Large time behavior of disturbed planar fronts in the Allen–Cahn equation

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1. Introduction

The Allen–Cahn equation is a well-known example of nonlinear parabolic equations in which solutions typically develop a transition layer that separates the spatial domain into different phase regions. The nature of this equation has been studied extensively since the pioneering work of Allen and Cahn [1] and that of Kawasaki and Ohta [9]. In the present paper, we consider the Cauchy problem for the Allen–Cahn equation

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Fig. 1. Planar wave $\Phi(y - ct)$ on $\mathbb{R}^2$.

\[ \begin{aligned}
u_t &= \Delta u + f(u), \quad x \in \mathbb{R}^{n-1}, \; y \in \mathbb{R}, \; t > 0, \\
u(x, y, 0) &= u_0(x, y), \quad x \in \mathbb{R}^{n-1}, \; y \in \mathbb{R},
\end{aligned} \tag{1} \tag{2} \]

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2} + \frac{\partial^2}{\partial y^2}$ and $n \geq 2$. We assume that the initial value $u_0$ is bounded and continuous on $\mathbb{R}^{n}$ and that the function $f$ is of the bistable type – or, more generally, a function satisfying the condition (F1) below – in which $u = \pm 1$ are both stable stationary states. A typical example is $f(u) = (1 - u^2)(u - a)$, $|a| < 1$.

In this paper we focus on the stability of the planar wave (see Fig. 1), which is a traveling wave of (1)-(2) given in the form $u(x, y, t) = \Phi(y - ct)$, where $c \in \mathbb{R}$ is a constant representing the speed and $\Phi(z)$ is what we call the profile function, which satisfies

\[ \begin{aligned}
\Phi''(z) + c\Phi'(z) + f(\Phi(z)) &= 0, \quad z \in \mathbb{R}, \\
\Phi(\pm \infty) &= \mp 1.
\end{aligned} \tag{3} \tag{4} \]

If $\Phi(z)$ satisfies (3)-(4), then its translation $\Phi(z - \xi)$ also satisfies (3)-(4) for any constant $\xi \in \mathbb{R}$. In order to avoid ambiguity, we impose the normalization condition

\[ \Phi(0) = 0. \tag{5} \]

It is well known that the pair $(c, \Phi)$ satisfying (3)-(5) is unique if it exists.

If $\Phi(y - ct)$ is a planar wave, then its translation $\Phi(y - ct + \xi)$ is also called a planar wave. The profile function satisfies $\Phi'(z) < 0$ for $z \in \mathbb{R}$ and there exist constants $C > 0$ and $\beta > 0$ such that

\[ \begin{aligned}
|\Phi'(z)|, |\Phi''(z)| &\leq Ce^{-\beta z} \quad \text{for } z > 0, \\
|\Phi'(z)|, |\Phi''(z)| &\leq Ce^{\beta z} \quad \text{for } z < 0.
\end{aligned} \]

For details, see [3,6] for instance. Note also that $c = 0$ if and only if $\int_{-1}^{1} f(s) \, ds = 0$. In this case $\Phi(y)$ is a stationary solution of (1)-(2).

Our objective in this paper is to study how a planar wave behaves when an arbitrarily large (but bounded) perturbation is given near the front region. Throughout this paper, we assume the following conditions:

(F1) $f \in C^1(\mathbb{R})$ satisfies

\[ \begin{aligned}
f(-1) &= f(1) = 0, \\
f'(-1) &< 0, \\
f'(1) &< 0.
\end{aligned} \]
and

\[ f(s) > 0 \quad \text{for} \ s \in (-\infty, -1) \cup (s_+, 1), \quad f(s) < 0 \quad \text{for} \ s \in (-1, s_-) \cup (1, \infty), \]

for some constants \( s_+ \) and \( s_- \) with \(-1 < s_- \leq s_+ < 1\).

(F2) There exist \( c \in \mathbb{R} \) and \( \Phi(z) \in C^2(\mathbb{R}) \) that satisfy (3)–(5).

In the special case where \( f \) is a bistable type nonlinearity, we have \( s_- = s_+ \). It is known that (F2) is automatically fulfilled if \( f \) is a bistable type nonlinearity.

The asymptotic stability of planar waves is studied in [7,8,11,16] in various topologies. In [7,8,16], it is assumed that initial perturbations are sufficiently small and decay to zero as \(|x| + |y| \to \infty\). The paper [11] proves asymptotic stability under any (possibly large) initial perturbations that decay to zero as \(|x| + |y| \to \infty\). The paper [11] also derives stability results for initial perturbations that are almost periodic in the \( x \)-direction and satisfy \( u_0(x, y) > s_+ \) for \( y < -1 \) and \( u_0(x, y) < s_- \) for \( y > 1 \).

All these results will be extended in Theorem 1.6 in the present paper. The paper [14] analyzes the large time behavior of the disturbed planar front in (1)–(2) on \( \mathbb{R}^2 \) and shows the similarity of its dynamics to that of the heat equation. Our Theorem 1.1 below also discusses the large time behavior of disturbed planar fronts, but it further clarifies their dynamics by revealing their relation to the mean curvature flow up to \( t = +\infty \).

Let us now state our main results.

**Theorem 1.1 (Large time behavior).** Let \( n \geq 2 \) and let (F1)–(F2) hold. Let \( u(x, y, t) \) be a solution of the problem (1)–(2) whose initial value \( u_0(x, y) \) is bounded and uniformly continuous, assume that it satisfies

\[
\liminf_{y \to -\infty} \inf_{x \in \mathbb{R}^{n-1}} u_0(x, y) > s_+, \quad \limsup_{y \to \infty} \sup_{x \in \mathbb{R}^{n-1}} u_0(x, y) < s_-.
\]

Then there exist a constant \( T > 0 \) and a smooth function \( \gamma(x, t) \) such that:

(i) for each \( t \in [T, \infty) \) and \( x \in \mathbb{R}^{n-1} \), one has \( u(x, y, t) = 0 \) if and only if \( y = \gamma(x, t) \);

(ii) it holds that

\[
\lim_{t \to \infty} \sup_{(x, y) \in \mathbb{R}^n} |u(x, y, t) - \Phi(y - \gamma(x, t))| = 0;
\]

(iii) for any \( \varepsilon > 0 \), there exists \( \tau_\varepsilon \in [T, \infty) \) such that the solution \( U(x, t) \) of the problem

\[
\begin{align*}
\frac{U_t}{\sqrt{1 + |\nabla_x U|^2}} & = \text{div} \left( \frac{\nabla_x U}{\sqrt{1 + |\nabla_x U|^2}} \right) + c, \quad x \in \mathbb{R}^{n-1}, \ t > 0, \\
U(x, 0) & = \gamma(x, \tau_\varepsilon), \quad x \in \mathbb{R}^{n-1},
\end{align*}
\]

satisfies

\[
\sup_{x \in \mathbb{R}^{n-1}, t \geq \tau_\varepsilon} |\gamma(x, t) - U(x, t - \tau_\varepsilon)| \leq \varepsilon.
\]

The symbol \( \nabla_x \) above denotes the \((n-1)\)-dimensional gradient. The statement (i) of Theorem 1.1 implies that the zero-level surface of \( u(x, y, t) \) has a graphical representation \( y = \gamma(x, t) \). The statement (ii) implies that the solution \( u(x, y, t) \) behaves like the function \( \Phi(y - \gamma(x, t)) \) for large \( t \), thus the large time behavior of the solution \( u(x, y, t) \) is basically determined by the position of the zero-level surface \( \gamma(x, t) \). Finally, the statement (iii) shows that the behavior of \( \gamma(x, t) \) can be approximated by the solution \( U(x, t) \) of the mean curvature flow on \( \mathbb{R}^{n-1} \) with a drift term \( c \). What is important
here is that this approximation is valid up to \( t = +\infty \). Previously such an approximation was known only for a finite time interval.

Our second main result is concerned with the asymptotic stability of planar waves. It is well known that planar waves are stable under bounded initial perturbations. This follows easily by combining the one-dimensional stability result of [6] and the comparison principle. However, as was shown in Proposition 1.9 of [11] and Theorem 2.1 of [14], planar waves are not necessarily stable with asymptotic phase if the initial perturbations are arbitrary. In fact, there are solutions that oscillate permanently between two planar waves. Thus it is important to specify the class of initial perturbations under which the planar waves are asymptotically stable. For this purpose, we introduce some notation.

**Definition 1.2** (Hull of a function). For a bounded continuous function \( g(x) : \mathbb{R}^m \to \mathbb{R} \), we define its **hull** \( \mathcal{H}_g \) by

\[
\mathcal{H}_g := \{ \sigma_a g \mid a \in \mathbb{R}^m \}^{L_\text{loc}^\infty(\mathbb{R}^m)},
\]

where \( \sigma_a \) denotes the shift operator defined by \( (\sigma_a g)(x) = g(x + a) \) and \( \overline{A}^X \) stands for the closure of a set \( A \) in the \( X \)-topology.

**Definition 1.3** (Hull of a function in the x-direction). For a bounded continuous function \( p(x, y) : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R} \), we define its **hull** \( \mathcal{H}_p \) by

\[
\mathcal{H}_p := \{ \sigma_a p \mid a \in \mathbb{R}^{n-1} \}^{L_\text{loc}^\infty(\mathbb{R}^n)},
\]

where \( \sigma_a \) denotes the shift operator defined by \( (\sigma_a p)(x, y) = p(x + a, y) \).

Throughout this paper we always consider the case where \( g(x) \) and \( p(x, y) \) are bounded and uniformly continuous. Thus the hull \( \mathcal{H}_g \) (resp. \( \mathcal{H}_p \)) is a compact set in \( L_\text{loc}^\infty(\mathbb{R}^m) \) (resp. \( L_\text{loc}^\infty(\mathbb{R}^n) \)).

**Definition 1.4** (Unique ergodicity). A bounded uniformly continuous function \( g(x) : \mathbb{R}^m \to \mathbb{R} \) is called **uniquely ergodic** if there exists a unique probability measure on \( \mathcal{H}_g \) that is \( \sigma_a \)-invariant for any \( a \in \mathbb{R}^m \).

**Definition 1.5** (Unique ergodicity in the x-direction). A bounded uniformly continuous function \( p(x, y) : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R} \) is called **uniquely ergodic in the x-direction** if there exists a unique probability measure on \( \mathcal{H}_p \) that is \( \sigma_a \)-invariant for any \( a \in \mathbb{R}^{n-1} \).

See Section 2 and Appendix A for more details on unique ergodicity. Now we state our second main result.

**Theorem 1.6** (Stability with asymptotic phase). In addition to the assumptions of Theorem 1.1, assume further that \( u_0(x, y) \) is uniquely ergodic in the x-direction. Then there exists a constant \( \mu \in \mathbb{R} \) such that

\[
\lim_{t \to \infty} \sup_{(x, y) \in \mathbb{R}^n} u(x, y, t) = \Phi(y - ct + \mu).
\]

The above theorem asserts that planar waves are stable with asymptotic phase under spatially ergodic perturbations. The following are special cases to which Theorem 1.6 applies (see Remark 2.4 for details):

(a) \( |u_0(x, y) - \Phi(y)| \to 0 \) as \( |x| + |y| \to \infty \).
(b) \( u_0(x, y) \) satisfies (6) and is periodic, quasi-periodic or almost periodic in the x-direction.
(c) $n = 3$ and $u_0(x, y) = \Phi(y) + h(y)g(x)$, where $g$ is a bounded uniformly continuous function on $\mathbb{R}^2$ whose level sets exhibit the Penrose tiling pattern (see Fig. 2) and $h(y)$ is any continuous function on $\mathbb{R}$ such that $|h(y)| \to 0$ as $y \to \pm \infty$.

Note that Theorem 1.6 in the above special case (a) extends the results of [7,8,11,16], which focus on the asymptotic stability of planar waves under spatially-decaying initial perturbations. Our earlier result [11, Theorem 1.7] deals with the case where the initial value $u_0(x, y)$ is almost periodic in the $x$-direction and monotone decreasing in the $y$-direction. Theorem 1.6 significantly improves this result in two respects. First, since Theorem 1.6 does not require monotonicity of initial value $u_0(x, y)$ in the $y$-direction, it allows the perturbed fronts to have non-graphical level surfaces. Secondly, as we will remark in Section 2.1 and Appendix A.2, the class of uniquely ergodic functions is wider than that of almost periodic functions. For the special case (c), see Example 2.3 (2) and Remark 2.4.

Theorem 1.6 will be proven by combining the following key observations:

1. $u(x, y, t)$ can be approximated by $\Phi(y - \gamma(x, t))$ for all large $t$, where $\gamma(x, t)$ is the zero-level surface of $u$ (Theorem 1.1 (i), (ii));
2. $\gamma(x, t)$ can be approximated by a solution of the mean curvature flow with a drift term (Theorem 1.1 (iii));
3. the ergodicity of the initial value $u_0(x, y)$ is inherited by the solution, therefore $\gamma(x, t)$ remains uniquely ergodic in $x$ for all large $t$ (Lemma 4.15);
4. solutions of the mean curvature flow – more precisely, its approximate equation (19) – with uniquely ergodic initial value converges to a drifting hyperplane uniformly as $t \to \infty$ (Lemma 3.11).

Finally, let us state a result on the inheritance of ergodicity that is used to derive the observation (3) above. This result applies to a large class of evolution equations and may be of interest to the reader in its own right.

**Inheritance of ergodicity**: Let $u$ be a solution of the evolution equation

\[
\begin{aligned}
\frac{du}{dt} &= A(u, t) \quad (t > 0), \\
u(0) &= \varphi \in X,
\end{aligned}
\]

where $X$ is a metric space on which some group action $\sigma_a : X \to X$ ($a \in \mathbb{R}^m$) is defined. Assume that this problem is well-posed and that it is “homogeneous” in the sense that if $u(t)$ is a solution,
then so is $\sigma_a u(t)$ for $a \in \mathbb{R}^m$. Then, if $\varphi$ is uniquely ergodic with respect to $\{\sigma_a\}_{a \in \mathbb{R}^m}$, so is $u(t)$ for each $t \geq 0$. (See Proposition 2.10 for details.)

This paper is organized as follows. In Section 2.1 we give basic remarks on ergodic functions. In Section 2.2, we prove the above-mentioned result on the inheritance of ergodicity (Proposition 2.10). This result will then be applied to the mean curvature flow (Corollary 2.11) and the Allen–Cahn equation (Corollary 2.12 and Lemma 4.15).

In Section 3, we analyze the mean curvature flow with a drift term and show that its solution converges to a drifting hyperplane as $t \to \infty$ if the initial value is uniquely ergodic (Theorem 3.1). Some technical lemmas given in Section 3 are valuable also in the analysis of the Allen–Cahn equation.

In Section 4, we study the problem (1)–(2) and complete the proof of Theorems 1.1 and 1.6. More precisely, in Section 4.1, we show the upper and lower bounds for the solution for large $t$. In Section 4.2, we recall a recent result of [2], which states that any entire solution of the Allen–Cahn equation lying between two planar waves is a planar wave (Lemma 4.5). This result will play a crucial role in analyzing basic properties of the $\omega$-limit points of a solution. In Sections 4.3 and 4.4, we give some estimates on the derivatives of the solution, and prove the statements (i) and (ii) of Theorem 1.1. In Sections 4.5 and 4.6, we construct supersolutions and subsolutions by using the solution of the mean curvature flow and complete the proof of Theorem 1.1. Finally, in Section 4.7, we prove Theorem 1.6.

In Appendix A, for the convenience of the reader, we review basic properties of ergodic functions. Among other things we prove the equivalence of different characterizations of unique (or strict) ergodicity as mentioned in Remark 2.1 and Proposition 2.7. We also show that almost periodicity implies strict ergodicity. Finally we give several examples of strictly ergodic functions including the Penrose tiling and aperiodic checker pattern.

Before ending this section, let us introduce some notation. For $\alpha \in (0, 1)$, $C^\alpha(\mathbb{R}^n)$ denotes the Hölder space, that is, the space of functions that are bounded and uniformly Hölder continuous on $\mathbb{R}^n$ with exponent $\alpha$. $C^{2+\alpha}(\mathbb{R}^n)$ denotes the space of functions with $u, u_{x_i}, u_{x_j}, u_{x_i x_j} \in C^\alpha(\mathbb{R}^n)$ for $i, j = 1, 2, \ldots, n - 1$. For the region $R_T = \mathbb{R}^n \times [0, T)$, $C^{\alpha, \alpha/2}(R_T)$ denotes the space of functions that are bounded and uniformly Hölder continuous with exponent $\alpha$ and $\alpha/2$ with respect to space variables $(x, y)$ and time variable $t$, respectively, on $R_T$. $C^{2+\alpha, 1+\alpha/2}(R_T)$ denotes the space of functions that satisfy $u, u_{x_i}, u_{x_j}, u_{x_i x_j} \in C^{\alpha, \alpha/2}(R_T)$. In what follows we always assume that $n \geq 2$ and that (F1)–(F2) hold.

2. Ergodicity in evolution equations

In a large class of evolution equations, the ergodicity of the initial data is inherited by the solution at later times. In this section we state this result in a rather general framework and apply it to the mean curvature flow and the Allen–Cahn equation. The results here will play a crucial role in the proof of Theorem 1.6.

2.1. Basic properties of unique ergodicity

In this subsection we discuss basic properties of uniquely ergodic functions and related topics. Further details will be given in Appendix A. We first deal with the standard notion of ergodicity on $\mathbb{R}^m$ and discuss ergodicity in the $x$-direction at the end of this subsection. We begin with the following remark.

Remark 2.1. A bounded uniformly continuous function $g(x): \mathbb{R}^m \to \mathbb{R}$ is uniquely ergodic if and only if, for any continuous map $\Psi: \mathcal{H}_g \to \mathbb{R}$, the following limit exists uniformly in $a \in \mathbb{R}^m$:

$$
\lim_{R \to \infty} \frac{1}{|B_R(a)|} \int_{B_R(a)} \Psi(\sigma_x g) \, dx, \quad (7)
$$
where $B_R(a)$ denotes the ball with radius $R$ centered at $a \in \mathbb{R}^m$ and $|B_R(a)|$ denotes its volume in $\mathbb{R}^m$. Furthermore, this limit is independent of $a$ and coincides with $\int_{\mathcal{H}_g} \Psi \, d\mu$, where $\mu$ is the (unique) invariant probability measure on $\mathcal{H}_g$. See Proposition 2.7 in the next subsection for details.

**Definition 2.2 (Recurrence and strict ergodicity).** A bounded uniformly continuous function $g(x) : \mathbb{R}^m \to \mathbb{R}$ is called recurrent if the shift dynamics on $\mathcal{H}_g$ is minimal; that is, if for any element $g^* \in \mathcal{H}_g$, there exists a sequence $\{a_j\} \subset \mathbb{R}^m$ such that

$$\sigma_{a_j} g^* \to g \text{ in } L^\infty_{\text{loc}}(\mathbb{R}^m) \text{ as } j \to \infty.$$  

A function is called strictly ergodic if it is recurrent and uniquely ergodic.

We recall that a set $D \subset \mathbb{R}^m$ is called relatively dense in $\mathbb{R}^m$ if there exists a constant $R > 0$ such that any ball in $\mathbb{R}^m$ with radius $R$ contains at least one point in the set $D$. It is then easily seen that a bounded uniformly continuous function $g(x) : \mathbb{R}^m \to \mathbb{R}$ is recurrent if and only if, for any $\varepsilon > 0$ and $M > 0$, the following set is relatively dense in $\mathbb{R}^m$:

$$D_{\varepsilon, M} := \left\{ a \in \mathbb{R}^m \mid \| \sigma_0 g - g \|_{L^\infty}(B_M) \leq \varepsilon \right\},$$

where $B_M := \{ x \in \mathbb{R}^m \mid |x| \leq M \}$.

A related but stronger property is almost periodicity. A bounded continuous function $g(x) : \mathbb{R}^m \to \mathbb{R}$ is almost periodic (in the sense of Bohr) if, for any $\varepsilon > 0$, the following set is relatively dense in $\mathbb{R}^m$:

$$D_{\varepsilon} := \left\{ a \in \mathbb{R}^m \mid \| \sigma_0 g - g \|_{L^\infty}(\mathbb{R}^m) \leq \varepsilon \right\}.$$  

This is equivalent to saying that the shift dynamics on $\mathcal{H}_g$ is minimal with respect to the $L^\infty(\mathbb{R}^m)$-topology. Since $D_\varepsilon \subset D_{\varepsilon, M}$ for every $M > 0$, this property is clearly stronger than recurrence. It is known that a bounded continuous function $g(x) : \mathbb{R}^m \to \mathbb{R}$ is almost periodic if and only if the hull $\mathcal{H}_g$ defined in Definition 1.2 is compact in $L^\infty(\mathbb{R}^m)$ (Bochner's criterion). In this case, the topology of $L^\infty_{\text{loc}}(\mathbb{R}^m)$ and that of $L^\infty(\mathbb{R}^m)$ are equivalent on $\mathcal{H}_g$, hence we have $\mathcal{H}_g = \{ \sigma_0 g | a \in \mathbb{R}^m \}_{L^\infty(\mathbb{R}^m)}$.

As is easily seen, any almost periodic function is strictly ergodic. For the reader's convenience, we will prove it in Proposition A.2 in Appendix A. In particular, any almost periodic function has the uniform mean in the sense that the following limit exists uniformly in $a \in \mathbb{R}^m$ and is independent of $a \in \mathbb{R}^m$:

$$\lim_{R \to \infty} \frac{1}{|B_R(a)|} \int_{B_R(a)} g(x) \, dx.$$  

Furthermore this limit coincides with $\int_{\mathcal{H}_g} \delta(z) \, d\mu$, where $\delta$ is the delta function and $\mu$ is the (unique) $\sigma_0$-invariant probability measure on $\mathcal{H}_g$. Summarizing, we have the following relation:

$$\mathcal{P} \subset \mathcal{QP} \subset \mathcal{AP} \subset \mathcal{SE} \subset \mathcal{UE}, \quad \mathcal{SE} = \mathcal{UE} \cap \mathcal{R}, \quad (8)$$

where $\mathcal{P}$, $\mathcal{QP}$, $\mathcal{AP}$, $\mathcal{SE}$, $\mathcal{UE}$, $\mathcal{R}$ denote, respectively, the sets of periodic functions, quasi-periodic functions, almost periodic functions, strictly ergodic functions, uniquely ergodic functions and recurrent functions.

**Example 2.3 (Uniquely ergodic functions on $\mathbb{R}^m$).** Let us give some examples of uniquely ergodic functions on $\mathbb{R}^m$.  

Remark 2.4

(1) A trivial example is a bounded continuous function that satisfies
\[ g(x) \to C_\infty \text{ as } |x| \to \infty \]
for some constant \( C_\infty \). In this case, as is easily seen, the quantity (7) in Remark 2.1 coincides with \( \Psi(C_\infty) \). Consequently, \( g \) is uniquely ergodic on \( \mathbb{R}^m \). The only \( \sigma_x \)-invariant probability measure on \( \mathcal{H}_g \) is the delta function \( \delta_{C_\infty} \) defined by
\[ \langle \delta_{C_\infty}, \Psi \rangle = \Psi(C_\infty) \text{ for } \Psi \in C(\mathcal{H}_g; \mathbb{R}) \].

Note that this function is not strictly ergodic unless \( g \equiv C_\infty \).

(2) Let \( m = 2 \) and let \( g \) be a bounded uniformly continuous function on \( \mathbb{R}^2 \) whose level sets exhibit the Penrose tiling pattern (Fig. 2, left). As we will see in Appendix A.3, such a function is strictly ergodic on \( \mathbb{R}^2 \) but not almost periodic.

(3) Another simple example is a function of the form \( g(x_1, x_2) = q(x_1) \), where \( q \) is a strictly ergodic function on \( \mathbb{R} \), such as
\[ q(x_1) = \sum_{j \in \mathbb{Z}} \text{sgn}(\cos 2\pi j \theta) \varphi(x_1 - j) \].

Here \( \theta \) is an irrational number, \( \text{sgn}(\cdot) \) is the sign function and \( \varphi \) is a continuous function such that \( \sum_{j \in \mathbb{Z}} \max_{z \in [0, 1]} |\varphi(z - j)| < \infty \) (with \( \varphi \not\equiv 0 \)). This is a slight generalization of the example in Veech [15], where \( \varphi(z) = (\sin \pi z / \pi z)^\alpha \) with \( \alpha > 1 \). Veech proposes this function as an example of almost automorphic function that is not almost periodic. It is easily seen that this function is also strictly ergodic because of the ergodicity of the Kornecker sequence \( (j\theta \text{ mod } 1)_{j \in \mathbb{Z}} \). The above function \( q \) exhibits an ergodic stripe pattern on the plane; see Fig. 2, right.

(4) A function of the form \( g(x_1, x_2) = q(x_1)q(x_2) \) with \( q \) as above is strictly ergodic on \( \mathbb{R}^2 \) and it exhibits an ergodic checker pattern. See Appendix A.4.

**Remark 2.4 (Uniquely ergodic functions in the direction x).** Let us give some examples of uniquely ergodic functions in the direction \( x \).

(1) A trivial example is a bounded continuous function that satisfies
\[ p(x, y) \to p_\infty(y) \text{ as } |x| \to \infty \text{ in } L^\infty_{loc}(\mathbb{R}^n) \]
for some function \( p_\infty \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \). The only \( \sigma_x \)-invariant probability measure on \( \mathcal{H}_p \) is the delta function \( \delta_{p_\infty} \) defined by
\[ \langle \delta_{p_\infty}, \Psi \rangle = \Psi(p_\infty) \text{ for any } \Psi \in C(\mathcal{H}_p; \mathbb{R}) \].

Thus Theorem 1.6 covers the case where \( |u_0(x, y) - \Phi(y)| \to 0 \) as \( |x| + |y| \to \infty \), as mentioned in the introduction.

(2) A function of the form \( p(x, y) = k(y) + h(y)g(x) \), where \( k(y) \), \( h(y) \) are any bounded continuous functions on \( \mathbb{R} \), is uniquely (resp. strictly) ergodic in the direction \( x \) if \( g(x) \) is a uniquely (resp. strictly) ergodic function on \( \mathbb{R}^{n-1} \). Some examples of such \( g(x) \) are given in Example 2.3 with \( m = n - 1 \). A typical situation to which Theorem 1.6 applies is the case \( u_0(x, y) = \Phi(y) + h(y)g(x) \), where \( \limsup_{|y| \to \infty} |h(y)| \) is relatively small.
2.2. Ergodicity in general evolution equations

In this subsection we prove a rather general statement on the inheritance of ergodicity. The proof is simple and elementary, but the result is useful. Let $X$ be a metric space with $\mathbb{R}^m$ action. This means that there exists a family of homeomorphisms $\sigma_a : X \to X$ ($a \in \mathbb{R}^m$) satisfying $\sigma_a \circ \sigma_b = \sigma_{a+b}$. What we have treated in Definitions 1.3 and 1.5 is a special case where $m = n - 1$, $X = C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, which is endowed with the topology of $L^\infty_{loc}(\mathbb{R}^n)$, and $\sigma_a$ is the translation operator $g(x, y) \mapsto g(x + a, y)$.

**Definition 2.5 (Hull of $g \in X$).** Given an element $g \in X$, we define its hull $\mathcal{H}_g$ by

$$
\mathcal{H}_g := \{ \sigma_a g \mid a \in \mathbb{R}^m \},
$$

where $\overline{A}$ stands for the closure of a set $A$ in the $X$-topology.

**Definition 2.6 (Unique ergodicity).** An element $g \in X$ is called uniquely ergodic with respect to $\{ \sigma_a \}_{a \in \mathbb{R}^m}$ if there exists a unique probability measure on $\mathcal{H}_g$ that is $\sigma_a$-invariant for any $a \in \mathbb{R}^m$.

**Proposition 2.7 (Equivalent characterizations).** For any element $g \in X$, the following conditions are mutually equivalent:

(a) $g$ is uniquely ergodic with respect to $\{ \sigma_a \}_{a \in \mathbb{R}^m}$.
(b) For any continuous map $\Psi : \mathcal{H}_g \to \mathbb{R}$, the following limit exists uniformly in $a \in \mathbb{R}^m$:

$$
\lim_{R \to \infty} \frac{1}{|B_R(a)|} \int_{B_R(a)} \Psi(\sigma_a g) \, dx.
$$

(9)

(c) For any continuous map $\Psi : \mathcal{H}_g \to \mathbb{R}$, the following limit exists uniformly in $z \in \mathcal{H}_g$ and is independent of $z$:

$$
\lim_{R \to \infty} \frac{1}{|B_R(0)|} \int_{B_R(0)} \Psi(\sigma_a z) \, dx.
$$

(10)

**Remark 2.8.** It is not important that the average in (9) is taken over a ball. It can be replaced by any other shape, such as a cube; see Remark A.1.

Note that the integrals (9) and (10) both coincide with $\int_{\mathcal{H}_g} \Psi \, d\mu$, where $\mu$ is the unique $\sigma_a$-invariant probability measure on $\mathcal{H}_g$. The above proposition is rather standard at least in the case $m = 1$. However, for the convenience of the reader, we give the proof of Proposition 2.7 in Appendix A.

**Lemma 2.9.** Let $X$, $Y$ be metric spaces with the following $\mathbb{R}^m$ actions:

$$
\sigma_a : X \to X \quad (a \in \mathbb{R}^m), \quad \tilde{\sigma}_a : Y \to Y \quad (a \in \mathbb{R}^m).
$$

Let $p : X \to Y$ be a continuous map such that $p(\sigma_a v) = \tilde{\sigma}_a p(v)$ for every $a \in \mathbb{R}^m$, $v \in X$. If $v$ is uniquely ergodic with respect to $\{ \sigma_a \}_{a \in \mathbb{R}^m}$, then $p(v)$ is uniquely ergodic with respect to $\{ \tilde{\sigma}_a \}_{a \in \mathbb{R}^m}$.

**Proof.** We put $w := p(v)$ and define

$$
\mathcal{T}_v := \{ \sigma_a v \mid a \in \mathbb{R}^m \}, \quad \mathcal{T}_w := \{ \tilde{\sigma}_a w \mid a \in \mathbb{R}^m \}.
$$
Then $\mathcal{H}_v = T_v^X$, $\mathcal{H}_w = T_w^Y$. By the equivariance $p(\sigma_\alpha v) = \tilde{\sigma}_\alpha p(v)$, $p$ maps $T_v$ onto $T_w$ continuously. Hence it maps $\mathcal{H}_v$ onto $\mathcal{H}_w$. Now let $\Psi : \mathcal{H}_w \to \mathbb{R}$ be any continuous map. Then $\Psi \circ p : \mathcal{H}_v \to \mathbb{R}$ is a continuous map. Therefore, by the unique ergodicity of $v$, the following limit exists uniformly in $a \in \mathbb{R}^m$:

$$\lim_{R \to \infty} \frac{1}{|B_R(a)|} \int_{B_R(a)} \Psi(p(\sigma_x v)) \, dx.$$

Since $p(\sigma_x v) = \tilde{\sigma}_x w$, the above limit is equal to

$$\lim_{R \to \infty} \frac{1}{|B_R(a)|} \int_{B_R(a)} \Psi(\tilde{\sigma}_x w) \, dx.$$

Thus, by Proposition 2.7, $w$ is uniquely ergodic with respect to $\{\tilde{\sigma}_\alpha\}_{\alpha \in \mathbb{R}^m}$. The proof of the lemma is complete. \( \square \)

Now we consider an evolution equation on a metric space $X$, which we write as

$$\begin{cases}
\frac{du}{dt} = A(u, t) \quad (t > 0), \\
u(0) = \varphi \in X.
\end{cases} \quad (11)$$

We denote by $u(t; \varphi)$ the solution of (11) with initial value $\varphi$. We assume that there is an $\mathbb{R}^m$ action on $X$, which we denote by $\sigma_\alpha : X \to X \, (\alpha \in \mathbb{R}^m)$.

**Proposition 2.10** (Inheritance of ergodicity). Assume that the following two conditions hold:

(a) [Well-posedness] For each fixed $t \geq 0$, the solution map $\varphi \mapsto u(t; \varphi) : X \to X$ is well defined and continuous;

(b) [Homogeneity] $u(t; \sigma_\alpha \varphi) = \sigma_\alpha u(t; \varphi)$ for every $\varphi \in X$ and $\alpha \in \mathbb{R}^m$.

Then, if the initial value $\varphi$ is uniquely ergodic with respect to $\{\sigma_\alpha\}_{\alpha \in \mathbb{R}^m}$, the same holds for $u(t; \varphi)$ for each fixed $t \geq 0$.

**Proof.** The conclusion follows from Lemma 2.9 by setting $X = Y$ and $p(\varphi) := u(t; \varphi)$. \( \square \)

The above proposition applies to a large class of evolution equations. The first example is the mean curvature flow with a drift term $c \in \mathbb{R}$. It is formulated in terms of the following Cauchy problem if the hypersurface is expressed as a graph on $\mathbb{R}^m$:

$$\begin{cases}
\frac{U_t}{\sqrt{1 + |\nabla_x U|^2}} = \text{div} \left( \frac{\nabla_x U}{\sqrt{1 + |\nabla_x U|^2}} \right) + c, \quad x \in \mathbb{R}^m, \, t > 0, \\
U(x, 0) = U_0(x),
\end{cases} \quad (12)$$

$$U(x, t) = U_0(x), \quad x \in \mathbb{R}^m, \quad (13)$$

where $\nabla_x$ denotes the $m$-dimensional gradient. For this problem, Proposition 2.10 gives the following result:

**Corollary 2.11** (Ergodicity in the mean curvature flow). Let $U_0(x)$ be a bounded Lipschitz continuous function on $\mathbb{R}^m$ that is uniquely ergodic. Then for each fixed $t \geq 0$, the solution $U(x, t)$ of (12)–(13) is uniquely ergodic on $\mathbb{R}^m$. 

Proof. Let

$$X = \{ w \in C(\mathbb{R}^m) \mid \|w\|_{L^\infty(\mathbb{R}^m)} \leq \|U_0\|_{L^\infty(\mathbb{R}^m)} \text{, Lip}(w) \leq \text{Lip}(U_0) \},$$

where Lip(w) denotes the Lipschitz coefficient of a function w(x). We endow X with the topology of $L^\infty_{loc}(\mathbb{R}^m)$, which makes X a complete metric space. Then the problem (12)–(13) is well-posed in X; see [5] for the case $c = 0$ and [4] for more general cases. Thus the unique ergodicity of $U(x, t)$ for each $t \geq 0$ follows from Proposition 2.10 by setting $\sigma_a : g(x) \mapsto g(x + a)$. □

Corollary 2.12 (Ergodicity in the Allen–Cahn equation). Let $u_0(x, y)$ be a uniformly continuous bounded function on $\mathbb{R}^{n-1} \times \mathbb{R}$ that is uniquely ergodic in the x-direction. Then for each fixed $t \geq 0$, the solution $u(x, y, t)$ of (1)–(2) is uniquely ergodic in the x-direction.

Proof. Define

$$X = \{ w \in C(\mathbb{R}^n) \mid \|w\|_{L^\infty(\mathbb{R}^n)} \leq \|u_0\|_{L^\infty(\mathbb{R}^n)} \},$$

and endow X with the $L^\infty_{loc}(\mathbb{R}^n)$-topology. Then the problem (1)–(2) is well-posed in X. Thus the conclusion follows from Proposition 2.10 by setting $\sigma_a : g(x, y) \mapsto g(x + a, y)$. □

As we will see in Lemma 4.15, the above result also implies that the zero-level surface of $u(x, y, t)$ is uniquely ergodic for all large $t \geq 0$.

3. Analysis of the mean curvature flow

In this section we focus on the mean curvature flow with a drift term, that is, the problem (12)–(13). We note that the function $U(x, t) = ct + \mu$ for any fixed $\mu \in \mathbb{R}$ satisfies Eq. (12) and represents a drifting hyperplane with constant speed c. Our objective in this section is to prove the following theorem concerning the asymptotic stability of such drifting hyperplanes.

Theorem 3.1 (Mean curvature flow with ergodic initial values). Let $m \geq 1$. Let $U(x, t)$ be a solution to the problem (12)–(13) whose initial value $U_0(x)$ is bounded, Lipschitz continuous and uniquely ergodic on $\mathbb{R}^m$. Then there exists a constant $\mu \in \mathbb{R}$ such that

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}^m} |U(x, t) - (ct + \mu)| = 0.$$

Remark 3.2. The assumption of Theorem 3.1 is satisfied if, in particular:

(a) $U_0(x) \to C_\infty$ as $|x| \to \infty$ for some constant $C_\infty$.
(b) $U_0$ is either periodic, quasi-periodic or almost periodic on $\mathbb{R}^m$.
(c) $m = 2$ and the level sets of $U_0(x)$ exhibit the Penrose tiling pattern.

See (8) and Example 2.3 in Section 2.1. Note that the cases (a) and (b) generalize the result of [12] to higher dimensions.

To prove the above result, we will first introduce the notion of $\omega$-limit points in Section 3.1. We then show uniform decay of $U_{x_i}$ and $U_{x_ix_j}$ in Section 3.2, which allows us to approximate (12) by a semilinear equation in Section 3.3. Finally we complete the proof of Theorem 3.1 in Section 3.4.

In what follows, we express solutions of (12)–(13) in a moving frame. Setting $v(x, t) = U(x, t) - ct$, the problem (12)–(13) is rewritten as
where $Q_k$ is a constant independent of $T > 0$.

Let $\{Q_k\}_{k=1,2,\ldots}$ be a sequence of compact subsets of $\mathbb{R}^m \times \mathbb{R}$ satisfying

$$Q_1 \subset Q_2 \subset \cdots \quad \text{and} \quad \lim_{k \to \infty} Q_k = \mathbb{R}^m \times \mathbb{R}.$$
This means \( v(x + x_i', t + t_i') \to w(x, t) \) as \( i \to \infty \) in \( C_{loc}^{2,1}(\mathbb{R}^m \times \mathbb{R}) \). The proof of the lemma is complete. \( \square \)

Since the convergence takes place in \( C_{loc}^{2,1}(\mathbb{R}^m \times \mathbb{R}) \), any \( \omega \)-limit point \( w(x, t) \) satisfies Eq. (14) on \( \mathbb{R}^m \times \mathbb{R} \). In other words it is an entire solution of (14). The following lemma is a modification of the result of Berestycki and Hamel [2, Theorem 3.1] on entire solutions of the Allen–Cahn equation:

**Lemma 3.6** (A Liouville type result). Let \( v(x, t) \) be a bounded function satisfying (14) on \( \mathbb{R}^m \times \mathbb{R} \) and assume that \( \nabla v \) is also bounded. Then there exists a constant \( \mu \in \mathbb{R} \) such that

\[
v(x, t) = \mu, \quad x \in \mathbb{R}^m, \ t \in \mathbb{R}.
\]

**Proof.** Fix \( a \in \mathbb{R}^m \) and \( T \in \mathbb{R} \) arbitrarily, and define a function \( v^s(x, t) \) by

\[
v^s(x, t) = v(x + a, t + T) + s.
\]

Since \( v(x, t) \) is bounded on \( \mathbb{R}^m \times \mathbb{R} \), we can define a constant \( s_* \) by

\[
s_* = \inf \left\{ s \in \mathbb{R} \mid v(x, t) \leq v^s(x, t) \text{ for all } (x, t) \in \mathbb{R}^m \times \mathbb{R} \right\}.
\]

Then \( v(x, t) \leq v^{s_*}(x, t) \) and there exists a sequence \( \{(x_i, t_i)\} \subset \mathbb{R}^m \times \mathbb{R} \) satisfying

\[
\lim_{i \to \infty} (v^{s_*}(x_i, t_i) - v(x_i, t_i)) = 0.
\]

Arguing as in the proof of Lemma 3.5, we can choose a subsequence \( \{(x_i', t_i')\} \) of \( \{(x_i, t_i)\} \) such that

\[
v(x + x_i', t + t_i') \to w(x, t) \quad \text{as } i \to \infty \text{ in } C_{loc}^{2,1}(\mathbb{R}^m \times \mathbb{R}),
\]

where \( w \) is some bounded entire solution on \( \mathbb{R}^m \times \mathbb{R} \). Then for each \( (x, t) \in \mathbb{R}^m \times \mathbb{R} \), we have

\[
w(x, t) = \lim_{i \to \infty} v(x + x_i', t + t_i')
\leq \lim_{i \to \infty} v^{s_*}(x + x_i', t + t_i') = w^{s_*}(x, t),
\]

where \( w^{s_*}(x, t) = w(x + a, t + T) + s_* \). In addition, we have

\[
w(0, 0) = \lim_{i \to \infty} v(x_i', t_i') = \lim_{i \to \infty} v^{s_*}(x_i', t_i') = w^{s_*}(0, 0).
\]

Thus, by the strong maximum principle, \( w \equiv w^{s_*} \). On the other hand, we have

\[
\sup_{(x, t) \in \mathbb{R}^m \times \mathbb{R}} w^{s_*}(x, t) = \sup_{(x, t) \in \mathbb{R}^m \times \mathbb{R}} \left( w(x + a, t + T) + s_* \right) = \sup_{(x, t) \in \mathbb{R}^m \times \mathbb{R}} w(x, t) + s_*.
\]

Combining these, we see that \( s_* = 0 \). Consequently, we obtain

\[
v(x, t) \leq v(x + a, t + T), \quad x \in \mathbb{R}^m, \ t \in \mathbb{R}.
\]

Since \( a \in \mathbb{R}^m \) and \( T \in \mathbb{R} \) are both arbitrary, \( v(x, t) \) is independent of \( a \) and \( T \). This completes the proof of the lemma. \( \square \)
Corollary 3.7 (Characterization of \( \omega \)-limit points). Let \( v(x, t) \) be a solution of (14)–(15) whose initial value \( U_0(x) \) is a bounded Lipschitz continuous function. Then any \( \omega \)-limit point of \( v \) is a constant.

**Proof.** Since every constant is a stationary solution of (14)–(15), the comparison principle implies

\[
\inf_{x \in \mathbb{R}^m} U_0(x) \leq v(x, t) \leq \sup_{x \in \mathbb{R}^m} U_0(x), \quad x \in \mathbb{R}^m, \ t \geq 0.
\]

Furthermore, by differentiating (14) by \( x = (x_1, \ldots, x_m) \) and applying the maximum principle, one easily finds that \( \text{Lip}(v(\cdot, t)) \leq \text{Lip}(U_0), \ t \geq 0 \). Consequently, any \( \omega \)-limit point of \( v \) is a bounded entire solution on \( \mathbb{R}^m \times \mathbb{R} \) with bounded gradient. Thus Lemma 3.6 gives the desired result. \( \Box \)

3.2. Uniform decay of derivatives

In this subsection we give uniform decay estimates for the derivatives of the solution \( U(x, t) \) of (12)–(13).

**Proposition 3.8** (Derivative decay). Let \( U(x, t) \) be a solution of (12)–(13) whose initial value \( U_0(x) \) is bounded and Lipschitz continuous on \( \mathbb{R}^m \). Then, for each \( 1 \leq i, j \leq m \),

\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}^m} |U_{x_i}(x, t)| = 0, \quad \lim_{t \to \infty} \sup_{x \in \mathbb{R}^m} |U_{x_ix_j}(x, t)| = 0.
\]

**Proof.** It suffices to show the same estimates for the solution \( v(x, t) \) of (14)–(15). We only show the decay estimate for \( |v_{x_i}| \), since that for \( |v_{x_ix_j}| \) can be obtained in the same way.

Assume that the estimate does not hold. Then there exist a constant \( \delta > 0 \) and a sequence \( \{(x_k, t_k)\} \) such that \( 0 < t_1 < t_2 < \cdots \to \infty \) and that

\[
|v_{x_i}(x_k, t_k)| \geq \delta, \quad \text{for all } k = 1, 2, \ldots.
\]

From Lemma 3.5, we can choose a subsequence \( \{(x_k', t_k')\} \) of \( \{(x_k, t_k)\} \) such that

\[
v(x + x_k', t + t_k') \to w(x, t) \quad \text{as } k \to \infty \text{ in } C^{2,1}_{\text{loc}}(\mathbb{R}^m \times \mathbb{R}),
\]

where \( w \) is an \( \omega \)-limit point of \( v \). Then it holds that

\[
|w_{x_i}(0, 0)| = \lim_{k \to \infty} v_{x_i}(x_k', t_k') \geq \delta.
\]

This, however, is impossible, since \( w(x, t) \) is a constant by Corollary 3.7. This contradiction proves the proposition. \( \Box \)

3.3. Approximation of the mean curvature flow

Our objective in this subsection is to prove the following lemma, which allows us to approximate the mean curvature flow by a semilinear equation under certain circumstances.

**Lemma 3.9** (Approximation of the mean curvature flow). Let \( U(x, t; \varphi) \) and \( V(x, t; \varphi) \) denote the solutions of the equations

\[
\frac{U_t}{\sqrt{1 + |\nabla U|^2}} = \text{div}\left( \frac{\nabla U}{\sqrt{1 + |\nabla U|^2}} \right) + c, \quad x \in \mathbb{R}^m, \ t > 0, \quad (17)
\]

\[
V_t = \Delta x V + \frac{c}{2} |\nabla x V|^2 + c, \quad x \in \mathbb{R}^m, \ t > 0, \quad (18)
\]
under the initial conditions \( U(\cdot, 0) = V(\cdot, 0) = \varphi \in W^{2,\infty}(\mathbb{R}^m) \). Then, for any constant \( \varepsilon > 0 \), there exists a constant \( \delta > 0 \) such that if \( \| \nabla_x \varphi \|_{W^{1,\infty}} \leq \delta \), it holds that

\[
\sup_{x \in \mathbb{R}^m} |U(x, t; \varphi) - V(x, t; \varphi)| \leq \varepsilon \quad \text{for all } t \geq 0.
\]

To prove this result, we prepare an auxiliary lemma concerning the decay estimates for the derivatives of the solution of (18).

**Lemma 3.10.** Let \( V(x, t) \) be a solution to the problem

\[
\begin{aligned}
V_t &= \Delta_x V + \frac{c}{2} |\nabla_x V|^2 + c, \quad x \in \mathbb{R}^m, \ t > 0, \\
V(x, 0) &= V_0(x), \quad x \in \mathbb{R}^m.
\end{aligned}
\]

Then the following estimates hold:

\[
\sup_{x \in \mathbb{R}^m} |V_{xi}(x, t)| \leq \min\{C_0 t^{-\frac{1}{2}}, C_1\},
\]

\[
\sup_{x \in \mathbb{R}^m} |V_{xixj}(x, t)| \leq \min\{C_0 t^{-1}, C_2\},
\]

\[
\sup_{x \in \mathbb{R}^m} |V_{xixjxk}(x, t)| \leq C_3 (1 + t)^{-\frac{3}{2}},
\]

\[
\sup_{x \in \mathbb{R}^m} |V_{xixjxk}(x, t)| \leq C_4 (1 + t)^{-\frac{3}{2}},
\]

for each \( 1 \leq i, j, k \leq m \), where \( C_0, C_1, C_2, C_3 \) and \( C_4 \) are positive constants such that

(i) \( C_0 \) depends only on \( c \) and \( \| V_0 \|_{L^\infty} \),

(ii) \( C_1 \) depends only on \( c, \| V_0 \|_{L^\infty} \) and \( \| \nabla_x V_0 \|_{L^\infty} \), and satisfies

\[ C_1 \to 0 \text{ as } \| \nabla_x V_0 \|_{L^\infty} \to 0. \]

(iii) \( C_2 \) depends only on \( c, \| V_0 \|_{L^\infty} \) and \( \| \nabla_x V_0 \|_{W^{1,\infty}} \), and satisfies

\[ C_2 \to 0 \text{ as } \| \nabla_x V_0 \|_{W^{1,\infty}} \to 0. \]

(vi) \( C_3 \) and \( C_4 \) depend only on \( c \) and \( \| V_0 \|_{W^{3,\infty}} \).

**Proof.** We first consider the case where \( c \neq 0 \). Define \( h(x, t) := \exp(c/2 \cdot (V(x, t) - ct)) \). Then it is easily seen that \( h \) is a solution of the following Cauchy problem for the linear heat equation:

\[
\begin{aligned}
h_t &= \Delta_x h, \quad x \in \mathbb{R}^m, \ t > 0, \\
h(x, 0) &= \exp\left(\frac{c}{2} V_0(x)\right), \quad x \in \mathbb{R}^m.
\end{aligned}
\]

Thus we have

\[
h(x, t) = \frac{1}{(4\pi t)^{\frac{m}{2}}} \int_{\mathbb{R}^m} \exp\left(-\frac{|x - \eta|^2}{4t}\right) \cdot \exp\left(\frac{c}{2} V_0(\eta)\right) d\eta.
\]
Using this expression, we easily find that

\[
\sup_{x \in \mathbb{R}^m} \lvert h_{x_i}(x, t) \rvert \leq \min \left\{ C \lVert h_0 \rVert_{L^\infty} \cdot t^{-\frac{1}{2}}, \lVert h_{0, x_i} \rVert_{L^\infty} \right\},
\]

\[
\sup_{x \in \mathbb{R}^m} \lvert h_{x_i x_j}(x, t) \rvert \leq \min \left\{ C \lVert h_0 \rVert_{L^\infty} \cdot t^{-1}, \lVert h_{0, x_i x_j} \rVert_{L^\infty} \right\},
\]

\[
\sup_{x \in \mathbb{R}^m} \lvert h_{x_i x_j x_k}(x, t) \rvert \leq \min \left\{ C \lVert h_0 \rVert_{L^\infty} \cdot t^{-\frac{3}{2}}, \lVert h_{0, x_i x_j x_k} \rVert_{L^\infty} \right\},
\]

where \( h_0(x) := h(x, 0) \) and \( C > 0 \) is a constant depending only on \( n \). Next, since \( V(x, t) = 2/c \cdot \log(h(x, t)) + ct \), we have

\[
V_{x_i} = \frac{2h_{x_i}}{ch}, \quad V_{x_i x_j} = \frac{2h_{x_i x_j}}{ch^2} - \frac{2h_{x_i} h_{x_j}}{ch^2},
\]

\[
V_{x_i x_j x_k} = \frac{2h_{x_i x_j x_k}}{ch} - \frac{2h_{x_i x_j} h_{x_k}}{ch^2} - \frac{2h_{x_i x_k} h_{x_j}}{ch^2} + \frac{4h_{x_i} h_{x_j} h_{x_k}}{ch^3},
\]

where we note that

\[
h(x, t) \geq \inf_{x \in \mathbb{R}^m} h(x, 0) = \inf_{x \in \mathbb{R}^m} \exp \left( \frac{c}{2} V_0(x) \right) > 0.
\]

Combining these formulas and the above estimates for \( h_{x_i} \), \( h_{x_i x_j} \) and \( h_{x_i x_j x_k} \), we obtain the desired estimates for \( V_{x_i} \), \( V_{x_i x_j} \) and \( V_{x_i x_j x_k} \), provided that \( c \neq 0 \). In the case where \( c = 0 \), the problem (19)–(20) itself is the Cauchy problem for the heat equation on \( \mathbb{R}^m \). Thus the desired estimates for \( V_{x_i} \), \( V_{x_i x_j} \) and \( V_{x_i x_j x_k} \) follow more directly. The estimate for \( V_{x_i t} \) then follows by simply differentiating Eq. (19) with respect to \( x_i \) and using the estimates for \( V_{x_i} \), \( V_{x_i x_j} \) and \( V_{x_i x_j x_k} \). This completes the proof of the lemma.

Now we complete the proof of Lemma 3.9.

**Proof of Lemma 3.9.** We construct supersolutions and subsolutions for (17) by using the solution \( V(x, t) \) of (18). For this purpose, we define

\[
L[\eta] := \frac{\eta_t}{\sqrt{1 + |\nabla_x \eta|^2}} - \text{div} \left( \frac{\nabla_x \eta}{\sqrt{1 + |\nabla_x \eta|^2}} \right) - c.
\]

We can rewrite it as

\[
L[\eta] = \frac{1}{\sqrt{1 + |\nabla_x \eta|^2}} \left( \eta_t - \Delta_x \eta + \sum_{i,j=1}^m \frac{\eta_{x_i} \eta_{x_j} \eta_{x_i x_j}}{1 + |\nabla_x \eta|^2} - c \sqrt{1 + |\nabla_x \eta|^2} \right)
\]

\[
= \frac{1}{\sqrt{1 + |\nabla_x \eta|^2}} \left( \eta_t - \Delta_x \eta - \frac{c}{2} |\nabla_x \eta|^2 - c \right)
\]

\[
+ \frac{1}{\sqrt{1 + |\nabla_x \eta|^2}} \left( \sum_{i,j=1}^m \frac{\eta_{x_i} \eta_{x_j} \eta_{x_i x_j}}{1 + |\nabla_x \eta|^2} + \frac{c |\nabla_x \eta|^4}{2(\sqrt{1 + |\nabla_x \eta|^2} + 1)^2} \right).
\]
Let \( V(x, t) \) be the solution of (18) and define \( V^+(x, t) := V(x, t) + p(t) \), where \( p(t) \) is a smooth function to be determined later. Then we have

\[
L[V^+] \geq \frac{1}{\sqrt{1 + |\nabla_x V|^2}} \left( p' - \sum_{i,j=1}^{m} |V_{x_i}V_{x_j}V_{x_i,x_j}| - |c||\nabla_x V|^4 \right).
\]

Let any \( \varepsilon > 0 \) be fixed. From Lemma 3.10, there exists a constant \( C > 0 \) depending only on \( c \) and \( \|\varphi\|_{L^\infty} \) such that \( \sum_{i,j=1}^{m} |V_{x_i}V_{x_j}V_{x_i,x_j}| + |c||\nabla_x V|^4 \leq Ct^{-2} \). Again by Lemma 3.10, there exists a constant \( \delta > 0 \) such that if \( \|\nabla_x \varphi\|_{W^{1,\infty}} \leq \delta \), the following estimate holds:

\[
\sum_{i,j=1}^{m} |V_{x_i}V_{x_j}V_{x_i,x_j}| + |c||\nabla_x V|^4 \leq \min \left\{ \frac{\varepsilon^2}{16C}, Ct^{-2} \right\}.
\]

Now we choose a smooth function \( q(t) \) satisfying

\[
\min \left\{ \frac{\varepsilon^2}{16C}, Ct^{-2} \right\} \leq q(t) \leq 2 \min \left\{ \frac{\varepsilon^2}{16C}, Ct^{-2} \right\},
\]

and define \( p(t) = \int_{0}^{t} q(s) \, ds \). Then we have \( 0 \leq p(t) \leq \varepsilon \) for \( t \geq 0 \) and

\[
L[V^+] \geq \frac{1}{\sqrt{1 + |\nabla_x V|^2}} \left( q' - \sum_{i,j=1}^{m} |V_{x_i}V_{x_j}V_{x_i,x_j}| - |c||\nabla_x V|^4 \right) \geq 0.
\]

That is, the function \( V^+(x, t) = V(x, t) + p(t) \) is a supersolution of (17). Consequently, the comparison principle implies that

\[
U(x, t) \leq V^+(x, t) = V(x, t) + p(t) \leq V(x, t) + \varepsilon, \quad x \in \mathbb{R}^m, \ t \geq 0.
\]

Similarly, we obtain \( U(x, t) \geq V(x, t) - \varepsilon \) by setting \( V^-(x, t) = V(x, t) - p(t) \), which is a subsolution of (17). This completes the proof of Lemma 3.9. \( \square \)

### 3.4. Proof of Theorem 3.1

In this subsection we complete the proof of Theorem 3.1. For this purpose, we provide an auxiliary lemma concerning the large time behavior of the solution of (19)–(20) with a uniquely ergodic initial value.

**Lemma 3.11.** Let \( V(x, t) \) be a solution to the problem (19)–(20) whose initial value \( V_0(x) \) is uniquely ergodic. Then there exists a constant \( \mu \in \mathbb{R} \) such that

\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}^m} |V(x, t) - (\mu + ct)| = 0.
\]

**Proof.** We first consider the case where \( c \neq 0 \). Let \( h(x, t) := \exp(c/2 \cdot (V(x, t) - ct)) \). Then \( h \) solves the Cauchy problem for the heat equation:

\[
\begin{cases}
  h_t = \Delta_x h, & x \in \mathbb{R}^m, \ t > 0, \\
  h(x, 0) = \exp \left( \frac{c}{2} V_0(x) \right), & x \in \mathbb{R}^m.
\end{cases}
\]
Consequently, we have
\[ V(x, t) = \frac{2}{c} \log \left( \int_{\mathbb{R}^m} G(x - \eta, t) \exp \left( \frac{c}{2} V_0(\eta) \right) d\eta \right) + ct, \]
where \( G(\xi, s) \) is the heat kernel on \( \mathbb{R}^m \) given by
\[ G(\xi, s) = \left( \frac{4\pi s}{} \right)^{-m/2} \exp(-|\xi|^2/4s). \]
Since \( V_0(x) \) is uniquely ergodic, by Remark 2.1, the function \( \exp(c/2 \cdot V_0(x)) \) has uniform mean in the sense that the following limit exists uniformly in \( a \in \mathbb{R}^m \) and is independent of \( a \):
\[ \mu^* := \lim_{R \to \infty} \frac{1}{|B_R(a)|} \int_{B_R(a)} \exp \left( \frac{c}{2} V_0(x) \right) dx. \]
This implies
\[ \int_{\mathbb{R}^m} G(x - \eta, t) \exp \left( \frac{c}{2} V_0(\eta) \right) d\eta \to \mu^* \quad \text{as} \quad t \to \infty, \]
uniformly in \( x \in \mathbb{R}^m \). Consequently, we obtain
\[ \lim_{t \to \infty} \sup_{x \in \mathbb{R}^m} \left| V(x, t) - \left( \frac{2}{c} \log \mu^* + ct \right) \right| = 0, \]
which proves the claim for the case \( c \neq 0 \). The case \( c = 0 \) can be shown similarly, since \( V(x, t) \) itself satisfies the heat equation. This completes the proof of the lemma. \( \square \)

Now we are ready to prove Theorem 3.1.

**Proof of Theorem 3.1.** For the solution \( U(x, t) \) of (12)–(13), we define \( V(x, t; \tau) \) as a function satisfying
\[
\begin{align*}
V_t &= \Delta_x V + \frac{c}{2} |\nabla_x V|^2 + c, \quad x \in \mathbb{R}^m, \ t > 0, \\
V(x, 0) &= U(x, \tau), \quad x \in \mathbb{R}^m.
\end{align*}
\]
Since Corollary 2.11 gives the unique ergodicity of \( U(x, \tau) \) at each \( \tau \geq 0 \), we see from Lemma 3.11 that there exists a constant \( \mu(\tau) \in \mathbb{R} \) such that
\[ \lim_{t \to \infty} \sup_{x \in \mathbb{R}^m} \left| V(x, t; \tau) - \left( \mu(\tau) + ct \right) \right| = 0. \]  
(21)

By Proposition 3.8 and Lemma 3.9, for any \( \varepsilon > 0 \), there exists a constant \( \tau_\varepsilon > 0 \) such that \( |U(x, t) - V(x, t - \tau_\varepsilon \tau_\varepsilon)| \leq \varepsilon \) for \( x \in \mathbb{R}^m, \ t \geq \tau_\varepsilon \); hence
\[
|U(x, t) - \left( \mu(\tau_\varepsilon) + c(t - \tau_\varepsilon) \right)| \leq |U(x, t) - V(x, t - \tau_\varepsilon \tau_\varepsilon)|
+ |V(x, t - \tau_\varepsilon \tau_\varepsilon) - \left( \mu(\tau_\varepsilon) + c(t - \tau_\varepsilon) \right)|
\leq \varepsilon + |V(x, t - \tau_\varepsilon \tau_\varepsilon) - \left( \mu(\tau_\varepsilon) + c(t - \tau_\varepsilon) \right)|,
\]
for any \( t \geq \tau_\varepsilon \). Combining this with (21), we see that
\[
\sup_{x \in \mathbb{R}^m} |U(x,t) - (\mu(t) + c(t - \tau_\varepsilon))| \leq 2\varepsilon,
\]
for all sufficiently large \( t > 0 \). This implies, in particular, that there exists a constant \( \mu \in \mathbb{R} \) such that \( (\mu(t) - c\tau_\varepsilon) \rightarrow \mu \) as \( \varepsilon \rightarrow 0 \). Consequently, we obtain
\[
\lim_{t \rightarrow \infty} \sup_{x \in \mathbb{R}^m} |U(x,t) - (\mu + ct)| = 0.
\]
This completes the proof of Theorem 3.1. \( \square \)

4. Analysis of the Allen–Cahn equation

Now we move on to the analysis of the Allen–Cahn equation and prove our main theorems, namely Theorems 1.1 and 1.6. As is mentioned in the introduction, the main novelty of Theorem 1.1 is the approximation of the Allen–Cahn equation by the mean curvature flow for up to \( t = \infty \). Once this is shown, Theorem 1.6 then easily follows from Theorem 3.1.

In Section 4.1, we give rough upper and lower bounds for the solution at large time. In Sections 4.2 and 4.3, we introduce the notion of \( \omega \)-limit points of the solution and derive useful estimates for the derivatives of the solution. In Section 4.4, we study basic properties of the zero-level surface of the solution. We then construct a fine set of supersolutions and subsolutions in Section 4.5, and give the proofs of the main theorems in Sections 4.6 and 4.7.

We will express solutions \( u(x,y,t) \) of (1)–(2) in a moving frame, so that the planar waves can be viewed as stationary states. Setting
\[
\begin{align*}
    u(x,y,t) &= w(x,z,t), \\
    z &= y - ct,
\end{align*}
\]
Eq. (1) is rewritten as
\[
w_t = \Delta w + cw_z + f(w), \quad x \in \mathbb{R}^{n-1}, \ z \in \mathbb{R}, \ t > 0,
\]
where \( \Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_{n-1}^2 + \partial^2/\partial z^2 \). We write \( w(x,z,t) \) as \( u(x,z,t) \) for simplicity and consider the problem of the form
\[
\begin{align*}
    \left\{ 
    &u_t = \Delta u + cu_z + f(u), \quad x \in \mathbb{R}^{n-1}, \ z \in \mathbb{R}, \ t > 0, \\
    &u(x,z,0) = u_0(x,z), \quad x \in \mathbb{R}^{n-1}, \ z \in \mathbb{R},
    \end{align*}
\]
Note that, for each constant \( \xi \in \mathbb{R} \), the function \( \Phi(z-\xi) \) is a stationary solution of this problem. Throughout this section, even if it is not mentioned specifically, we always assume that the initial value \( u_0 \) is bounded and uniformly continuous on \( \mathbb{R}^n \), and satisfies
\[
\liminf_{z \rightarrow -\infty} \inf_{x \in \mathbb{R}^{n-1}} u_0(x,z) > s_+, \quad \limsup_{z \rightarrow \infty} \sup_{x \in \mathbb{R}^{n-1}} u_0(x,z) < s_-, \quad (24)
\]
where \( s_\pm \) are the constants in the assumption (F1).

4.1. Upper and lower bounds at large time

Our objective in this subsection is to prove the following preliminary estimates:

**Lemma 4.1 (Upper and lower bounds at large time).** Let \( u(x,z,t) \) be a solution of (22)–(23). Then there exist constants \( z_+, z^- \in \mathbb{R} \) such that
\[
\liminf_{t \to \infty} \inf_{x \in \mathbb{R}^{n-1}} u(x, z, t) \geq \Phi(z - z_+), \quad \text{uniformly in } z \in \mathbb{R}, \quad (25)
\]
\[
\limsup_{t \to \infty} \sup_{x \in \mathbb{R}^{n-1}} u(x, z, t) \leq \Phi(z - z_-), \quad \text{uniformly in } z \in \mathbb{R}. \quad (26)
\]

To prove this, we recall a well-known lemma by Fife and McLeod [6] for the one-dimensional problem:

\[
\begin{align*}
\begin{cases}
  u_t = u_{zz} + cu_z + f(u), & z \in \mathbb{R}, \ t > 0, \\
  u(z, 0) = u_0(z), & z \in \mathbb{R}.
\end{cases} 
\end{align*} \quad (27)
\]
\[
\begin{align*}
\begin{cases}
  u_0 = u_0^+ + u_0^- \quad (28)
\end{cases}
\end{align*}
\]

**Lemma 4.2.** (See [6].) Let \( s_{\pm} \) be as in the assumption (F1). Then, for any \( \delta_1 \in (0, s_- + 1) \) and any \( \delta_2 \in (0, 1 - s_+) \), there exist constants \( \beta > 0 \) and \( C \geq 1 \) depending only on \( \delta_1, \delta_2 \) and \( f \) such that the functions \( u^+ (z, t) \) and \( u^- (z, t) \) given by

\[
\begin{align*}
u^+ (z, t) &= \Phi (z - C\delta_1 (1 - e^{-\beta t})) + \delta_1 e^{-\beta t}, \\
u^- (z, t) &= \Phi (z + C\delta_2 (1 - e^{-\beta t})) - \delta_2 e^{-\beta t},
\end{align*}
\]

are a supersolution and a subsolution, respectively. More precisely,

\[
\begin{align*}
L[u^+] := u^+_t - u^+_{zz} - cu^+_z - f(u^+) &> 0, \\
L[u^-] := u^-_t - u^-_{zz} - cu^-_z - f(u^-) &\leq 0.
\end{align*}
\]

Note that every solution \( u(z, t) \) to the one-dimensional problem (27)–(28) is also a solution to the multi-dimensional problem (22)–(23) with \( n \geq 2 \).

**Proof of Lemma 4.1.** We only show the upper bound (26), since the other is similar. Let \( u^+ (z, t) \) be as in Lemma 4.2. Then it suffices to show that there exist constants \( T > 0, z_0 \in \mathbb{R} \) and \( \delta_1 \in (0, s_- + 1) \) such that

\[
u(x, z, T) \leq \Phi (z - z_0) + \delta_1 = u^+ (z - z_0, 0), \quad (x, z) \in \mathbb{R}^n. \quad (29)
\]

Indeed, the comparison principle and (29) give \( u(x, z, t) \leq u^+ (z - z_0, t - T) \) for \( t \geq T \), which yields (26) by letting \( t \to \infty \).

Now, in order to show (29), we choose a constant \( \delta_1 \in (0, s_- + 1) \) satisfying

\[
\limsup_{z \to \infty} \sup_{x \in \mathbb{R}^{n-1}} u_0(x, z) < -1 + \delta_1.
\]

Then, since \( f(s) < 0 \) for \( s > 1 \) by the assumption (F1), we see from the comparison principle that

\[
u(x, z, T) \leq 1 + \frac{\delta_1}{2} \quad \text{for } (x, z) \in \mathbb{R}^n \quad (30)
\]

for all sufficiently large \( T > 0 \). Next we show that

\[
\limsup_{z \to \infty} \sup_{x \in \mathbb{R}^{n-1}} u(x, z, T) < -1 + \delta_1 \quad (31)
\]

for each \( T > 0 \). For this purpose, choose constants \( \beta, M \) such that
Lemma 4.5. Let \( u_0(x, z) \leq \beta < -1 + \delta_1 \) and that
\[
u_0(x, z) \leq \beta + Me^{-cz}, \quad (x, z) \in \mathbb{R}^n.
\]
Then the function \( u(x, z) = \beta + Me^{-c(z-aT)} \) is a supersolution of (22) if \( a > 0 \) is chosen sufficiently large. Hence \( u(x, z, T) \leq \beta + Me^{-c(z-aT)} \). This proves (31). The assertion (29) then follows immediately by combining (30) and (31). This completes the proof of the lemma. \( \square \)

4.2. \( \omega \)-limit points in the Allen–Cahn equation

In this subsection, we first introduce the notion of \( \omega \)-limit points of the solution \( u(x, z, t) \) of (22)–(23), where we consider a sequence both in \( x \) and \( t \), similarly to Definition 3.3. We then show that any \( \omega \)-limit point is a planar wave under the assumption (24).

Definition 4.3 (\( \omega \)-limit point). A function \( w(x, z, t) \) defined on \( \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \) is called an \( \omega \)-limit point of the solution \( u(x, z, t) \) of (22)–(23) if there exists a sequence \( \{(x_i, t_i)\} \) such that \( 0 < t_1 < t_2 < \cdots \to \infty \) and that
\[
u(x + x_i, z, t + t_i) \to w(x, z, t) \quad \text{as} \quad i \to \infty \quad \text{in} \quad C^{2,1}_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}).
\]

Remark 4.4 (Construction of \( \omega \)-limit point). Let \( u(x, z, t) \) be a solution of (22)–(23). Then for any sequence \( \{(x_i, t_i)\} \) with \( 0 < t_1 < t_2 < \cdots \to \infty \), there exist a subsequence \( \{(x'_i, t'_i)\} \) and an \( \omega \)-limit point \( w(x, z, t) \) of \( u(x, z, t) \) such that
\[
u(x + x'_i, z, t + t'_i) \to w(x, z, t) \quad \text{as} \quad i \to \infty \quad \text{in} \quad C^{2,1}_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}).
\]

Indeed, since \( u_0 \) is bounded on \( \mathbb{R}^n \), the assumption (F1) and the comparison principle imply that \( u(x, z, t) \) is bounded on \( \mathbb{R}^n \times [0, \infty) \). Therefore, by standard parabolic estimates, the solution \( u \) belongs to \( C^{2+\alpha,1+\alpha/2}_{\text{loc}}(\mathbb{R}^n \times [\delta, T]) \) for any \( 0 < \delta < T \). Furthermore
\[
\|u\|_{C^{2+\alpha,1+\alpha/2}_{\text{loc}}(\mathbb{R}^n \times [\delta, T])} \leq C,
\]
where \( C > 0 \) is a constant independent of \( T > 0 \). Thus the \( \omega \)-limit point can be constructed in a way similar to the proof of Lemma 3.5.

Berestycki and Hamel [2] recently obtained the following result that states that any entire solution of the Allen–Cahn equation lying between two planar waves is itself a planar wave. This result turns out to be exceedingly useful for our analysis.

Lemma 4.5. (See [2, Theorem 3.1].) Let \( u(x, z, t) \) be a function that is defined on \( \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \) and satisfies
\[
u_t = \Delta u + cu_z + f(u), \quad (x, z) \in \mathbb{R}^n, \quad t \in \mathbb{R}.
\]
Assume further that there exist two constants \( z_\alpha, z^\alpha \in \mathbb{R} \) such that
\[
\Phi(z - z_\alpha) \leq u(x, z, t) \leq \Phi(z - z^\alpha), \quad (x, z) \in \mathbb{R}^n, \quad t \in \mathbb{R}.
\]
Then there exists a constant \( z_0 \in [z_\alpha, z^\alpha] \) such that
\[
u(x, z, t) = \Phi(z - z_0), \quad (x, z) \in \mathbb{R}^n, \quad t \in \mathbb{R}.
\]
Though we do not repeat the proof of the above result here, we note that our proof of Lemma 3.6 in the previous section is largely adopted from the argument in [2].

**Corollary 4.6 (Characterization of \( \omega \)-limit points).** Let \( u(x, z, t) \) be a solution of (22)–(23). Then any \( \omega \)-limit point \( w(x, z, t) \) of \( u \) is a planar wave, that is, there exists a constant \( z_0 \in \mathbb{R} \) such that

\[
w(x, z, t) = \Phi(z - z_0), \quad (x, z) \in \mathbb{R}_n, \quad t \in \mathbb{R}.
\]

**Proof.** By Lemma 4.1, any \( \omega \)-limit point \( w \) of \( u \) satisfies

\[
\Phi(z - z_\omega) \leq w(x, z, t) \leq \Phi(z - z^\omega), \quad (x, z) \in \mathbb{R}_n, \quad t \in \mathbb{R}
\]

for some constants \( z_\omega, z^\omega \in \mathbb{R} \). Thus the conclusion follows immediately from Lemma 4.5. \( \square \)

### 4.3. Estimates of the derivatives

In this subsection we derive estimates for the derivatives of the solution of (22)–(23). For this purpose, Corollary 4.6 in the previous subsection plays a crucial role.

**Lemma 4.7 (Monotonicity in \( z \)).** Let \( u(x, z, t) \) be a solution of (22)–(23). Then for any constant \( R > 0 \), there exists a constant \( T > 0 \) such that

\[
\inf_{x \in \mathbb{R}^{n-1}, |z| \leq R, t \geq T} -u_z(x, z, t) > 0.
\]

**Proof.** If the above claim does not hold, then there exists a sequence \( \{(x_k, z_k, t_k)\} \) such that \( \{z_k\} \subset [-R, R] \), \( 0 < t_1 < t_2 < \cdots \to \infty \), and that

\[
\liminf_{k \to \infty} u_z(x_k, z_k, t_k) \geq 0.
\]

Replacing \( \{(x_k, z_k, t_k)\} \) by its subsequence if necessary, we may assume without loss of generality that \( z_k \) converges to some limit \( z_\infty \in [-R, R] \) and that

\[
u(x + x_k, z, t + t_k) \to w(x, z, t) \quad \text{as} \quad k \to \infty \quad \text{in} \quad C^{2,1}_\text{loc}(\mathbb{R}_n \times \mathbb{R}),
\]

where \( w \) is an \( \omega \)-limit point of \( u \). Hence

\[
w_z(0, z_\infty, 0) = \lim_{k \to \infty} u_z(x_k, z_k, t_k) \geq 0.
\] (33)

On the other hand, Corollary 4.6 shows that \( w(x, z, t) = \Phi(z - z_0) \) for some \( z_0 \in \mathbb{R} \). This contradicts (33) since \( \Phi'(z) < 0 \). The proof of the lemma is complete. \( \square \)

**Corollary 4.8 (Monotonicity in \( z \) around the zero-level).** Let \( u(x, z, t) \) be a solution of (22)–(23). Then there exists a constant \( T > 0 \) such that

\[
\inf_{(x, z, t) \in D} -u_z(x, z, t) > 0,
\] (34)

where \( D = \{(x, z, t) \in \mathbb{R}_n \times [T, \infty) \ | \ |u(x, z, t)| \leq 1/2\}. \)

Corollary 4.8 follows immediately by combining Lemma 4.7 with the upper and lower bounds for \( u(x, z, t) \) given in Lemma 4.1. The next lemma is concerned with the uniform decay estimate for \( |u_{x_i}| \) and \( |u_{x_i x_j}| \). The proof is similar to that of Lemma 4.7.
Fig. 3. Perturbed fronts with non-graphical and graphical zero-level surfaces.

**Lemma 4.9** *(Decay of x-derivatives).* Let \( u(x, z, t) \) be a solution of (22)–(23). Then for any constant \( R > 0 \), it holds that

\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}^{n-1}, |z| \leq R} |u_{x_i}(x, z, t)| = 0, \quad \lim_{t \to \infty} \sup_{x \in \mathbb{R}^{n-1}, |z| \leq R} |u_{x_i x_j}(x, z, t)| = 0,
\]

for each \( 1 \leq i, j \leq n-1 \).

**Proof.** We only show the decay estimate for \( |u_{x_i}| \), since that for \( |u_{x_i x_j}| \) can be obtained in a similar way. If the estimate does not hold, there exist a constant \( \delta > 0 \) and a sequence \( \{(x_k, z_k, t_k)\} \) such that \( \{z_k\} \subset [-R, R] \), \( 0 < t_1 < t_2 < \cdots \to \infty \), and

\[
|u_{x_i}(x_k, z_k, t_k)| \geq \delta, \quad \text{for all } k = 1, 2, \ldots.
\]

Replacing \( \{(x_k, z_k, t_k)\} \) by its subsequence if necessary, we may assume without loss of generality that (32) holds for some \( \omega \)-limit point \( w \) and that \( \{z_k\} \) converges to some limit \( z_* \) as \( k \to \infty \). This implies

\[
\left| w_{x_i}(0, z_*, 0) \right| = \left| \lim_{k \to \infty} u_{x_i}(x_k, z_k, t_k) \right| \geq \delta. \tag{35}
\]

On the other hand, since \( w(x, z, t) \) is a planar wave by Corollary 4.6, we have \( w_{x_i} \equiv 0 \). This contradicts (35), completing the proof of the lemma. \( \square \)

**4.4. Zero-level surface of the solution**

From Corollary 4.8 and Lemma 4.9, we can derive the following lemma that shows that the zero-level surface of the solution \( u(x, z, t) \) has a graphical representation \( z = \Gamma(x, t) \) for all large \( t \) (see Fig. 3). In what follows we write \( \Gamma(x, t) \) for the zero-level surface of \( u(x, z, t) \) of (22)–(23), while that for \( u(x, y, t) \) of (1)–(2) is denoted by \( \gamma(x, t) \) as in Theorem 1.1. Clearly \( \gamma(x, t) = \Gamma(x, t) + ct \).

**Lemma 4.10** *(Zero-level surface).* Let \( u(x, z, t) \) be a solution of (22)–(23) and let \( T > 0 \) be as defined in Corollary 4.8. Then there exists a smooth bounded function \( \Gamma(x, t) \) such that

\[
u(x, z, t) = 0 \quad \text{if and only if} \quad z = \Gamma(x, t), \tag{36}
\]

for any \( (x, t) \in \mathbb{R}^{n-1} \times [T, \infty) \). Furthermore the following estimates hold:
(i) For each $1 \leq i, j \leq n - 1$,

$$
\lim_{t \to -\infty} \sup_{x \in \mathbb{R}^{n-1}} |\Gamma_{x_i}(x, t)| = 0, \quad \lim_{t \to \infty} \sup_{x \in \mathbb{R}^{n-1}} |\Gamma_{x_i x_j}(x, t)| = 0.
$$

(ii) There exists a constant $M > 0$ such that, for each $1 \leq i, j \leq n - 1$,

$$
\sup_{x \in \mathbb{R}^{n-1}} |\Gamma_{x_i x_j x_k}(x, t)| \leq M, \quad \text{for } t \geq T.
$$

**Proof.** Since $D := \{(x, z, t) \in \mathbb{R}^n \times [T, \infty); \ |u(x, z, t)| \leq 1/2\}$ is bounded in the $z$-direction by virtue of Lemma 4.1 and the fact that $\Phi(\pm \infty) = \mp 1$, we can define a bounded function $\Gamma(x, t)$ satisfying (36) thanks to Corollary 4.8. Here $\Gamma(x, t)$ is smooth by the implicit function theorem, since $u(x, z, t)$ is smooth for $t > 0$. Differentiating the formula $u(x, \Gamma(x, t), t) = 0$ by $x_i$ and $x_j$, we have $u_{x_i} + u_z \Gamma_{x_i} = 0$ and $u_{x_i x_j} + u_{zz} \Gamma_{x_i} \Gamma_{x_j} + u_z \Gamma_{x_i x_j} = 0$; hence

$$
\Gamma_{x_i} = -\frac{u_{x_i}}{u_z}, \quad \Gamma_{x_i x_j} = -\frac{u_{x_i x_j}}{u_z} - \frac{u_{xz} \Gamma_{x_i} \Gamma_{x_j}}{u_z}.
$$

This, along with Corollary 4.8 and Lemma 4.9, gives the desired decay estimates for $|\Gamma_{x_i}|$ and $|\Gamma_{x_i x_j}|$. The estimate (37) follows by differentiating the latter formula of (38) with $x_k$ and applying parabolic estimates for $u(x, z, t)$ up to the third order. This completes the proof. $\square$

Finally, the following lemma, which is again a consequence of Corollary 4.6, shows that the large time behavior of the solution can be essentially determined by the zero-level surface $\Gamma(x, t)$.

**Lemma 4.11.** Let $u(x, z, t)$ be a solution of (22)–(23) and let $\Gamma(x, t)$ be as defined in Lemma 4.10. Then it holds that

$$
\lim_{t \to \infty} \sup_{(x, z) \in \mathbb{R}^n} |u(x, z, t) - \Phi(z - \Gamma(x, t))| = 0.
$$

**Proof.** If the above claim does not hold, there exist a constant $\delta > 0$ and a sequence $\{(x_k, z_k, t_k)\}$ such that $0 < t_1 < t_2 < \cdots \to \infty$ and that

$$
|u(x_k, z_k, t_k) - \Phi(z_k - \Gamma(x_k, t_k))| \geq \delta.
$$

On the other hand, by virtue of Lemma 4.1 and the boundedness of $\Gamma(x, t)$, we can choose constants $R > 0$ and $T > 0$ such that

$$
\sup_{x \in \mathbb{R}^{n-1}, |z| \geq R, t \geq T} |u(x, z, t) - \Phi(z - \Gamma(x, t))| < \delta.
$$

This means that $\{z_k\}$ is bounded. Arguing as in the proof of Lemma 4.7, we can choose a subsequence of $\{(x_k, z_k, t_k)\}$, which we denote again by $\{(x_k, z_k, t_k)\}$, such that (32) holds for some $\omega$-limit point $w$ and that the following limits exist:

$$
z_\infty := \lim_{k \to \infty} z_k, \quad \gamma_\infty := \lim_{k \to \infty} \Gamma(x_k, t_k).
$$

This and (39) imply

$$
|\omega(w, z_\infty, 0) - \Phi(z_\infty - \gamma_\infty)| = \lim_{k \to \infty} |u(x_k, z_k, t_k) - \Phi(z_k - \Gamma(x_k, t_k))| \geq \delta.
$$

(40)
On the other hand, since we have $\Phi(0) = 0$ and

$$w(0, \gamma_\infty, 0) = \lim_{k \to \infty} u(x_k, \Gamma(x_k, t_k), t_k) = 0,$$

Corollary 4.6 implies $w(x, z, t) \equiv \Phi(z - \gamma_\infty)$. This contradicts (40), which completes the proof of the lemma. 

By setting $y = z + ct$ and $\gamma(x, t) = \Gamma(x, t) + ct$, we obtain the statements (i), (ii) of Theorem 1.1 from Lemmas 4.10 and 4.11. Thus it remains to prove the statement (iii) concerning the large time behavior of $\Gamma(x, t)$. This will be done in Sections 4.5 and 4.6 by constructing suitable supersolutions and subsolutions.

### 4.5. Construction of supersolutions and subsolutions

In this subsection we construct supersolutions and subsolutions of (22)–(23). For this purpose, we consider the problem of the form

$$\begin{cases}
V_t = \Delta_x V + \frac{c}{2} |\nabla_x V|^2, & x \in \mathbb{R}^{n-1}, t > 0, \\
V(x, 0) = V_0(x), & x \in \mathbb{R}^{n-1},
\end{cases}$$

(41)

where $\Delta_x$ and $\nabla_x$ denote the $(n-1)$-dimensional Laplacian and the $(n-1)$-dimensional gradient, respectively. Note that all the estimates in Lemma 3.10 are also valid for the solutions of the problem (41)–(42).

**Lemma 4.12 (Supersolution).** For any constants $M > 0$ and $\varepsilon \in (0, 1]$, there exist a constant $\delta > 0$ and smooth functions $p(t), q(t)$ satisfying

$$p(0) > 0, \quad q(0) = 0, \quad 0 \leq p(t), q(t) \leq \varepsilon \quad \text{for } t \geq 0,$$

(43)

such that, if $V(x, t)$ is any solution of (41)–(42) with $\|V_0\|_{W^{1,\infty}} \leq M$ and $\|\nabla_x V_0\|_{W^{1,\infty}} \leq \delta$, then the function $u^+(x, z, t)$ defined by

$$u^+(x, z, t) = \Phi \left( \frac{z - V(x, t)}{\sqrt{1 + |\nabla_x V|^2}} - q(t) \right) + p(t),$$

satisfies

$$L[u^+] := u^+_t - \Delta u^+ - cu^+_x - f(u^+) \geq 0, \quad (x, z) \in \mathbb{R}^n, t > 0.$$

**Proof.** We divide the proof into four steps.

**Step 1.** We define $u^+(x, z, t)$ as above, where $q(t), p(t)$ are bounded functions to be determined later. For simplicity, we set

$$\eta(x, z, t) = \frac{z - V(x, t)}{\sqrt{1 + |\nabla_x V(x, t)|^2}},$$

Then, by using the relation $\Phi'' + c\Phi' + f(\Phi) = 0$, we have
\[ L[u^+] = (\eta_t - \Delta_x \eta - c \eta_z + c)\Phi' + (1 - |\nabla_x \eta|^2 - \eta_z^2)\Phi'' - q'(t)\Phi' + p'(t) - f(\Phi + p(t)) + f(\Phi). \]

By rewriting the above expression in terms of \( V \), we obtain
\[ L[u^+] = (I_0 - I_2)\Phi' + (I_1 - 3I_3)\eta\Phi' - 2I_2\eta\Phi'' - I_3\eta^2\Phi'' + J, \]

where \( I_0, I_1, I_2, I_3 \) and \( J \) are functions given by
\[
I_0 = -\frac{V_t + c}{\sqrt{1 + |\nabla_x V|^2}} + \text{div}\left(\frac{\nabla_x V}{\sqrt{1 + |\nabla_x V|^2}}\right) + c, \\
I_1 = \sum_{i=1}^{n-1} \frac{V_i V_{xi} V_{xii}}{1 + |\nabla_x V|^2} + \sum_{i,j=1}^{n-1} \frac{V_i^2 V_{xi} V_{xij}}{1 + |\nabla_x V|^2}, \\
I_2 = \sum_{i,j=1}^{n-1} \frac{V_i V_x V_{xij}}{(1 + |\nabla_x V|^2)^{\frac{3}{2}}}, \\
I_3 = \sum_{i=1}^{n-1} \left(\frac{\sum_{j=1}^{n-1} V_j V_{xij}}{1 + |\nabla_x V|^2}\right)^2, \\
J = \left(-\Phi \frac{q'(t)}{p(t)} + \frac{p'(t)}{p(t)} - \int_0^1 f'(\Phi + \tau p(t)) \, d\tau \right) \cdot p(t).
\]

Since \( V(x,t) \) satisfies Eq. (41), we have
\[
I_0 = \frac{1}{\sqrt{1 + |\nabla_x V|^2}} \left(-V_t + \Delta_x V + \frac{c}{2} |\nabla_x V|^2 \right) \\
- \frac{c|\nabla_x V|^4}{2\sqrt{1 + |\nabla_x V|^2}^2(\sqrt{1 + |\nabla_x V|^2} + 1)^2} - \sum_{i,j=1}^{n-1} \frac{V_i V_x V_{xij}}{1 + |\nabla_x V|^2}^{\frac{3}{2}} \\
= - \frac{c|\nabla_x V|^4}{2\sqrt{1 + |\nabla_x V|^2}^2(\sqrt{1 + |\nabla_x V|^2} + 1)^2} - \sum_{i,j=1}^{n-1} \frac{V_i V_x V_{xij}}{1 + |\nabla_x V|^2}^{\frac{3}{2}}.
\]

**Step 2.** Now we estimate \( L[u^+] \) except for \( J \) by
\[ I = (I_0 - I_2)\Phi' + (I_1 - 3I_3)\eta\Phi' - 2I_2\eta\Phi'' - I_3\eta^2\Phi''. \]

Since \( \Phi'(z) \) and \( \Phi''(z) \) decay to zero exponentially as \( |z| \to \infty \) and \( q(t) \) is assumed to be bounded, the following functions are all bounded:
\[ \eta\Phi'(\eta + q(t)), \quad \eta\Phi''(\eta + q(t)), \quad \eta^2\Phi''(\eta + q(t)). \]

Now we choose a constant \( C_1 > 0 \) arbitrarily. Then from the above boundedness and Lemma 3.10, we can choose a constant \( C_2 \geq 1 \) depending only on \( c \) and \( M \), and a constant \( \delta > 0 \) depending only on \( C_1 \) such that if \( \|\nabla_x V_0\|_{W^{1,\infty}} \leq \delta \), it holds that
\[ |I(x,z,t)| \leq P(t), \quad \text{where} \quad P(t) = \min\{C_2t^{-2}, C_1\}. \]
Step 3. Now we determine the constant $C_1$ and the smooth functions $p(t)$ and $q(t)$. By the assumption (F1), we can choose a constant $K \in (0, 1]$ such that

$$-f'(s) \geq 2K > 0, \quad s \in [-1 - \varepsilon, -1 + 2\varepsilon] \cup [1 - \varepsilon, 1 + \varepsilon].$$

We define the constants $C_1 > 0$ and $C_0 \geq 1$ by

$$C_1 = \frac{K^2 \varepsilon^2}{16C_2C_0^2}, \quad C_0 = \max\left\{1, \frac{2K + \|f'\|_L^\infty(-1, 1)}{\min_{\Phi \in [-1+\varepsilon, 1-\varepsilon]} |\Phi'|}\right\}.$$

We choose functions $p(t), q(t) \in C^\infty[0, \infty)$ satisfying

$$P(t) \leq Kp(t) \leq 2P(t), \quad K|p'(t)| \leq 2|P'(t)|, \quad q(t) = C_0 \int_0^t p(s) \, ds.$$

Then (43) holds, since we have

$$p(0) \geq \frac{K\varepsilon^2}{16C_2C_0^2} > 0, \quad 0 < p(t) \leq \frac{K\varepsilon^2}{8C_2C_0^2} \leq \varepsilon, \quad 0 \leq q(t) \leq C_0 \int_0^\infty p(s) \, ds \leq \varepsilon.$$

Step 4. Now we complete the proof. Since we have $|J(x, z, t)| \leq P(t)$ by Step 2, it suffices to show the inequality $J(x, z, t) \geq P(t)$. When $\Phi \in [-1, -1 + \varepsilon] \cup [1 - \varepsilon, 1]$, noting that $0 < p(t) \leq \varepsilon$, we have

$$J(x, z, t) \geq \left(\frac{p'(t)}{p(t)} - \int_0^1 f'(\Phi + \tau p(t)) \, d\tau\right) \cdot p(t) \geq Kp(t) \geq P(t),$$

since we have

$$\sup_{t \geq 0} \frac{|p'(t)|}{p(t)} \leq \sup_{t \geq 0} \frac{2|p'(t)|}{Kp(t)} \leq \sup_{t \geq 0} \frac{2|p'(t)|}{P(t)} = \frac{K\varepsilon}{2C_2C_0} \leq K.$$

On the other hand, when $\Phi \in [-1 + \varepsilon, 1 - \varepsilon]$, we have

$$J(x, z, t) \geq \left(-\Phi' + \frac{p'(t)}{p(t)} - \|f'\|_L^\infty(-1, 1)\right) \cdot p(t)$$

$$\geq \left(-C_0\Phi - K - \|f'\|_L^\infty(-1, 1)\right) \cdot p(t)$$

$$\geq Kp(t) \geq P(t).$$

Thus we obtain $L[u^+] \geq 0$, which completes the proof. \(\square\)

We can also construct subsolutions. Since the proof is similar to that of Lemma 4.12, we omit the proof of Lemma 4.13 below.

Lemma 4.13 (Subsolution). For any constants $M > 0$ and $\varepsilon \in (0, 1]$, there exist a constant $\delta > 0$ and smooth functions $p(t), q(t)$ satisfying

$$p(0) > 0, \quad q(0) = 0, \quad 0 \leq p(t), q(t) \leq \varepsilon \quad \text{for } t \geq 0.$$
such that, if $V(x, t)$ is any solution of (41)–(42) with $\|V_0\|_{W^{3, \infty}} \leq M$ and $\|\nabla_x V_0\|_{W^{1, \infty}} \leq \delta$, then the function $u^{-}(x, z, t)$ defined by

$$ u^{-}(x, z, t) = \Phi \left( \frac{z - V(x, t)}{\sqrt{1 + |\nabla_x V|^2}} + q(t) \right) - p(t), $$

satisfies

$$ L[u^{-}] := u^{-}_t - \Delta u^{-} - cu^{-}_z - f(u^{-}) \leq 0, \quad (x, z) \in \mathbb{R}^n, \quad t > 0. $$

### 4.6. Proof of Theorem 1.1

In this subsection we complete the proof of Theorem 1.1 by proving the statement (iii). On the formal level, it is well known that the motion of the zero-level surface $\Gamma(x, t)$ can be approximated by the mean curvature flow with a drift term. However, the point of statement (iii) is that this approximation remains valid up to $t = +\infty$. In order to prove this assertion, we first show that $\Gamma(x, t)$ can be approximated by a solution of (41).

**Lemma 4.14 (Approximation of $\Gamma(x, t)$).** Let $u(x, z, t)$ be a solution of (22)–(23) and let $\Gamma(x, t)$ be as defined in Lemma 4.10. Then for any $\varepsilon > 0$, there exists a constant $\tau_\varepsilon > 0$ such that the function $V(x, t)$ defined by

$$ \begin{cases} V_t = \Delta_x V + \frac{c}{2} |\nabla_x V|^2, & x \in \mathbb{R}^{n-1}, \quad t > 0, \\ V(x, 0) = \Gamma(x, \tau_\varepsilon), & x \in \mathbb{R}^{n-1}, \end{cases} $$

satisfies

$$ \sup_{x \in \mathbb{R}^{n-1}} \bigg| \Gamma(x, t) - V(x, t - \tau_\varepsilon) \bigg| \leq \varepsilon, \quad t \geq \tau_\varepsilon. $$

**Proof.** From Corollary 4.8 and Lemma 4.10, we first choose constants $T > 0$, $M > 0$ and $K > 0$ such that, for $D := \{(x, z, t) \in \mathbb{R}^n \times [T, \infty) \mid |u(x, z, t)| \leq 1/2\}$, it holds that

$$ \sup_{t \geq T} \|\Gamma(\cdot, t)\|_{W^{3, \infty}} \leq M, \quad \inf_{(x, z) \in D} -u_z(x, z, t) \geq K. $$

For the constants $M$ and $\hat{\varepsilon} := 1/(\|\Phi\|_{L^\infty} + 1) \cdot \min\{K \varepsilon, 1/2\}$, we choose a constant $\delta > 0$ and functions $p(t), q(t)$ satisfying

$$ p(0) > 0, \quad q(0) = 0, \quad 0 \leq p(t), q(t) \leq \hat{\varepsilon} \quad \text{for} \quad t \geq 0, $$

as in Lemma 4.12. Then, from Lemma 4.10, there exists a constant $\tau_\varepsilon \in [T, \infty)$ such that $\|\nabla_x \Gamma(\cdot, \tau_\varepsilon)\|_{W^{1, \infty}} \leq \delta$. Furthermore, by choosing $\tau_\varepsilon$ larger if necessary, we have

$$ u(x, z, \tau_\varepsilon) \leq \Phi \left( z - \Gamma(x, \tau_\varepsilon) \right) + \frac{p(0)}{2} \leq \Phi \left( \frac{z - \Gamma(x, \tau_\varepsilon)}{\sqrt{1 + |\nabla_x \Gamma(x, \tau_\varepsilon)|^2}} \right) + p(0), \quad (46) $$

where the first inequality is given by Lemma 4.11, and the second inequality comes from the smallness of $|\nabla_x \Gamma|$ given in Lemma 4.10. For such $\tau_\varepsilon$, we define $V(x, t)$ as a function satisfying (44)–(45). Then Lemma 4.12 implies that the function $u^{+}(x, z, t)$ given by

$$ u^{+}(x, z, t) = \Phi \left( \frac{z - V(x, t - \tau_\varepsilon)}{\sqrt{1 + |\nabla_x V|^2}} - q(t - \tau_\varepsilon) \right) + p(t - \tau_\varepsilon), $$

Lemma 4.13, we can show (Ergodicity of zero-level surface). Let \( u \)

\begin{align*}
\text{Lemma 4.15} \\
\text{Proof of Theorem 1.1.} \\
\text{Thus we only show the statement (iii).} \\
\text{2} \\
\text{approximated by the solution} \\
\text{This means that the zero-level surface} \\
\text{This implies} \Gamma(x, t) \leq V(x, t - \tau_\varepsilon) + \varepsilon \text{ for } t \geq \tau_\varepsilon. \text{Similarly, by using the subsolution } u^-(x, z, t) \text{ given in Lemma 4.13, we can show } \Gamma(x, t) \geq V(x, t - \tau_\varepsilon) - \varepsilon \text{ for } t \geq \tau_\varepsilon. \text{This completes the proof of the lemma.} \\
\text{Now we are ready to complete the proof of Theorem 1.1.} \\
\text{Proof of Theorem 1.1.} \text{The statements (i) and (ii) of Theorem 1.1 are derived directly from Lemmas 4.10 and 4.11, respectively. Thus we only show the statement (iii).} \\
\text{By Lemma 4.14, the large time behavior of the zero-level surface } \Gamma(x, t) \text{ of the solution } u(x, z, t) \text{ of (22)–(23) is approximated by the solution } V(x, t) \text{ of the equation} \\
\text{This means that the zero-level surface } \gamma(x, t) = \Gamma(x, t) + ct \text{ of the solution } u(x, y, t) \text{ of (1)–(2) can be approximated by the solution } \tilde{V}(x, t) \text{ of the equation} \\
\text{Thus the statement (iii) of Theorem 1.1 follows from Lemma 3.9. This completes the proof of Theorem 1.1.} \\
\text{4.7. Proof of Theorem 1.6} \\
\text{In this section we prove Theorem 1.6 concerning the stability of the planar wave under ergodic perturbations. We first show that the zero-level surface } \Gamma(x, t) \text{ remains uniquely ergodic for all large } t \geq 0. \\
\text{Lemma 4.15 (Ergodicity of zero-level surface). Let } u(x, z, t) \text{ be a solution of (22)–(23) and assume that } u_0(x, z) \text{ is uniquely ergodic in the x-direction. Then the zero-level surface } \Gamma(x, t) \text{ defined in Lemma 4.10 is uniquely ergodic for each } t \geq T, \text{ where } T > 0 \text{ is the constant given in Lemma 4.10.}
**Proof.** We put \( u_1(x, z) := u(x, z, t) \) and \( \Gamma_t^n(x) := \Gamma^n(x, t) \) for any fixed \( t \geq T \). We define their hulls \( \mathcal{H}_{u_1} \) and \( \mathcal{H}_{\Gamma_t^n} \) by

\[
\mathcal{H}_{u_1} := \{ \sigma u_1 \mid a \in \mathbb{R}^{n-1} \}_{L^\infty}, \quad \mathcal{H}_{\Gamma_t^n} := \{ \sigma \Gamma_t^n \mid a \in \mathbb{R}^{n-1} \}_{L^\infty}.
\]

Next we denote by \( p(\sigma u_1) \) the zero-level surface of \( \sigma u_1 \). Then clearly \( p(\sigma u_1) = \sigma \Gamma_t^n \), thus we have \( p(\sigma u_1) = \sigma p(u_1) \) for \( a \in \mathbb{R}^{n-1} \). Furthermore, for any sequence \( \{a_i\} \subset \mathbb{R}^{n-1} \) such that \( \sigma u_1 \) is convergent, we can easily see from (34), (36) and the implicit function theorem that \( \lim_{u \to 0} \sigma \Gamma_t^n \) is the zero-level surface of \( \lim_{i \to \infty} \sigma_i u_1 \). Consequently, \( p \) can be extended to a continuous map \( p: \mathcal{H}_{u_1} \to \mathcal{H}_{\Gamma_t^n} \) with the property \( p(\sigma u) = \sigma p(u) \) for any \( a \in \mathbb{R}^{n-1} \), \( u \in \mathcal{H}_{u_1} \). One can now apply Lemma 2.9 with \( X = \mathcal{H}_{u_1}, \ Y = \mathcal{H}_{\Gamma_t^n} \), to conclude that the unique ergodicity of \( \Gamma_t^n \) follows from that of \( u_1 \) given in Corollary 2.12. This completes the proof. \( \Box \)

Now we complete the proof of Theorem 1.6.

**Proof of Theorem 1.6.** Let \( T > 0 \) and \( \Gamma^n(x, t) \) be as defined in Lemma 4.10. Then \( \Gamma^n(x, t) \) is uniquely ergodic for each \( t \geq T \) from Lemma 4.15. Thus Theorems 1.1 and 3.1 give the desired conclusion. The proof of the theorem is complete. \( \Box \)

**Appendix A. Basic properties of ergodic functions**

**A.1. Equivalent characterizations of unique ergodicity**

Here we give the proof of Proposition 2.7, which shows the equivalence of various characterizations of unique ergodicity, thereby confirming the claim in Remark 2.1.

**Proof of Proposition 2.7.** The proof is rather standard, but we give it here for the clarity of the paper. The assertion (c) \( \Rightarrow \) (b) is obvious, once we set \( z = \sigma g \) in (10). To prove (b) \( \Rightarrow \) (c), we define

\[
m(a, R) := \frac{1}{|B_R(a)|} \int_{B_R(a)} \Psi(\sigma x g) \, dx = \frac{1}{|B_R(0)|} \int_{B_R(0)} \Psi(\sigma x \sigma g) \, dx.
\]

Then, for any \( a, b \in \mathbb{R}^m \),

\[
|m(a, R) - m(b, R)| \leq \frac{1}{|B_R(a)|} \int_{B_R(a) \cap B_R(b)} \left| \Psi(\sigma x g) \right| \, dx,
\]

where \( A \cup B := (A \setminus B) \cup (B \setminus A) \). Hence

\[
\lim_{R \to \infty} \sup |m(a, R) - m(b, R)| \leq \lim_{R \to \infty} \sup \frac{|B_R(a) \cap B_R(b)|}{|B_R(a)|} \|\Psi\|_{L^\infty} = 0,
\]

which shows that the limit in (9) is independent of \( a \in \mathbb{R}^m \). Let \( m^* \) denote this limit. We choose \( z \in \mathcal{H}_g \) arbitrarily and let \( \{a_k\} \) be a sequence in \( \mathbb{R}^m \) such that \( \sigma_{a_k} g \to z \) as \( k \to \infty \). Then we have

\[
\left| \frac{1}{|B_R(0)|} \int_{B_R(0)} \Psi(\sigma x g) \, dx - m^* \right| = \left| \lim_{k \to \infty} m(a_k, R) - m^* \right| \leq \sup_{a \in \mathbb{R}^m} |m(a, R) - m^*|.
\]

Since \( m(a, R) \to m^* \) as \( R \to \infty \) uniformly in \( a \in \mathbb{R}^m \), the right-hand side of the above inequality tends to 0 as \( R \to \infty \). This implies (c).
Next we prove (c) ⇒ (a). Let $\mu_1$ and $\mu_2$ be any $[\sigma_a]$-invariant probability measures on $\mathcal{H}_g$ and let $\Psi : \mathcal{H}_g \to \mathbb{R}$ be any continuous map. By the $[\sigma_a]$-invariance of $\mu_1$, we have

$$\int_{\mathcal{H}_g} \Psi(z) \, d\mu_1(z) = \int_{\mathcal{H}_g} \Psi(\sigma_a z) \, d\mu_1(z) \quad \text{for any } x \in \mathbb{R}^m,$$

where $i = 1, 2$. Then Fubini’s theorem gives

$$\int_{\mathcal{H}_g} \Psi(z) \, d\mu_1(z) = \frac{1}{|B_R(0)|} \int_{B_R(0)} \int_{\mathcal{H}_g} \Psi(\sigma_a z) \, d\mu_1(z) \, dx$$

$$= \int_{\mathcal{H}_g} \frac{1}{|B_R(0)|} \int_{B_R(0)} \Psi(\sigma_a z) \, dx \, d\mu_1(z).$$

Letting $R \to \infty$, we obtain

$$\int_{\mathcal{H}_g} \Psi(z) \, d\mu_1(z) = \int_{\mathcal{H}_g} m_\infty(\Psi) \, d\mu_1(z) = m_\infty(\Psi) \quad \text{for } i = 1, 2,$$

where $m_\infty(\Psi)$ denotes the limit in (10). Thus we have $\int_{\mathcal{H}_g} \Psi(z) \, d\mu_1(z) = \int_{\mathcal{H}_g} \Psi(z) \, d\mu_2(z)$ for any continuous function $\Psi : \mathcal{H}_g \to \mathbb{R}$, which implies $\mu_1 = \mu_2$. The assertion (c) ⇒ (a) is proven.

It remains to prove (a) ⇒ (b). Assume (a) and let $\mu$ be the unique $[\sigma_a]$-invariant probability measure on $\mathcal{H}_g$. If (b) does not hold, then there exists a continuous map $\Psi : \mathcal{H}_g \to \mathbb{R}$ such that the limit in (10) does not exist uniformly in $z \in \mathcal{H}_g$, or the limit is not independent of $z$. In either case, we can find sequences $\{(z_k, R_k)\}$ and $\{((\tilde{z}_k, \tilde{R}_k)\}$ in $\mathcal{H}_g \times (0, \infty)$ such that $R_k, \tilde{R}_k \to \infty$ as $k \to \infty$ and that

$$\lim_{k \to \infty} m(z_k, R_k, \Psi_0) \neq \lim_{k \to \infty} m(\tilde{z}_k, \tilde{R}_k, \Psi_0), \quad (49)$$

where

$$m(z, R, \Psi_0) := \frac{1}{|B_R(0)|} \int_{B_R(0)} \Psi_0(\sigma_a z) \, dx.$$

Since $|m(z_k, R_k, \Psi)| \leq \|\Psi\|_{L^\infty(\mathcal{H}_g)}$ for any continuous function $\Psi : \mathcal{H}_g \to \mathbb{R}$, the sequence of functionals $\Psi \to m(z_k, R_k, \Psi)$ $(k = 1, 2, 3, \ldots)$ is uniformly bounded. Hence it has a subsequence, denoted again by $m(z_k, R_k, \cdot)$, that converges in the dual space of $C(\mathcal{H}_g; \mathbb{R})$ in the weak* sense. The limit is a Radon measure, which we denote by $\mu_1$. Thus

$$\lim_{k \to \infty} m(z_k, R_k, \Psi) = \int_{\mathcal{H}_g} \Psi(z) \, d\mu_1(z).$$

It is clear that $\int_{\mathcal{H}_g} d\mu_1(z) = 1$ and that $\int_{\mathcal{H}_g} \Psi(z) \, d\mu_1(z) \geq 0$ for any $\Psi \geq 0$. Therefore $\mu_1$ is a probability measure. Furthermore, the same estimate as in (47) gives

$$\left| m(z_k, R_k, \Psi) - m(z_k, R_k, \Psi \circ \sigma_a) \right| \leq \frac{1}{|B_{R_k}(0)|} \|\Psi\|_{L^\infty(\mathcal{H}_g)} \left| B_{R_k}(0) \cap B_{R_k}(a) \right|,$$
where \((\Psi \circ \sigma_a)(z) := \Psi(\sigma_a z)\). Letting \(k \to \infty\), we obtain

\[
\int_{\mathcal{H}_g} \Psi(z) \, d\mu_1(z) = \int_{\mathcal{H}_g} \Psi(\sigma_a z) \, d\mu_1(z)
\]

for any \(a \in \mathbb{R}^n\) and any \(\Psi \in C(\mathcal{H}_g; \mathbb{R})\); hence \(\mu_1\) is \(\{\sigma_a\}\)-invariant. The assumption (a) then implies \(\mu_1 = \mu\), therefore

\[
\lim_{k \to \infty} m(z_k, R_k, \Psi) = \int_{\mathcal{H}_g} \Psi(z) \, d\mu(z) \quad \text{for } \Psi \in C(\mathcal{H}_g; \mathbb{R}).
\]

Similarly we obtain

\[
\lim_{k \to \infty} m(\tilde{z}_k, \tilde{R}_k, \Psi) = \int_{\mathcal{H}_g} \Psi(z) \, d\mu(z) \quad \text{for } \Psi \in C(\mathcal{H}_g; \mathbb{R})
\]

which contradicts (49). This proves \((a) \Rightarrow (b)\), and the proof of the proposition is complete. \(\Box\)

**Remark A.1.** It is clear that the above proof remains valid if, in the definition of \(m(a, R)\), the ball \(B_R(a)\) is replaced by

\[
D_R(a) := \{Rx + a \mid x \in D_1\},
\]

where \(D_1\) is a bounded measurable set of any shape (such as a cube). The same is true with the expression (7). In other words, the shape of the region over which the average is taken is not relevant in the characterization of unique ergodicity.

**A.2. Almost periodicity and unique ergodicity**

It is well known that almost periodic functions are uniquely ergodic. For the convenience of the reader, we give an elementary proof of this fact for functions of the form \(g(x, y)\).

**Proposition A.2.** Any function \(g(x, y) : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}\) that is almost periodic in the \(x\)-direction is uniquely ergodic in the \(x\)-direction.

**Proof.** Let \(\Psi\) be any continuous function on \(\mathcal{H}_g\). For \(a \in \mathbb{R}^{n-1}\) and \(R > 0\), we define

\[
m(a, R) := \frac{1}{|D_R(a)|} \int_{D_R(a)} \Psi(\sigma_x g) \, dx,
\]

where \(D_R(a)\) is the cube of size \(2R\) centered at \(a = (a_1, \ldots, a_{n-1}) \in \mathbb{R}^{n-1}\), that is,

\[
D_R(a) = \{x \in \mathbb{R}^{n-1} \mid |x_i - a_i| \leq R, \ i = 1, 2, \ldots, n-1\}.
\]

By Proposition 2.7 and Remark A.1, it suffices to show that \(m(a, R)\) converges to a certain value as \(R \to \infty\) uniformly in \(a \in \mathbb{R}^{n-1}\).

We first show that the almost periodicity of \(g\) implies

\[
\lim_{R \to \infty} |m(a, R) - m(0, R)| = 0 \quad \text{uniformly in } a \in \mathbb{R}^{n-1}.
\] (50)
Let $\varepsilon > 0$ be arbitrarily fixed. Since $\Psi$ is continuous on the compact set $\mathcal{H}_g$, it is uniformly continuous. Thus there exists $\delta( \varepsilon ) > 0$ such that $| \Psi(z) - \Psi(z') | < \varepsilon$ for any $z, z' \in \mathcal{H}_g$ with $\| z - z' \|_{L^\infty(\mathbb{R}^n)} < \delta( \varepsilon )$. Moreover, by the almost periodicity of $g$, we can find $L( \varepsilon ) > 0$ such that, for any $a \in \mathbb{R}^{n-1}$, the following set is not empty:

$$\left\{ b \in \mathbb{R}^{n-1} \mid \| \sigma_b g - g \|_{L^\infty(\mathbb{R}^n)} < \delta( \varepsilon ) \right\} \cap B_{L(\varepsilon)}(a).$$

Now choose $a \in \mathbb{R}^{n-1}$ arbitrarily. Then there exists $b \in B_{L(\varepsilon)}(a)$ such that $\| \sigma_b g - g \|_{L^\infty(\mathbb{R}^n)} \leq \delta( \varepsilon )$ and thus that $\| \sigma_a \sigma_b g - \sigma_x g \|_{L^\infty(\mathbb{R}^n)} \leq \delta( \varepsilon )$ for any $x \in \mathbb{R}^{n-1}$. Hence

$$\| m( a, R ) - m( 0, R ) \| \leq \| m(a, R) - m(b, R) \| + \| m(b, R) - m(0, R) \| \leq \frac{|D_R(a) \cap D_R(b)|}{|D_R(a)|} \| \Psi \|_{L^\infty} \max_{x \in B_R(0)} | \Psi( \sigma_b \sigma_x g) - \Psi( \sigma_x g) |.$$

The first term of the right-hand side is of $O( L( \varepsilon ) / R )$ as $R \to \infty$ and the second term of the right-hand side is less than $\varepsilon$. Thus we obtain (50).

Once we have (50), to prove the proposition, it suffices to show that the following limit exists

$$\lim_{R \to \infty} m(0, R).$$

Given any $\varepsilon > 0$, by (50), we can choose $R(\varepsilon) > 0$ such that

$$| m( a, R(\varepsilon) ) - m( 0, R(\varepsilon) ) | < \varepsilon \quad \text{for any } a \in \mathbb{R}^{n-1}.$$  \hfill (51)

Let $k \in \mathbb{N}$ be arbitrary. Then the cube $D_{kR(\varepsilon)}(a)$ can be divided into $k^{n-1}$ cubes of size $2R(\varepsilon)$, each of which is a translation of $D_{R(\varepsilon)}(a)$. Thus (51) applies to each of the small cubes. Collecting $k^{n-1}$ cubes and averaging them, we obtain

$$| m( 0, kR(\varepsilon) ) - m( 0, R(\varepsilon) ) | < \varepsilon \quad \text{for any } k \in \mathbb{N}.$$  \hfill (52)

Now we choose a sufficiently large $R > 0$ and write it as

$$R = kR(\varepsilon) + r \quad (0 \leq r < R(\varepsilon)).$$

Then it is easily seen that

$$| m( 0, R ) - m( 0, kR(\varepsilon) ) | \leq \frac{1}{|D_R(0)|} \int_{D_R(0) \setminus D_{kR(\varepsilon)}(0)} \Psi( \sigma_x g ) \, dx - \left( 1 - \frac{|D_{kR(\varepsilon)}(0)|}{|D_R(0)|} \right) m( 0, kR(\varepsilon) )$$

\begin{align*}
&\leq \left( 1 - \frac{|D_{kR(\varepsilon)}(0)|}{|D_R(0)|} \right) \left( \| \Psi \|_{L^\infty} + m( 0, kR(\varepsilon) ) \right) \\
&\leq \left( 1 - \frac{(kR(\varepsilon))^n - 1}{(kR(\varepsilon) + r)^{n-1}} \right) \left( \| \Psi \|_{L^\infty} + m( 0, R(\varepsilon) ) + \varepsilon \right).
\end{align*}

If we let $R \to \infty$, then $k \to \infty$, therefore $| m(0, R) - m(0, kR(\varepsilon)) | \to 0$. Combining this with (52), we obtain

$$\limsup_{R \to \infty} | m(0, R) - m(0, R(\varepsilon)) | \leq \varepsilon.$$

Since $\varepsilon$ is arbitrary, we see that $\lim_{R \to \infty} m(0, R)$ exists. The proof of the proposition is complete. \qed
A.3. Penrose pattern

In this appendix we give more details of the Penrose pattern mentioned in Example 2.3 (2).

Let \( \mathcal{P} = \{p_1, \ldots, p_m\} \) be a finite set of closed convex polygons in \( \mathbb{R}^2 \) that are translationally incongruent. A tiling \( Z \) is a set of spatial translations of elements of \( \mathcal{P} \) such that the union of all the elements of \( Z \) covers \( \mathbb{R}^2 \) and that no two elements of \( Z \) intersect each other except possibly on the boundary. We call each element of \( \mathcal{P} \) the prototile.

Let \( \mathcal{Z}_\mathcal{P} \) denote the set of all the tilings composed of the prototiles in \( \mathcal{P} \). We assume that \( \mathcal{Z}_\mathcal{P} \) is non-empty. Then the group of translations on \( \mathbb{R}^2 \) naturally acts on \( \mathcal{Z}_\mathcal{P} \). Following [13], we define a topology in \( \mathcal{Z}_\mathcal{P} \) as follows: a sequence \( Z_k \) \((k = 1, 2, 3, \ldots)\) in \( \mathcal{Z}_\mathcal{P} \) converges to an element \( Z_\infty \in \mathcal{Z}_\mathcal{P} \) if and only if, for any \( R > 0 \), the sequence \( \partial Z_k \cap \{x \in \mathbb{R}^2 : |x| < R\} \) converges to \( \partial Z_\infty \cap \{x \in \mathbb{R}^2 : |x| < R\} \) in the Hausdorff distance. Here, for a tiling \( Z \), \( \partial Z \) denotes the union of all the boundaries of the tiles that comprise \( Z \). With this topology, \( \mathcal{Z}_\mathcal{P} \) becomes a metric space. (The topology of \( \mathcal{Z}_\mathcal{P} \) is defined in a slightly different way in [13], but it is obviously equivalent to the one above.)

Next, given a tiling \( Z \in \mathcal{Z}_\mathcal{P} \), we consider a function \( g(x) \) on \( \mathbb{R}^2 \) whose level sets exhibit the tiling pattern of \( Z \) in the following sense:

(G1) \( g = \alpha \) on \( \partial Z \) for some constant \( \alpha \in \mathbb{R} \), and \( g \neq \alpha \) on \( \mathbb{R}^2 \setminus \partial Z \).

(G2) For any elements \( q, q' \in Z \) that are translations of the same prototile in \( \mathcal{P} \), we have

\[
g(x) = g(x + a) \quad \text{for } x \in q,
\]

where \( a \in \mathbb{R}^2 \) is the vector that translates \( q \) onto \( q' \).

The second condition is equivalent to saying that, for each prototype \( p_j \in \mathcal{P} \) \((j = 1, 2, \ldots, m)\), a function element \( \varphi_j(x) \) is defined, so that, for every tile \( q \in Z \), the restriction of \( g \) onto the region \( q \) is a translation of the corresponding function element. It follows from (G1), (G2) that, for each \( a \in \mathbb{R}^2 \),

\[
\sigma_a g = g \quad \Leftrightarrow \quad \tilde{\sigma}_a Z = Z,
\]

where \( \sigma_a \) denotes the shift operator \( g(x) \mapsto g(x + a) \) and

\[
\tilde{\sigma}_a Z := \{\sigma_a q \mid q \in Z\}, \quad \sigma_a q = \{x \in \mathbb{R}^2 \mid x + a \in q\}.
\]

Now we consider Penrose tilings. A Penrose tiling consists of 10 prototiles that are 2\(k\pi/5\) rotations \((k = 0, 1, \ldots, 4)\) of two types of tiles – the fat rhombus (with interior angles 4\(\pi/5\), 6\(\pi/5\)) and the thin one (with interior angles 2\(\pi/5\), 8\(\pi/5\)) – as illustrated in Fig. 2. It is known that there are uncountably many incongruent Penrose tilings, all of which are aperiodic.

**Proposition A.3.** Any continuous function on \( \mathbb{R}^2 \) whose level sets exhibit the Penrose tiling pattern in the sense (G1), (G2) above is strictly ergodic but not almost periodic.

**Proof.** Let \( \mathcal{P} \) denote the set of Penrose prototiles and \( \mathcal{Z}_\mathcal{P} \) the set of all Penrose tilings. We denote by \( \{\sigma_a\}_{a \in \mathbb{R}^2} \) the group of shift operators acting on \( \mathcal{C}(\mathbb{R}^2) \) and by \( \{\tilde{\sigma}_a\}_{a \in \mathbb{R}^2} \) the group of translations acting on \( \mathcal{Z}_\mathcal{P} \) as defined above.

Now let \( Z_0 \) be any Penrose tiling and \( g \) be a continuous function on \( \mathbb{R}^2 \) that exhibit the pattern \( Z_0 \). More generally, we associate with each prototype in \( \mathcal{P} \) a function element as mentioned in the remark after (G2) above. Then for every tiling \( Z \in \mathcal{Z}_\mathcal{P} \) we can associate a continuous function \( g(x; Z) \), thus the above function \( g(x) \) is written as \( g(x; Z_0) \).

It is easily seen that

\[
\sigma_a g(x; Z) := g(x + a; Z) = g(x; \tilde{\sigma}_a Z),
\]
and that $\sigma_ag(\cdot, Z)$ converges locally uniformly to some function on $\mathbb{R}^2$ if and only if $\tilde{\sigma}_aZ$ converges in $\mathcal{Z}_P$. Therefore, the association $Z \mapsto g(\cdot, Z)$ defines a continuous map from $\mathcal{Z}_P$ into $C(\mathbb{R}^2)$. Furthermore, it is known that every Penrose tiling $Z \in \mathcal{Z}_P$ is strictly ergodic; see [13] and the references therein. Consequently, by Lemma 2.9, $g(x, Z_0)$ is a strictly (hence uniquely) ergodic function.

It remains to show that $g(x, Z_0)$ is not almost periodic. Suppose that $g$ is almost periodic. Then for any $\varepsilon > 0$, there exists a relatively dense set $D_\varepsilon \subset \mathbb{R}^2$ such that

$$\|\sigma_ag - g\|_{L^\infty(\mathbb{R}^2)} < \varepsilon \quad \text{for } a \in D_\varepsilon,$$

or, equivalently,

$$\|g(\cdot, Z_0) - g(\cdot, \tilde{\sigma}_a Z_0)\|_{L^\infty(\mathbb{R}^2)} < \varepsilon \quad \text{for } a \in D_\varepsilon.$$

If we choose $\varepsilon > 0$ sufficiently small, then we see from the conditions (G1), (G2) that the above inequality holds if and only if $Z_0$ and $\tilde{\sigma}_a Z_0$ are very close to each other in the Hausdorff distance. Considering that $Z_0$ is composed of non-overlapping tiles that are translations of a finite set of prototiles, we easily find that $Z_0$ and $\tilde{\sigma}_a Z_0$ can be very close to each other if and only if the latter coincides with the former after a small adjustment of the position. This means that $Z_0 = \tilde{\sigma}_b Z_0$, where $b$ is a point very close to $a$. Since such points $a$ are distributed relatively densely on $\mathbb{R}^2$, we can choose $a$ far enough from the origin so that $b \neq 0$. This implies that $Z_0$ is periodic, which is impossible as the Penrose tiling is known to be aperiodic. This contradiction shows that $g(x, Z_0)$ is not an almost periodic function. □

**Remark A.4.** There are also the so-called “marked” Penrose tilings, in which both the fat and thin rhombi have special markings so that they and their turnaround (by the angle $\pi$) are not identified. In this case, their $2k\pi/10$ rotations ($k = 0, 1, \ldots, 9$) are all different, thus $\mathcal{P}$ consists of 20 prototiles. The proposition remains valid for marked Penrose tilings provided that we slightly alter the conditions on $g$ to reflect the markings of the rhombi.

### A.4. Product of ergodic functions

In Example 2.3 (4), we stated that a function of the form $g(x_1, x_2) = q(x_1)q(x_2)$ is uniquely ergodic on $\mathbb{R}^2$ if $q$ is a uniquely ergodic function on $\mathbb{R}$. More generally we have the following result whose proof is rather elementary:

**Proposition A.5.** Let $q_1, q_2$ be uniquely ergodic functions on $\mathbb{R}^k, \mathbb{R}^{m-k}$, respectively, where $1 \leq k \leq m - 1$. Then the function $g(x_1, \ldots, x_m) := q_1(x_1, \ldots, x_k)q_2(x_{k+1}, \ldots, x_m)$ is uniquely ergodic on $\mathbb{R}^m$.

**Proof.** For each $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, we set $\tilde{x} := (x_1, \ldots, x_k), \hat{x} := (x_{k+1}, \ldots, x_m)$. Then

$$\sigma_ag(x) := g(x + a) = q_1(\tilde{x} + \tilde{a})q_2(\hat{x} + \hat{a}) = \sigma_a q_1(\tilde{x}) \sigma_a q_2(\hat{x}).$$

We first prove the unique ergodicity of the vector-valued function $q(x) := (q_1(\tilde{x}), q_2(\hat{x})): \mathbb{R}^k \times \mathbb{R}^{m-k} \to \mathbb{R}^2$. It is clear that $\mathcal{H}_q = \mathcal{H}_{q_1} \times \mathcal{H}_{q_2}$. Let $\mu$ be any shift-invariant probability measure on $\mathcal{H}_q$. Then, for any $\Psi_1 \in C(\mathcal{H}_{q_1}; \mathcal{R})$ and $\Psi_2 \in C(\mathcal{H}_{q_2}; \mathcal{R})$, the function $\psi(z) := \Psi_1(\tilde{z}) \Psi_2(\hat{z})$ belongs to $C(\mathcal{H}_q; \mathcal{R}) = C(\mathcal{H}_{q_1} \times \mathcal{H}_{q_2}; \mathcal{R})$. If we fix $\Psi_2$, then

$$\Psi_1 \mapsto \int_{\mathcal{H}_{q_1} \times \mathcal{H}_{q_2}} \Psi_1(\tilde{z}) \Psi_2(\hat{z}) \, d\mu.$$
defines a shift-invariant functional on \( C(\mathcal{H}_q; \mathbb{R}) \). By the unique ergodicity of \( q_1 \), there exists a constant \( C(\Psi_2) \) depending on \( \Psi_2 \) such that

\[
\int_{\mathcal{H}_q \times \mathcal{H}_q} \Psi_1(\tilde{z})\Psi_2(\tilde{z}) \, d\mu = C(\Psi_2) \int_{\mathcal{H}_q} \Psi_1(\tilde{z}) \, d\mu_1, 
\]

where \( \mu_1 \) is the unique shift-invariant probability measure on \( \mathcal{H}_q \). Similarly, there exists a constant \( C(\Psi_1) \) depending on \( \Psi_1 \) such that

\[
\int_{\mathcal{H}_q \times \mathcal{H}_q} \Psi_1(\tilde{z})\Psi_2(\tilde{z}) \, d\mu = C(\Psi_1) \int_{\mathcal{H}_q} \Psi_2(\tilde{z}) \, d\mu_2, 
\]

where \( \mu_2 \) is the unique shift-invariant probability measure on \( \mathcal{H}_q \). Combining these, we see that there exists a constant \( C \), independent of the choice of \( \Psi_1 \) and \( \Psi_2 \), such that

\[
\int_{\mathcal{H}_q \times \mathcal{H}_q} \Psi_1(\tilde{z})\Psi_2(\tilde{z}) \, d\mu = C \left( \int_{\mathcal{H}_q} \Psi_1(\tilde{z}) \, d\mu_1 \right) \left( \int_{\mathcal{H}_q} \Psi_2(\tilde{z}) \, d\mu_2 \right). 
\] (54)

Setting \( \Psi_1 \equiv 1 \), \( \Psi_2 \equiv 1 \), we see that \( C = 1 \). Consequently, for any continuous function of the form \( \Psi_1(\tilde{z})\Psi_2(\tilde{z}) \), the left-hand side of (54) does not depend on the choice of the shift-invariant probability measure \( \mu \) on \( \mathcal{H}_q \). Since the linear combinations of such functions form a dense subset of \( C(\mathcal{H}_q; \mathbb{R}) \), this implies the uniqueness of the shift-invariant probability measure on \( \mathcal{H}_q \). In other words, \( q(z) = (q_1(\hat{x}), q_2(\hat{x})) \) is uniquely ergodic on \( \mathbb{R}^m \). Finally, since the association \((r_1(\hat{x}), r_2(\hat{x})) \mapsto r_1(\hat{x})r_2(\hat{x})\) defines a continuous map from \( \mathcal{H}_q \) onto \( \mathcal{H}_g \), the conclusion of the proposition follows from Lemma 2.9.

References