goethals for
pointing out the connection with Coding Theory and for further explanations and references, and to T. Beth for some other help.

As this paper is to be regarded as a sequel to [3], we will describe the sets of odd type in dimensions two and three. We have already stated that, if $\mathcal{H}$ and $\mathcal{H}'$ are of odd type in $\Pi = PG(d, 4)$, so is $\mathcal{H} \cup \mathcal{H}'$. This may be seen in the following way. Firstly, we note that $\mathcal{H} \cup \mathcal{H}'$ consists of the points in both $\mathcal{H}$ and $\mathcal{H}'$ as well as the points in neither $\mathcal{H}$ nor $\mathcal{H}'$. So, if $l$ is any line and $|l \cap \mathcal{H} \cap \mathcal{H}'|$ is odd, then $|l \cap (\mathcal{H} \setminus \mathcal{H}')|$, $|l \cap (\mathcal{H}' \setminus \mathcal{H})|$, and $|l \cap (\Pi \setminus (\mathcal{H} \cup \mathcal{H}'))|$ are all even, whence $|l \cap (\mathcal{H} \cup \mathcal{H}')|$ is odd; similarly, if $|l \cap \mathcal{H} \cap \mathcal{H}'|$ is even, then $|l \cap (\Pi \setminus (\mathcal{H} \cup \mathcal{H}'))|$ is odd and so is $|l \cap (\mathcal{H} \cup \mathcal{H}')|$.

Another point of view is to write $\mathcal{H} = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ and $\mathcal{H}' = (\beta_1, \beta_2, \ldots, \beta_r)$, where $\alpha_i = 0$ or 1 as $P_i$ is or is not in $\mathcal{H}$ (this is consistent with $\mathcal{H}^e$ as above) and similarly for $\mathcal{H}'$. Then $\alpha_i + \beta_i = 0 \pmod{2}$ if and only if $\alpha_i = \beta_i = 0$ or $\alpha_i = \beta_i = 1$. So, under $\cap$, the set of odd type form a vector space $O$ over $GF(2)$. Incidentally, it also follows that $\sum \alpha_i = \sum \beta_i = 0 \pmod{2}$.

The Hermitian varieties in $\Pi$ (including $\Pi$ itself) form a vector subspace $H$ of $O$ under the addition of the corresponding Hermitian forms. For, if $F$ is a Hermitian form and $X$ the vector of a point, then $F(X) = 0$ or 1 as the point $P(X)$ lies on the variety $V(F)$ or not. If $G$ is another Hermitian form, then $(F + G)(X) = 0$ if and only if $F(X) = G(X) = 0$ or $F(X) = G(X) = 1$. Hence $H$ is a vector subspace of $O$. Since

$$|H| = 2^{d+1} \cdot 4^{\binom{d+1}{2}} = 2^{(d+1)^2},$$

it follows that $\dim H = (d + 1)^2$. In particular, when $d = 2$, $\dim O = 11$ and $\dim H = 9$; when $d = 3$, $\dim O = 24$ and $\dim H = 16$.

### 2. Dimension Theorems of Coding Theory

The $h$-weight $w_h(N)$ of an integer $N$ with respect to a given prime $p$ is defined as follows: $w_h(N) = r$ if

$$N = N_0 + N_1 + \cdots + N_r,$$

where

(i) $N_0 \geq 0$ and $N_i > 0$ for $1 \leq i \leq r$;
(ii) $N_i$ is a multiple of $p^i - 1$ for $1 \leq i \leq r$;
(iii) if $N_i = \sum a_{ij} p^j$ with $0 \leq a_{ij} < p$, then $\sum_i a_{ij} < p$;
(iv) $r$ is the largest integer such that (i)–(iii) hold.
In our case we are specifically interested in the case \( h = p = 2 \). Here \( p^h - 1 = 3 \). So, to find the 2-weight of a number, we look at its binary representation and count as many pairs of adjoining ones as possible, whose removal still leaves a multiple of three (when \( N_0 = 0 \)). An extra zero on the right makes no difference to this process. So \( w_2(2N) = w_2(N) \). Table 1 gives the examples required: in each case \( N_0 = 0 \).

**Theorem 1.** The dimension of a cyclic code is the number of roots of its parity-check polynomial.

*Proof.* See MacWilliams and Sloane [4, p. 218].

Let \( M \) be the incidence matrix of points and \( s \)-subspaces in \( PG(d, q) \), \( q = p^h \), \( p \) prime. Then \( C^*(s, d, q) \) is the code generated by the columns of \( M \) and \( C(s, d, q) \) the dual or orthogonal code, called the projective geometry code. In the cases considered above \( E = C(1, d, 4) \). Since there is a projectivity permuting the points of \( PG(d, q) \) in a single cycle [2, p. 74] the codes \( C^*(s, d, q) \) and \( C(s, d, q) \) are cyclic.

**Theorem 2.** Let \( \alpha \) denote a primitive root of \( GF(q^{d+1}) \), \( q = p^h \), and let \( \beta = \alpha^{q-1} \). Then the roots of the parity-check polynomial of the code \( C(s, d, q) \) are \( \beta^n \), where \( 1 \leq n \leq (q^{d+1} - 1)/(q - 1) - 1 \) and \( w_2(n(q - 1)) \leq s \) with respect to \( p \).

*Proof.* See Peterson and Weldon [6, Chap. 10].

**Corollary 1.** \( \dim C(1, 2, 4) = 11 \).

*Proof.* From the theorem, let \( \alpha \) be a primitive root of \( GF(4^3) \) and let \( \beta = \alpha^3 \). Then the required roots are \( \beta^n \), where \( 1 \leq n \leq 20 \) and \( w_2(3n) \leq 1 \). A set of representatives for \( Z_{21}\setminus\{0\} \) is

\[
2^i \quad i = 0, 1, \ldots, 5; \\
3 \cdot 2^i \quad i = 0, 1, 2; \\
5 \cdot 2^i \quad i = 0, 1, \ldots, 5; \\
7 \cdot 2^i \quad i = 0, 1; \\
9 \cdot 2^i \quad i = 0, 1, 2.
\]

As \( w_2(N) = w_2(2N) \), from Table 1 the required roots are \( \beta^{2^i} (i = 0, 1, \ldots, 5) \), \( \beta^{3 \cdot 2^i} (i = 0, 1, 2) \), \( \beta^{5 \cdot 2^i} (i = 0, 1) \), where \( \beta^{21} = 1 \). Hence, from Theorem 1, \( \dim C(1, 2, 4) = 11 \).

**Corollary 2.** \( \dim C(1, 3, 4) = 24 \).
SETS OF EVEN TYPE IN $PG(3, 4)$

Table 1

<table>
<thead>
<tr>
<th>$N$</th>
<th>Binary representation of $N$ as $\sum N_i$</th>
<th>$w_2(N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>11</td>
<td>1</td>
</tr>
<tr>
<td>9</td>
<td>1001</td>
<td>1</td>
</tr>
<tr>
<td>15</td>
<td>1100</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>+0011</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>10101</td>
<td>1</td>
</tr>
<tr>
<td>27</td>
<td>11000</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>+00011</td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>100100</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>+000011</td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>100001</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>+001100</td>
<td></td>
</tr>
<tr>
<td>63</td>
<td>110000</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>+001100</td>
<td></td>
</tr>
<tr>
<td></td>
<td>+000011</td>
<td></td>
</tr>
<tr>
<td>87</td>
<td>1010100</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>+0000011</td>
<td></td>
</tr>
<tr>
<td>111</td>
<td>1100000</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>+0001100</td>
<td></td>
</tr>
<tr>
<td></td>
<td>+0000011</td>
<td></td>
</tr>
</tbody>
</table>

Proof. Let $\alpha$ be a primitive root of $GF(4^4)$ and let $\beta = \alpha^3$. The required roots are $\beta^n$, where $1 \leq n \leq 84$ and $w_2(3n) \leq 1$.

A set of representatives for $Z_{80}\setminus\{0\}$ is

\[2^1, 3 \cdot 2^1, 5 \cdot 2^1, 7 \cdot 2^1, 9 \cdot 2^1, 13 \cdot 2^1, 15 \cdot 2^1, 21 \cdot 2^1, 29 \cdot 2^1, 37 \cdot 2^1 \quad (i = 0, 1, \ldots, 7);\]

\[17 \cdot 2^1 \quad (i = 0, 1, 2, 3).\]

Again from Table 1, as $w_2(N) = w_2(2N)$, the required roots are $\beta^{2^i}, \beta^{3 \cdot 2^i}, \beta^{7 \cdot 2^i} (i = 0, 1, \ldots, 7)$, where $\beta^{8^i} = 1$.

Theorem 3. The weight of each code word in $C(s, d, q)$ is a multiple of $p^h$, where $\lambda = \nu/(p - 1) - 1$ and $\nu$ is the smallest number of roots of the parity-check polynomial whose product is one.

Proof. See McEliece [5].

Corollary 1. Each code word in $C(1, 2, 4)$ has even weight; that is, each set of even type in $PG(2, 4)$ has an even number of points.
Proof. With the notation of Theorem 2, Corollary 1, \(1 = \beta^{31} = \beta^{7} \cdot \beta^{14}\). So \(\nu = 2, \lambda = 1\) and \(p^4 = 2\).

In fact it follows easily without Theorem 3 that every code word in \(C(1, d, 4)\) has even weight. However, the following corollary is by no means obvious without the theorem.

**Corollary 2.** Each code word in \(C(1, 3, 4)\) has weight divisible by four; that is, the number of points in each set of even type in \(\text{PG}(3, 4)\) is a multiple of four.

**Proof.** With the notation of Theorem 2, Corollary 2, consider the smallest number of \(\beta^{8i}, \beta^{3 \cdot 2i}, \beta^{7 \cdot 2i}\) \((i = 0, 1, \ldots, 7)\) whose product is one, where \(\beta^{88} = 1\). For \(i, j \in \{0, 1, \ldots, 7\}\), modulo 85,

\[
2^i + 2^j \equiv 0, \quad 2^i + 3 \cdot 2^j \equiv 0, \quad 2^i + 7 \cdot 2^j \equiv 0, \quad 3 \cdot 2^i + 7 \cdot 2^j \equiv 0.
\]

So \(\nu > 2\). However \(7 + 14 + 64 = 85\). So \(\beta^{7} \cdot \beta^{14} \cdot \beta^{34} = 1\). Hence \(\nu = 3, \lambda = 2\) and \(p^4 = 4\). 

### 3. Sets of Odd Type in PG(2, 4)

For a fixed set \(\mathcal{H}\) of odd type in \(\text{PG}(d, 4)\), let \(\tau_r\) be the number of lines \(r\)-secant to it. Let \(\nu_r\) denote the number of lines \(r\)-secant to its complement \(\mathcal{H}^c\). Then \(\nu_0 = \tau_5, \nu_2 = \tau_3, \nu_4 = \tau_1\). The projective group of \(\mathcal{H}\) is the group of projectivities fixing \(\mathcal{H}\).

**Theorem 4.** Sets of even type in \(\text{PG}(2, 4)\) form the binary \((21, 11)\) projective geometry code \(C(1, 2, 4)\). The projectively distinct sets \(\mathcal{H}\) of odd type are given in Table 2, where \(|\mathcal{H}| = k\) and \(G\) is the projective group of \(\mathcal{H}\).

<table>
<thead>
<tr>
<th>Type</th>
<th>Description</th>
<th>(k)</th>
<th>(21 - k)</th>
<th>Number</th>
<th>(G)</th>
<th>(\tau_1)</th>
<th>(\tau_3)</th>
<th>(\tau_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>(\mathbb{H}_{s,4}): Hermitian curve</td>
<td>9</td>
<td>12</td>
<td>280</td>
<td>(\text{PGU}(3, 4))</td>
<td>9</td>
<td>12</td>
<td>0</td>
</tr>
<tr>
<td>II</td>
<td>(\text{PG}(2, 2)): subplane</td>
<td>7</td>
<td>14</td>
<td>360</td>
<td>(\text{PGL}(3, 2))</td>
<td>14</td>
<td>7</td>
<td>0</td>
</tr>
<tr>
<td>III</td>
<td>Oval + external line</td>
<td>11</td>
<td>10</td>
<td>1008</td>
<td>(\text{A}_5)</td>
<td>5</td>
<td>15</td>
<td>1</td>
</tr>
<tr>
<td>IV</td>
<td>Complement of an oval</td>
<td>15</td>
<td>6</td>
<td>168</td>
<td>(\text{A}_5)</td>
<td>0</td>
<td>15</td>
<td>6</td>
</tr>
<tr>
<td>V</td>
<td>(\mathbb{H}_{s,2,4}): three concurrent lines</td>
<td>13</td>
<td>8</td>
<td>210</td>
<td>((\mathbb{Z}_9 \times S_3)\mathbb{Z}_4^2)</td>
<td>2</td>
<td>16</td>
<td>3</td>
</tr>
<tr>
<td>VI</td>
<td>(\mathbb{I}_1): a single line</td>
<td>5</td>
<td>16</td>
<td>21</td>
<td>(\text{GL}(2, 4)\mathbb{Z}_4^2)</td>
<td>20</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>VII</td>
<td>(\text{PG}(2, 4))</td>
<td>21</td>
<td>0</td>
<td>1</td>
<td>(\text{PGL}(3, 4))</td>
<td>0</td>
<td>0</td>
<td>21</td>
</tr>
</tbody>
</table>

\(2048 = 2^{11}\)
SETS OF EVEN TYPE IN $PG(3, 4)$

Proof. This follows almost immediately from the classification of $k_{n,2,q}$ Tallini Scafati [7]. There, the first type is a Hermitian arc, which for $q = 4$ is a set of nine points such that every line is a 1-secant or a 3-secant. But, it is easy to show that such a set is necessarily a $U_{2,4}$; for example, see [2, Theorem 11.1.1, Corollary]. See also [3] for the list and existence of the $k_{n,2,q}$. \\

Remarks. (i) Types I, II, III, IV are non-singular; types V, VI, VII are singular.

(ii) Types I, V, VI, VII are Hermitian and their total number is $512 = 2^9$, in accordance with the fact that $\dim H = 9$ as in Section 1.

(iii) Each type is determined by the number of points it contains.

4. SETS OF ODD TYPE IN $PG(3, 4)$

The main tool for classification of sets $\mathcal{X}$ of odd type in $PG(3, 4)$ is the classification in $PG(2, 4)$, since any plane section of $\mathcal{X}$ is still of odd type.

A singular set of odd type in $PG(3, 4)$ is the cone joining a point $P_0$ to a section of odd type in a plane not through $P_0$. These cones will be written $\Pi_0\mathcal{X}_1, ..., \Pi_0\mathcal{X}_{\text{VII}}$, where, for example, $\mathcal{X}_I$ is a set of odd type I in a plane and $\Pi_0\mathcal{X}_I$ consists of the points on the joins of $\Pi_0$ to the points of $\mathcal{X}_I$.

From [3], we require Theorems 5–7, where $\mathcal{X}$ is a non-singular $k_{3,3,4}$.

The plane sections of particular types are numbered as in Table 2.

THEOREM 5. If $\mathcal{X}$ has a section of type IV and no section of type I or II, then it has one section of type VII, $k = 53$ and $\mathcal{X}$ is the projection $\mathcal{X}_3$ of a non-singular quadric in $PG(4, 4)$.

Proof. See [3, Sects. 5 and 6].

THEOREM 6. If $\mathcal{X}$ has a section of type VII, then $k = 53$ and $\mathcal{X} = \mathcal{X}_3$.

Proof. See [3, Sects. 5 and 6].

THEOREM 7. If $\mathcal{X}$ has no section of type IV, then $k = 37$ or $k = 45$. If $k = 45$, then $\mathcal{X}$ is the non-singular Hermitian surface $\mathcal{H}_{3,4}$. If $k = 37$, then $\mathcal{X} = \mathcal{X}^*$, which comprises a line $l$ and 4 pairs $(l_i, l'_i), i = 1, ..., 4$, such that $l, l_i, l'_i$ are concurrent and coplanar and that a set of 4 lines one from each pair have only $l$ as transversal.

Proof. See [3, Sects. 7 and 8].
It now remains to consider non-singular sets of odd type which contain a section of type IV, no section of type VII and at least one section of type I or II.

As above, \( \tau_r \) is the total number of \( r \)-secants to a fixed set \( \mathcal{H} \) of odd type. Also, let \( \rho_r = \rho_r(P) \) and \( \sigma_r = \sigma_r(Q) \) be the respective number of \( r \)-secants through a point \( P \) on \( \mathcal{H} \) and a point \( Q \) off \( \mathcal{H} \). Then

\[
\rho_1 + \rho_3 + \rho_5 = 21, \\
2\rho_3 + 4\rho_5 = k - 1; \\
\sigma_1 + \sigma_3 = 21, \\
\sigma_1 + 3\sigma_3 = k.
\]

Hence

\[
\sigma_1 = (63 - k)/2, \\
\sigma_3 = (k - 21)/2.
\]

Therefore

\[
\tau_1 = (85 - k)\sigma_1/4 = (85 - k)(63 - k)/8, \\
\tau_3 = (85 - k)\sigma_3/2 = (85 - k)(k - 21)/4, \\
\tau_5 = 35 - \tau_1 - \tau_3.
\]

Let \( \mathcal{H}, \mathcal{H}' \) be sets of odd type with \( k = | \mathcal{H} |, k' = | \mathcal{H}' |, t = | \mathcal{H} \cap \mathcal{H}' |. Then

\[
| \mathcal{H} \setminus \mathcal{H}' | = 85 - k - k' + 2t.
\]

**Theorem 8.** Let \( \mathcal{H} \) be a non-singular \( k_{3,3,4} \) with a section of type IV by a plane \( \pi \) but no section of type VII, and containing no line other than those in \( \pi \). Then \( k = 33 \) and \( \mathcal{H}' = \Pi_0 \mathcal{H}_{1V} \cup \pi \).

**Proof.** Let \( l \) be a line of \( \mathcal{H} \) in \( \pi \). Every plane through \( l \) other than \( \pi \) is of type III or VI. If there are \( n_3 \) of type III and \( n_6 \) of type VI, then

\[
15 + 6n_3 = k, \quad n_3 + n_6 = 4.
\]

From Theorem 3, Corollary 2, we know that 4 divides \( 85 - k \). Hence \( k = 1 \pmod{4} \) and \( n_3 \) is odd. Therefore \( n_3 = 1 \) or 3. If \( n_3 = 1 \), then \( k = 21 \) and, from (5), \( \tau_3 = 0 \). So every line meeting \( \mathcal{H} \) in two points lies in it, whence \( \mathcal{H} \) is a plane: a contradiction. Thus \( n_3 = 3 \) and \( k = 33 \).

Let \( l' \) be a 3-secant in \( \pi \). Every plane other than \( \pi \) through \( l' \) is of type I or II. If there are \( n_1 \) and \( n_2 \) of these planes respectively, then

\[
15 + 6n_1 + 4n_2 = 33, \quad n_1 + n_2 = 4.
\]
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<tr>
<th>Type</th>
<th>$\mathcal{X}$</th>
<th>$k$</th>
<th>$85 - k$</th>
<th>Number</th>
<th>9</th>
<th>7</th>
<th>11</th>
<th>15</th>
<th>13</th>
<th>5</th>
<th>21</th>
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<th>$\tau_3$</th>
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<td>1 + 12</td>
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<td>27</td>
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</table>
Therefore \( n_1 = 1 \) and \( n_2 = 3 \). Hence the total numbers of sections of \( \mathcal{X} \) of different types are as follows:

\[
N_I = 15, \quad N_{II} = 45, \quad N_{III} = 18, \quad N_{IV} = 1, \quad N_{VI} = 6.
\]

Consider the lines through a point off \( \mathcal{X} \). From (4), \( \sigma_1 = 15 \) and \( \sigma_2 = 6 \). So, in particular, through a point \( Q_1 \) of the oval \( \mathcal{M} = \pi \setminus \mathcal{X} \), there are six 3-secants: five of these are in \( \pi \) leaving one, \( l_1 \), not in \( \pi \) (which must meet the three ovals in the planes of type III through \( l \)). Hence all planes through \( l_1 \) have two 3-secants through \( Q_1 \) and hence are of type I.

Take \( Q_1 \) on \( l' \) and let \( Q_2 \) be the other point of \( l' \cap \mathcal{M} \). Then there exists a unique 3-secant \( l_2 \) through \( Q_2 \) not in \( \pi \). Both planes \( l' \cap l_1 \) and \( l' \cap l_2 \) are of type I. But \( n_1 = 1 \), whence \( l' \), \( l_1 \), \( l_2 \) are coplanar. So \( l_1 \) and \( l_2 \) meet. Hence the six 3-secants not in \( \pi \) through the six points of \( \mathcal{M} \) are concurrent at a point \( Q \).

Now, the triangle \( Q_1Q_2Q \), with sides \( l_2 \), \( l_1 \), \( l' \) all 3-secants of a Hermitian curve \( \mathcal{U}_2,4 \), has \( Q_1 \) and \( Q_2 \) off \( \mathcal{U}_2,4 \). If \( Q \) were on \( \mathcal{U}_2,4 \), then \( l_1 \), \( l_2 \) and the lines joining \( Q \) to the points of \( l' \cap \mathcal{U}_2,4 \) would be five 3-secants of \( \mathcal{U}_2,4 \) through \( Q \), a contradiction. So \( Q \) is not on \( \mathcal{U}_2,4 \) and not on \( \mathcal{X} \).

Thus \( \mathcal{X} \) consists of the 18 points on the six 3-secants of \( \mathcal{X} \) joining \( Q \) to the points of \( \mathcal{M} = \pi \setminus \mathcal{X} \) and of the 15 points of \( \mathcal{M}' = \pi \cap \mathcal{X} \); that is, \( \mathcal{X} = Q,\mathcal{M}', \pi \).

**Theorem 9.** Let \( \mathcal{X} \) be a non-singular \( k_{3,3,4} \) containing a section of type IV by a plane \( \pi \), a line \( l \) not in \( \pi \), a section of type I or II by a plane \( \pi' \), but no section of type VII. Then \( k = 41, 45 \) or 49.

**Proof.** Since every line of \( \pi \) lies in \( \mathcal{X} \) or is a 3-secant, the plane sections through \( l \) are of type III, IV or V. Let the respective numbers be \( m_3, m_4, m_5 \). Then

\[
5 + 6m_3 + 10m_4 + 8m_5 = k, \quad m_3 + m_4 + m_5 = 5.
\]

As the planes through \( l \) and a line of \( \mathcal{X} \) in \( \pi \) are of type IV or V, so

\[
m_4 \mid m_5 \geq 2.
\]

Hence \( k \geq 39 \). If \( m_3 \geq 1 \), then \( k \leq 51 \). However, by Theorem 3, Corollary 2, we have that \( k \equiv 1 \pmod{4} \). Hence \( k = 41, 45 \) or 49.

Suppose therefore that \( m_3 = 0 \) for every line of \( \mathcal{X} \), not in \( \pi \). Now assume that \( \mathcal{X} \) has a plane \( \pi'' \) of type VI. Then \( \pi \cap \pi'' \) is a line \( l' \) of \( \mathcal{X} \) and every line of \( \mathcal{X} \) not in \( \pi \) meets \( l' \). Let \( l'' \) be the other line of \( \mathcal{X} \) in \( \pi \) through \( P = l' \cap \pi'' \). Then a plane through \( l \) other than \( ll' \) and \( ll'' \) meets \( \pi'' \) in a 1-secant and \( \pi \) in a...
3-secant through $P$, and so is of type III: so for some line of $\mathcal{H}$ not in $\pi$, $m_\alpha \geq 1$. Hence $\mathcal{H}$ has no section of type VI.

Now, let $l'$ be a line of $\mathcal{H}$ in $\pi$ and let $l' \cap \pi' = P'$. Let $l''$ be the other line of $\mathcal{H}$ through $P'$ in $\pi$ and let $l_1$ be a unisecant in $\pi'$ through $P'$. Then the planes $l'l_1$ and $l'l_2$ are of type III or V, and the other planes through $l_1$ are of type I or II. So, with $n_1, n_2, n_3, n_4$ planes of the corresponding types through $l_1$,

$$1 + 8n_1 + 6n_2 + 10n_3 + 12n_4 = k,$$

$$n_1 + n_2 = 3, \quad n_3 + n_4 = 2.$$ 

So $39 \leq k \leq 49$. Since $k \equiv 1 \pmod{4}$, so $k = 41, 45$ or 49.

**Theorem 10.** Sets of even type in $PG(3,4)$ form the binary $(85,24)$ projective geometry code $C(1,3,4)$. The projectively distinct sets of $\mathcal{K}$ of odd type are given in Table 3, where $|\mathcal{H}| = k$, $N_1, \ldots, N_{VI}$ are the numbers of plane sections of the corresponding types (above which are written the numbers of points in such sections), and $\tau_i$ is the number of $i$-secants.

**Proof.** Types 1–7 are the singular types, obtained by joining a point $P$ to all the points of a plane set of odd type. Types 8–11 are given by Theorems 5–8. Then, for any set $\mathcal{K}$ of type 1–11, the sets $\mathcal{K} \cup \Pi_2$ for various planes $\Pi_2$ only give the three new types 12–14. Details of the sets $\mathcal{K}_a$ and $\mathcal{K}_s$ are given in [3]. Now, the calculation of the number of each type is straightforward. As this gives a total number of $2^{24}$, Theorem 2, Corollary 2 tells us that the list is complete. Each set of type 11–14 is of the form $\mathcal{K} \cup \Pi_2$, where the type of $\Pi_2 \cap \mathcal{K}$ is given by the upper index on $\Pi_2$.

**Remarks.** (i) Types 1, 2, 3, 4 and 8 are Hermitian and total $2^{16}$ in accordance with $\dim H = 16$, Section 1.

(ii) The sets are listed in descending order of the dimension of their space of singular points: three for type 1, two for type 2, one for type 3, zero for types 4–7 and minus one for types 8–14.

(iii) $\mathcal{K}_{3,4}$ is the only non-singular $k_{3,3,4}$ with exactly two types of plane section.

(iv) As $k$ determines $\tau_1$, $\tau_3$, $\tau_5$, these numbers do not distinguish projectively distinct $\mathcal{K}$ with the same number of points.

(v) The number of projectively distinct $\mathcal{K}$ for each $k$ is as follows:

<table>
<thead>
<tr>
<th>$k$</th>
<th>85</th>
<th>61</th>
<th>53</th>
<th>49</th>
<th>45</th>
<th>41</th>
<th>37</th>
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<td>24</td>
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