Schubert calculus and equivariant cohomology of Grassmannians

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Abstract

We give a description of equivariant cohomology of Grassmannians that places the theory into a general framework for cohomology theories of Grassmannians. As a result we obtain a formalism for equivariant cohomology where the basic results of equivariant Schubert calculus, the basis theorem, Pieri’s formula and Giambelli’s formula can be obtained from the corresponding results of the general framework by a change of basis.

In order to show that our formalism reflects the geometry of Grassmannians we relate our theory to the treatment of equivariant cohomology of Grassmannians by A. Knutson and T. Tao.

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0. Introduction

During the last years there has appeared a rich literature on the equivariant cohomology ring of Grassmannians. We recommend the notes [6] by W. Fulton for references to the literature as well as for a presentation of equivariant cohomology both from a geometric and algebraic point of view. In this article we give an alternative description of equivariant cohomology of Grassmannians that places the theory into a general framework for cohomology theories of Grassmannians. As a result we obtain a formalism for equivariant cohomology where the basic results...
of equivariant Schubert calculus, the basis theorem, Pieri’s formula and Giambelli’s formula can be obtained from the corresponding results of the general framework by a change of basis.

More precisely, we gave in [17] a general formalism of Schubert calculus for grassmannians consisting of a factorization algebra \( \text{Fact}_A^I(p) \) of a polynomial \( p(T) \) with coefficients in a ring \( A \), and a structure as an \( \text{Fact}_A^I(p) \)-module on the exterior power \( \bigwedge^I_A A[T]/(p) \). In this article we show that when \( p(T) \) is a general factorial power that is, \( p(T) := (T | y)^n = (T - y_1) \cdots (T - y_n) \) with \( y_1, \ldots, y_n \) in \( A \), we obtain the equivariant cohomology of grassmannians from the general formalism by replacing the basis \( 1, \xi, \ldots, \xi^{n-1} \) of \( A[\xi] := A[T]/(p) \), where \( \xi \) is the class of \( T \), by the general factorial powers \( (\xi | y)^0, (\xi | y)^1, \ldots, (\xi | y)^n-1 \). The reader should also consult Fulton’s notes [6] for the correspondence between the classical and equivariant theories.

In particular Fulton explains in lecture 7 of the notes how the equivariant Giambelli formula for grassmannians amount to a degeneracy formula [10] in algebraic geometry.

Our work builds upon earlier results. In [18] we gave a formalism where the ring of symmetric functions \( A[T_1, \ldots, T_n]^\text{sym} \) in \( n \) variables operates on the exterior power \( \bigwedge^n_A A[T] \). We proved that even in this generality there exists a Schubert calculus having a basis theorem, a Pieri formula and a Giambelli formula. In [16] we saw that we from this formalism obtain a corresponding equivariant Schubert calculus by replacing the basis \( 1, T, T^2, \ldots \) of \( A[T] \) by the basis \( (T | y)^0, (T | y)^1, (T | y)^2, \ldots \) of general factorial powers. In this article we specialize the equivariant Schubert calculus of [16] to an equivariant Schubert calculus for grassmannians. In this way we obtain the Pieri and the Giambelli formulae in equivariant cohomology by specialization from the corresponding results in [16].

In order to show that our formalism reflects the geometry of grassmannians we must prove that the expressions given by the equivariant Giambelli formula correspond to the classes in equivariant cohomology given by Schubert schemes in the grassmannian. This we do by relating our theory to the formalism for equivariant cohomology of A. Knutson and T. Tao [13]. We note that our approach has several advantages. In [13] there are two proofs of the existence of Schubert classes, one topological and one combinatorial. Both are “top down,” that is, they start from the most equivariant case. In our theory the existence is part of the general framework of factorial Schur functions and requires no separate proof. Our approach is also computationally efficient.

We note that we obtain the full Pieri formula, and not only the Chevalley formula for divisors (see e.g. [13–15,20,23,25] for various forms of the latter formula). It should also be pointed out that our equivariant Giambelli formula is an unshifted version of that of Mihalcea [21], that is, using only general factorial powers and not their shifted counterparts.

To see the connections of our general formalism with geometry we consider a locally free \( \mathcal{O}_X \)-module \( \mathcal{E} \) on a base scheme \( X \) and let the grassmannian be \( \text{Grass}_X^l(\mathcal{E}) \) parameterizing \( l \)-quotients of \( \mathcal{E} \). Then the ring \( A \) is the bivariant Chow ring of \( X \), when a bivariant theory exists, and \( p(T) \) is the Chern polynomial of \( \mathcal{E} \). Different choices of \( p(T) \) and different presentations of \( \text{Fact}_A^I(p) \) give various cohomology theories of grassmannians, like the classical cohomology, equivariant cohomology, quantum cohomology and equivariant quantum cohomology (see Section 7). For example, by one of the standard constructions of factorization algebras in terms of generators and relations the factorization algebra of the polynomial \( p(T) = (T - y_1) \cdots (T - y_n) + (-1)^n q \) becomes presented in the same form as the equivariant quantum cohomology given by A. Givental and B. Kim in [9,11,12], and by A. Astashkevich and V. Sadov [1]. Another representation gives the form of the equivariant quantum cohomology for grassmannians which is the main result of [21]. The latter presentation of the factorization algebra with \( q = 0 \) gives equivariant cohomology, and when \( p(T) = X^n + (-1)^n q \) we get the small quantum cohomology of E. Witten [27] (see also A. Bertam [2] and L. Gatto [7], both containing the Giambelli formula).
More details on earlier work are given in the introduction of [16]. As in [16] we want to emphasize the influence by the work of Gatto [7]. Our treatment of equivariant Schubert calculus was initiated by the desire to make explicit the existence results of [8]. The explicit equivariant Pieri formula was first given in [26], although in a different language (see also [8]). The use of factorial Schur functions similar to the treatment here can be found in [21], where the importance of Giambelli type formulae is manifested.

1. Factorization and splitting algebras

We construct splitting and factorization algebras of polynomials and give the properties of such algebras that we shall need in the sequel. For alternative approaches and further results see [3,17,24].

1.1. Notation. We fix positive integers $k, l, n$ with $n = k + l$. Let $A$ be a commutative ring with unit and denote by $A[T]$ and $A[T_1, \ldots, T_n]$ the polynomial rings in 1, respectively $n$, independent variables over $A$. We denote by $A[T_1, \ldots, T_n]_{\text{sym}}$ the subring of $A[T_1, \ldots, T_n]$ of symmetric functions and by $c_i(T_1, \ldots, T_n)$, respectively $s_i(T_1, \ldots, T_n)$, the elementary symmetric, respectively complete symmetric, functions in $T_1, \ldots, T_n$.

1.2. Definition. Let $p(T)$ be a monic polynomial in $A[T]$ of degree $n$. We denote by $\text{Split}_p$ and $\text{Fact}_p^l$ the covariant functors from $A$-algebras to sets that to every $A$-algebra $B$ associate the sets

$$\text{Split}_p(B) = \{\text{splittings } p(T) = (T - b_1) \cdots (T - b_n) \text{ over } B[T]\},$$

respectively,

$$\text{Fact}_p^l(B) = \{\text{factorizations } p(T) = q(T)r(T) \text{ in } B[T], \text{ with } q \text{ monic of degree } l\}.$$ 

Moreover, the functors associate to every $A$-algebra homomorphism $\varphi : B \to C$ the natural maps $\text{Split}_p(B) \to \text{Split}_p(C)$, respectively $\text{Fact}_p^l(B) \to \text{Fact}_p^l(C)$, obtained by using $\varphi$ on the coefficients of the polynomials $T - b_1, \ldots, T - b_n, q(T), r(T)$ involved in the factorizations.

An algebra $\text{Split}_A(p)$ that represents the functor $\text{Split}_p$ is called a splitting algebra for $p$ over $A$, and when $p(T) = (T - \xi_1) \cdots (T - \xi_n)$ is the universal splitting over $\text{Split}_A(p)$ we call $\xi_1, \ldots, \xi_n$ the universal roots. Moreover, an algebra $\text{Fact}_A^l(p)$ representing the functor $\text{Fact}_p^l$ is called a factorization algebra for $p$ over $A$ in a factor of degree $l$.

Two representants are naturally isomorphic and the isomorphism will map the universal roots in one representant to the universal roots in the other. It is also clear that a permutation of the universal roots will be universal roots.

1.3. Proposition. Let $p(T) = T^n - c_1(p)T^{n-1} + \cdots + (-1)^nc_n(p)$ be a polynomial in $A[T]$.

(1) Denote by $\mathcal{I}$ the ideal in $A[T_1, \ldots, T_n]$ generated by the relations we obtain by setting the coefficient of each of the powers $1, T, \ldots, T^{n-1}$ equal in both sides of the equation $p(T) = (T - T_1) \cdots (T - T_n)$, that is, $\mathcal{I}$ is generated by the elements $c_i(p) - c_i(T_1, \ldots, T_n)$ for $i = 1, \ldots, n$. Then the residue algebra $A[T_1, \ldots, T_n]/\mathcal{I}$ is a splitting algebra for $p$ over $A$ and the residue classes $\xi_1, \ldots, \xi_n$ of $T_1, \ldots, T_n$ in $A[T_1, \ldots, T_n]/\mathcal{I}$ are universal roots.
(2) Let \( X_i = c_i(T_1, \ldots, T_l) \) for \( i = 1, \ldots, l \), and \( Y_i = c_i(T_{i+1}, \ldots, T_n) \) for \( i = 1, \ldots, k \), with \( X_i = 0 \) for \( i > 1 \) and \( Y_i = 0 \) for \( i > k \). Denote by \( \mathfrak{J} \) the ideal in \( A[X_1, \ldots, X_l, Y_1, \ldots, Y_k] \) generated by the relations we get by setting the coefficients of each of the powers \( 1, T, \ldots, T^{n-1} \) equal in both sides of the equation
\[
p(T) = (T^l - X_1 T^{l-1} + \cdots + (-1)^l X_l)(T^k - Y_1 T^{k-1} + \cdots + (-1)^k Y_k),
\]
that is, \( \mathfrak{J} \) is generated by the elements \( c_i(p) - (X_i + X_{i-1} Y_1 + \cdots + X_1 Y_{i-1} + Y_i) \) for \( i = 1, \ldots, n \). Then \( A[T_1, \ldots, T_n]/\mathfrak{J} \) is a factorization algebra for \( p \) over \( A \) in a factor of degree 1, and the universal factorization is \( p(T) = q(T) r(T) \) where the coefficients of \( 1, T, \ldots, T^{l-1} \) of \( q \) are the residue classes of \( X_i, X_{i-1}, \ldots, X_1 \), respectively.

**Proof.** Both assertions are clear from the definitions of splitting and factorization algebras. \( \square \)

1.4. **Lemma.** Let \( B \) be an \( A \)-algebra that is free as an \( A \)-module with basis \( 1 = b_1, b_2, \ldots, b_m \). For every ideal \( \mathfrak{J} \) in \( A \) we have \( \mathfrak{J} = A \cap \mathfrak{J} B \), and \( B/\mathfrak{J} B \) is a free \( A/\mathfrak{J} \)-module with a basis given by the residue classes of \( 1 = b_1, b_2, \ldots, b_m \) in \( B/\mathfrak{J} B \).

**Proof.** We have \( B = \bigoplus_{i=1}^m A b_i \), and thus \( \mathfrak{J} B = \bigoplus_{i=1}^m \mathfrak{J} b_i \). Consequently \( A \cap \mathfrak{J} B = \mathfrak{J} b_1 = \mathfrak{J} \). Moreover we have natural isomorphisms of \( A \)-modules \( B/\mathfrak{J} B \cong \bigoplus_{i=1}^m A b_i / \bigoplus_{i=1}^m \mathfrak{J} b_i \cong \bigoplus_{i=1}^m A b_i/\mathfrak{J} b_i \) that give the second part of the lemma. \( \square \)

1.5. **Proposition.** Let \( \text{Split}_A(p) \) and \( \text{Fact}_A^\mathfrak{J}(p) \) be the splitting and factorization algebras of \( p \) over \( A \) defined in Proposition 1.3. There is an injective \( A \)-algebra homomorphism
\[
\text{Fact}_A^\mathfrak{J}(p) \rightarrow \text{Split}_A(p)
\]
determined by mapping the class of \( X_i = c_i(T_1, \ldots, T_l) \) to \( c_i(\xi_1, \ldots, \xi_l) \) for \( i = 1, \ldots, l \), and the class of \( Y_i = c_i(T_{i+1}, \ldots, T_n) \) to \( c_i(\xi_{i+1}, \ldots, \xi_n) \) for \( i = 1, \ldots, k \). The image is the \( A \)-algebra generated by \( c_1(\xi_1, \ldots, \xi_l), \ldots, c_l(\xi_1, \ldots, \xi_l) \), or equivalently, the \( A \)-algebra generated by the elements \( s_1(\xi_1, \ldots, \xi_l), \ldots, s_l(\xi_1, \ldots, \xi_l) \).

**Proof.** With the notation of Proposition 1.3 we have that the polynomial ring \( B := A[X_1, \ldots, X_l, Y_1, \ldots, Y_k] \) is contained in \( A[T_1, \ldots, T_n] \). Moreover we have \( T^l - X_1 T^{l-1} + \cdots + (-1)^l X_l = \prod_{i=1}^l (T - T_i) \), and \( T^k - Y_1 T^{k-1} + \cdots + (-1)^k Y_k = \prod_{i=1}^k (T - T_{i+1}) \). By definition the ideals \( \mathfrak{J} \) and \( \mathfrak{J} \) are generated by the relations we obtain by comparing the coefficients of 1, \( T, \ldots, T^{n-1} \) in \( p(T) = \prod_{i=1}^l (T - T_i) \), respectively in \( p(T) = (T^l - X_1 T^{l-1} + \cdots + (-1)^l X_l)(T^k - Y_1 T^{k-1} + \cdots + (-1)^k Y_k) = \prod_{i=1}^l (T - T_i) \prod_{i=1}^k (T - T_{i+1}) \). Consequently \( \mathfrak{J} \) and \( \mathfrak{J} \) are generated by the same elements in \( B \). Hence it follows from Lemma 1.4 that \( \mathfrak{J} = B \cap \mathfrak{J} A[T_1, \ldots, T_n] = B \cap \mathfrak{J} \), that is, we have an injection \( \text{Fact}_A^\mathfrak{J}(p) = B/\mathfrak{J} \rightarrow A[T_1, \ldots, T_n]/\mathfrak{J} = \text{Split}_A(p) \) that maps the class of \( X_i = c_i(T_1, \ldots, T_l) \) to \( c_i(\xi_1, \ldots, \xi_l) \), for \( i = 1, \ldots, l \), and the class of \( Y_i = c_i(T_{i+1}, \ldots, T_n) \) to \( c_i(\xi_{i+1}, \ldots, \xi_n) \) for \( i = 1, \ldots, k \).

Finally we note that since \( q(T) = \prod_{i=1}^l (T - \xi_i) \) divides \( p(T) \) over the ring \( C := A[c_1(\xi_1, \ldots, \xi_l), \ldots, c_l(\xi_1, \ldots, \xi_l)] \) we obtain that \( C \) also contains the coefficients \( c_1(\xi_{i+1}, \ldots, \xi_n), \ldots, c_k(\xi_{i+1}, \ldots, \xi_n) \) of \( r(T) = p(T)/q(T) \), and thus the image of \( \text{Fact}_A^\mathfrak{J}(p) \) is in \( C \). From the relations between the elements \( c_i(\xi_1, \ldots, \xi_l) \) and \( s_i(\xi_1, \ldots, \xi_l) \) that we obtain from the equations
1 = q(T^{-1})/q(T^{-1}) = T^i q(T^{-1}) (1 + s_1(\xi_1, \ldots, \xi_l) T + s_2(\xi_1, \ldots, \xi_l) T^2 + \cdots) we conclude that the algebra \( C \) is also generated by the elements \( s_1(\xi_1, \ldots, \xi_l), s_2(\xi_1, \ldots, \xi_l) \). \( \square \)

1.6. Proposition. Let \( p \) be a monic polynomial of degree \( n \) in \( A[T] \), and let \( C := \text{Fact}_A^1(p) \) be the splitting algebra for \( p \) over \( A \) of Proposition 1.3 with universal splitting \( p(T) = q(T)r(T) \). We consider \( \text{Split}_A(p) \) as a \( C \)-algebra via the homomorphism \( C \to \text{Split}_A(p) \) of Proposition 1.5. Let \( \xi_1, \ldots, \xi_l \) and \( \eta_1, \ldots, \eta_l \) be the universal roots in \( \text{Split}_C(q) \), respectively \( \text{Split}_C(r) \). Then there is a canonical isomorphism of \( C \)-algebras

\[
\text{Split}_A(p) \to \text{Split}_C(q) \otimes_C \text{Split}_C(r)
\]

that maps \( \xi_i \) to \( \xi_i \otimes 1 \) for \( i = 1, \ldots, l \), and to \( 1 \otimes \eta_{i-l} \) for \( i = l+1, \ldots, n \).

The \( A \)-algebra \( \text{Split}_A(p) \) is free as an \( A \)-module generated by the \( n! \) elements \( \xi_1^{h_1} \cdots \xi_n^{h_n} \) for \( 0 \leq h_i \leq n - i \), and \( \text{Split}_A(p) \) is a free module over the image of \( \text{Fact}_A^1(p) \) generated by the \( k! \) elements \( \xi_1^{h_1} \cdots \xi_l^{h_l} \xi_{l+1}^{h_{l+1}} \cdots \xi_n^{h_n} \) with \( 0 \leq h_i \leq l - i \) for \( i = 1, \ldots, l \) and \( 0 \leq h_{l+i} \leq k - i \) for \( i = 1, \ldots, k \). Consequently the last part of the proposition follows from the first part.

Proof. That \( \text{Split}_A(p) \) is a free module over \( A \) with the basis given in the proposition is well known (see [3,24]). It can also easily be deduced from the main theorem for symmetric functions. Similarly \( \text{Split}_C(q) \) and \( \text{Split}_C(r) \) have the bases \( \xi_1^{h_1} \cdots \xi_l^{h_l} \) with \( 0 \leq h_i \leq l - i \) for \( i = 1, \ldots, l \), respectively \( \eta_1^{h_1} \cdots \eta_k^{h_k} \) with \( 0 \leq h_i \leq k - i \) for \( i = 1, \ldots, k \). Consequently the last part of the proposition follows from the first part.

The existence of the homomorphism of the proposition follows from the universal property of \( \text{Split}_A(p) \). In fact, consider \( q \) and \( r \) as polynomials over the \( A \)-algebra \( D := \text{Split}_C(q) \otimes_C \text{Split}_C(r) \) via the canonical homomorphism \( C \to D \), and \( p \) as a polynomial over \( D \) via the composite homomorphism \( A \to C \to D \). Then we have a splitting \( p(T) = q(T)r(T) = (T - \xi_1 \otimes 1) \cdots (T - \xi_l \otimes 1)(T - 1 \otimes \eta_1) \cdots (T - 1 \otimes \eta_k) \) of \( p \) over \( D \), and consequently a homomorphism as in the proposition.

To prove that the homomorphism is bijective we give an inverse. Such an inverse is determined by two \( C \)-algebra homomorphisms \( \text{Split}_C(q) \to \text{Split}_A(p) \) and \( \text{Split}_C(r) \to \text{Split}_A(p) \). The existence of such homomorphisms follows from the universal properties of \( \text{Split}_C(q) \), respectively \( \text{Split}_C(r) \), since, in the notation of Propositions 1.3 and 1.5, we have \( q(T) = T^l - c_1(\xi_1, \ldots, \xi_l) T^{l-1} + \cdots + (-1)^l c_l(\xi_1, \ldots, \xi_l) = (T - \xi_1) \cdots (T - \xi_l) \) and \( r(T) = T^k - c_1(\xi_{l+1}, \ldots, \xi_n) T^{k-1} + \cdots + (-1)^k c_k(\xi_{l+1}, \ldots, \xi_n) = (T - \xi_{l+1}) \cdots (T - \xi_n) \) in \( \text{Split}_A(p) \). \( \square \)

2. Equivariant Schubert calculus on \( \bigwedge^l A[\xi] \)

We recall some results on general Schubert calculus on exterior products \( \bigwedge^l A[T] \) from [18] and [16], and show how these can be specialized to give a general version of Schubert calculus on \( \bigwedge^l A[T]/(p) \), where \( p(T) \) is a monic polynomial of degree \( n \). This specialization is also performed in [17], but we give a different approach and proof, from that of [17].

We also introduce factorial Schur functions (see [16,19, I.3,20], [21], or [23]) and show how the general version of Schubert calculus can be interpreted in terms of these Schur functions to give the equivariant Schubert calculus on grassmannians.
2.1. Notation and recall of equivariant Schubert calculus on $\bigwedge^l A[T]$. We identify $\bigotimes^l A[T]$ with $A[T_1, \ldots, T_l]$. For every $A[T]$-module $M$ the tensor product $\bigotimes^l M$ thus becomes an $A[T_1, \ldots, T_l]$-module and, in particular, a module over the symmetric functions $A[T_1, \ldots, T_l]^{\text{sym}}$. It is easy to verify, and is in fact part of a general principle (see [18, Proposition 1.6]), that there is a unique $A[T_1, \ldots, T_l]^{\text{sym}}$-module structure on $\bigwedge^l M$ such that the canonical surjection $\bigotimes^l M \rightarrow \bigwedge^l M$ is $A[T_1, \ldots, T_l]^{\text{sym}}$-linear. When $M = A[T]$, the module structure is explained below.

For every collection of polynomials $f_1, \ldots, f_l$ in $A[T]$ we write

$$(f_i(T_j)) = \begin{pmatrix} f_1(T_1) & \cdots & f_1(T_l) \\ \vdots & & \vdots \\ f_l(T_1) & \cdots & f_l(T_l) \end{pmatrix}.$$ 

Moreover, let $e_i = \cdots + b_{i-1} T^{-1} + \cdots + b_{i-l+1} T^{-l+1} + b_{i-l} T^{-l} + \cdots$ for $i = 1, \ldots, l$ be a collection of Laurent series with coefficients in a ring $B$. We write

$$\text{Res}(e_1, \ldots, e_l) = \det(b_{i-j}) = \det \begin{pmatrix} b_{1-1} & \cdots & b_{1-l} \\ \vdots & & \vdots \\ b_{l-1} & \cdots & b_{l-l} \end{pmatrix}.$$ 

Note that $\text{Res}(e_1, \ldots, e_l) = 0$ if at least one of the $e_1, \ldots, e_l$ is in $B[T]$.

Assume that $l \geq 2$. Since $\det(f_i(T_j))$ is alternating in $T_1, \ldots, T_l$ for a given collection of polynomials $f_1, \ldots, f_l$ in $A[T]$ we have that $\Delta := \prod_{1 \leq i < j \leq l} (T_i - T_j)$ divides $\det(f_i(T_j))$ and with $Q(T) = (T - T_1) \cdots (T - T_l)$ we obtain [16, Proposition 1.4]

$$\text{Res}(f_1/Q, \ldots, f_l/Q) = \det(f_i(T_j))/\Delta,$$

with $1/Q(T) = T^{-l} + s_1(T_1, \ldots, T_l)/T^{-l-1} + s_2(T_1, \ldots, T_l)/T^{-l-2} + \cdots$. Moreover we proved in [16, Proposition 1.5] that, with the $A[T_1, \ldots, T_l]^{\text{sym}}$-module structure on $\bigwedge^l A[T]$ mentioned above, we have an isomorphism of $A[T_1, \ldots, T_l]^{\text{sym}}$-modules

$$\sigma^{\text{sym}} : \bigwedge^l A[T] \rightarrow A[T_1, \ldots, T_l]^{\text{sym}}$$

determined by $\sigma^{\text{sym}}(f_1 \wedge \cdots \wedge f_l) = \text{Res}(f_1/Q, \ldots, f_l/Q) = \det(f_i(T_j))/\Delta$.

Let $y_1, y_2, \ldots$ be elements in $A$. We write, using a notation similar to the one used in [19, I.3, 20],

$$(T|y)^i = (T - y_1) \cdots (T - y_i) \quad \text{for } i = 1, 2, \ldots,$$

and $(T|y)^0 = 1$. Then $(T|y)^0, (T|y)^1, \ldots$ is an $A$-module basis for $A[T]$, and $\Delta = \det((T_j|y)^{l-i})$, where the indices satisfy $1 \leq i, j \leq l$.

For every partition $b$: $b_1 \geq \cdots \geq b_l$ we write

$$s_b(T_1, \ldots, T_l|y) = \text{Res}((T|y)^{b_1+l-1}/Q, \ldots, (T|y)^{b_l}/Q)$$

$$= \det((T_j|y)^{b_1+l-i})/\det((T_j|y)^{l-i}).$$
The polynomials \( s_b(T_1, \ldots, T_l|y) \) are called factorial Schur functions. When \( y = 0 \) we obtain the Schur functions. Geometrically the factorial Schur functions correspond to the equivariant Schubert classes, as is shown in Section 6.

Let \( \text{Split}_A(p) = A[\xi_1, \ldots, \xi_n] \) be a splitting algebra for the monic polynomial \( p \) in \( A[T] \) over \( A \) with universal roots \( \xi_1, \ldots, \xi_n \). Moreover, we let \( \text{Fact}_A^l(p) \) be the factorization algebra in \( \text{Split}_A(p) \) generated by the elements \( c_i(\xi_1, \ldots, \xi_l) \) for \( i = 1, \ldots, l \). The universal splitting we denote by \( p(T) = q(T)r(T) \) with

\[
q(T) = T^l - c_1(\xi_1, \ldots, \xi_l)T^{l-1} + \cdots + (-1)^lc_l(\xi_1, \ldots, \xi_l) = (T - \xi_1) \cdots (T - \xi_l).
\]

We write

\[
\]

where \( \xi \) is the residue class of \( T \) and denote by

\[
\rho_p : A[T] \to A[\xi]
\]

the residue homomorphism.

As we mentioned above we have unique \( A[T_1, \ldots, T_l]^\text{sym} \)-module structures on \( \bigwedge^l A[T] \) and \( \bigwedge^l A[\xi] \) such that the residue homomorphisms \( \bigotimes^l A[T] \to \bigwedge^l A[T] \to \bigwedge^l A[\xi] \) are \( A[T_1, \ldots, T_l]^\text{sym} \)-linear. We denote by

\[
\rho : A[T_1, \ldots, T_l]^\text{sym} \to \text{Fact}_A^l(p)
\]

the residue homomorphism given by \( \rho(c_i(T_1, \ldots, T_l)) = c_i(\xi_1, \ldots, \xi_l) \) for \( i = 1, \ldots, l \), and consider \( \text{Fact}_A^l(p) \) as an \( A[T_1, \ldots, T_l]^\text{sym} \)-algebra via \( \rho \).

2.2. Lemma. Let \( Q(T) = (T - T_1) \cdots (T - T_l) \). For all collections \( f_1, \ldots, f_l \) of polynomials in \( A[T] \) we have

\[
\text{Res}(f_1/Q, \ldots, f_l/Q)(\xi_1, \ldots, \xi_l) = \text{Res}(f_1/q, \ldots, f_l/q).
\]

Proof. The rational function \( f_i(T)/Q(T) \) is a Laurent series with coefficients in \( A[T_1, \ldots, T_l]^\text{sym} \) and \( f_i(T)/q(T) \) is a Laurent series with coefficients in the algebra \( A[c_1(\xi_1, \ldots, \xi_l), \ldots, c_l(\xi_1, \ldots, \xi_l)] \). It is clear that we have, for all \( f \) in \( A[T] \), an equality \( (f/Q)(\xi_1, \ldots, \xi_l) = f/q \). The lemma is an immediate consequence of the latter equality. \( \square \)

The next result is the main result of [17]. However, we provide a different proof from that of [17].

2.3. Theorem. (See [17].) Let \( p \) be a monic polynomial in \( A[T] \) of degree \( n \) and let \( p = qr \) be the universal factorization of \( p \) over \( \text{Fact}_A^l(p) \) with \( q \) of degree \( l \). Consider \( \bigwedge^l A[\xi] \) as an \( A[T_1, \ldots, T_l]^\text{sym} \)-module as in 2.1.

1. The \( A[T_1, \ldots, T_l]^\text{sym} \)-module structure on \( \bigwedge^l A[\xi] \) factors via the residue morphism \( \rho : A[T_1, \ldots, T_l]^\text{sym} \to \text{Fact}_A^l(p) \) into a \( \text{Fact}_A^l(p) \)-module structure on \( \bigwedge^l A[\xi] \).
(2) For every collection \( f_1, \ldots, f_l \) of polynomials in \( A[T] \) we have a \( \text{Fact}_A^l(p) \)-module isomorphism

\[
\sigma_{\text{fact}} : \bigwedge^l A[\xi] \to \text{Fact}_A^l(p)
\]

determined by mapping \( f_1(\xi) \wedge \cdots \wedge f_l(\xi) \) to \( \text{Res}(f_1/q, \ldots, f_l/q) \).

In particular the elements \( s_b(\xi_1, \ldots, \xi_l|y) \) for all partitions \( b : k \geq b_1 \geq \cdots \geq b_l \) is a basis of the \( A \)-module \( \text{Fact}_A^l(p) \).

**Proof.** The composite of the homomorphism \( \sigma_{\text{sym}} : \bigwedge^l A[T] \to A[T_1, \ldots, T_l]^{\text{sym}} \) of 2.1 and the residue homomorphism \( \rho : A[T_1, \ldots, T_l]^{\text{sym}} \to \text{Fact}_A^l(p) \) gives an \( A[T_1, \ldots, T_l]^{\text{sym}} \)-module homomorphism \( \rho \sigma_{\text{sym}} : \bigwedge^l A[T] \to \text{Fact}_A^l(p) \) determined by \( \rho \sigma_{\text{sym}}(f_1 \wedge \cdots \wedge f_l) = \text{Res}(f_1/Q, \ldots, f_l/Q)(\xi_1, \ldots, \xi_l) \). It follows from Lemma 2.2 that the image of \( f_1 \wedge \cdots \wedge f_l \) is equal to \( \text{Res}(f_1/q, \ldots, f_l/q) \). To prove that the homomorphism \( \sigma_{\text{fact}} \) exists we must consequently prove that \( \rho \sigma_{\text{sym}} \) factors via the residue homomorphism \( \bigwedge^l A[T] \to \bigwedge^l A[\xi] \). However, it follows from the exact sequence \( 0 \to (p(T)) \to A[T] \to A[\xi] \to 0 \) that the kernel of the \( A \)-module homomorphism \( \bigwedge^l \rho_p : \bigwedge^l A[T] \to \bigwedge^l A[\xi] \) is generated by the elements of the form \( f_1 \wedge \cdots \wedge f_l \) with \( f_i(T) = p(T) \) for some \( i \). For such an element we have \( f_i(T)/q(T) = p(T)/q(T) = r(T) \) and thus \( \text{Res}(f_1/q, \ldots, f_l/q) = 0 \). Hence \( \sigma \rho_{\text{sym}} : \bigwedge^l A[T] \to \text{Fact}_A^l(p) \) factors via \( \bigwedge^l \rho_p : \bigwedge^l A[T] \to \bigwedge^l A[\xi] \) into an \( A \)-module homomorphism \( \sigma_{\text{fact}} : \bigwedge^l A[\xi] \to \text{Fact}_A^l(p) \), as we wanted to prove.

Since \( \bigwedge^l \rho_p : \bigwedge^l A[T] \to \bigwedge^l A[\xi] \) and \( \rho \sigma_{\text{sym}} : \bigwedge^l A[T] \to \text{Fact}_A^l(p) \) both are \( A[T_1, \ldots, T_l]^{\text{sym}} \)-module homomorphisms the same is true for the homomorphism \( \sigma_{\text{fact}} : \bigwedge^l A[\xi] \to \text{Fact}_A^l(p) \). Hence we have proved the existence of an \( A \)-module homomorphism \( \sigma_{\text{fact}} \) as in (2), and that it is an \( A[T_1, \ldots, T_l]^{\text{sym}} \)-module homomorphism.

We next prove that \( \sigma_{\text{fact}} \) is an isomorphism. Since \( \rho \) is surjective and \( \sigma_{\text{sym}} \) is an isomorphism the homomorphism \( \rho \sigma_{\text{sym}} : \bigwedge^l A[T] \to \text{Fact}_A^l(p) \) is surjective. Thus the homomorphism \( \sigma_{\text{fact}} : \bigwedge^l A[\xi] \to \text{Fact}_A^l(p) \) is surjective. Since \( \bigwedge^l A[\xi] \) is a free \( A \)-module with a basis \( \xi^{h_1} \wedge \cdots \wedge \xi^{h_l} \) for all \( n > h_1 > \cdots > h_l \geq 0 \), we conclude that \( \text{Fact}_A^l(p) \) is generated by the \( \binom{n}{l} \) elements \( \text{Res}(T^{h_1}/q, \ldots, T^{h_l}/q) \) with \( n > h_1 > \cdots > h_l \geq 0 \). However, it follows from Proposition 1.6 that \( \text{Split}_A(p) = A[\xi_1, \ldots, \xi_l] \) is a free \( A \)-module of rank \( n! \), and that it is a free \( \text{Fact}_A^l(p) \)-module of rank \( l!k! \). Consequently the elements \( \text{Res}(T^{h_1}/q, \ldots, T^{h_l}/q) \), for all \( n > h_1 > \cdots > h_l \geq 0 \) must be a basis for \( \text{Fact}_A^l(p) \) as an \( A \)-module. In particular \( \text{Fact}_A^l(p) \) is a free \( A \)-module of rank \( \binom{n}{l} \). We conclude that \( \sigma_{\text{fact}} : \bigwedge^l A[\xi] \to \text{Fact}_A^l(p) \) is an isomorphism of \( A \)-modules and thus an isomorphism of \( A[T_1, \ldots, T_l]^{\text{sym}} \)-modules, where \( \text{Fact}_A^l(p) \) is an \( A[T_1, \ldots, T_l]^{\text{sym}} \)-module via the residue map \( \rho \). Hence the \( A[T_1, \ldots, T_l]^{\text{sym}} \)-module structure on \( \bigwedge^l A[\xi] \) factors via the residue homomorphism \( \rho : A[T_1, \ldots, T_l]^{\text{sym}} \to \text{Fact}_A^l(p) \) of Section 2.1 into a \( \text{Fact}_A^l(p) \)-module structure on \( \bigwedge^l A[\xi] \). We have proved assertion (1) and the first part of assertion (2) of the theorem.

Let \( (\xi|y)^l \) be the image of \( (T|y)^l \) by the residue map \( \rho \). The last part of assertion (2) follows since \( (\xi|y)^{h_1+l-1} \wedge \cdots \wedge (\xi|y)^{h_l} \), for all partitions \( b : k \geq b_1 \geq \cdots \geq b_l \), form a basis for \( \bigwedge^l A[\xi] \), and we have an equality \( \text{Res}((T|y)^{h_1+l-1}/q, \ldots, (T|y)^{h_l}/q) = s_b(\xi_1, \ldots, \xi_l|y) \) by the definition of the element \( s_b(T_1, \ldots, T_l|y) \) and Lemma 2.2. □
2.4. Remark. The above theory when used to the generalized factorial powers \( p(T) = (T|y)^n \) gives the equivariant Schubert calculus. In the next result, and the following two sections we have \( p(T) = (T|y)^n \).

2.5. Equivariant Schubert calculus on \( \bigwedge^l A[\xi] \). Let \( (T|y)^n = q(T)r(T) \) be the universal factorization over \( \text{Fact}^l_A((T|y)^n) \) with \( q \) of degree \( l \).

(1) (Poincaré duality) The \( \text{Fact}^l_A((T|y)^n) \)-module \( \bigwedge^l A[\xi] \) is free of rank 1 with basis \( (\xi|y)^{l-1} \wedge \cdots \wedge (\xi|y)^0 \).

(2) (The determinantal formula) For every collection \( f_1, \ldots, f_l \) of polynomials in \( A[T] \) we have

\[
\sum_{\sigma \in S_l} (-1)^{\sigma} f_{\sigma(1)}(\xi) \cdots f_{\sigma(l)}(\xi) = \text{Res}(f_1/q, \ldots, f_l/q)((\xi|y)^{l-1} \wedge \cdots \wedge (\xi|y)^0).
\]

(3) (The equivariant Giambelli–Gatto formula) For every partition \( b: b_1 \geq \cdots \geq b_l \) we have

\[
(\xi|y)^{b_1+1-l} \wedge \cdots \wedge (\xi|y)^{b_l} = s_b(\xi, \ldots, \xi_l|y)((\xi|y)^{l-1} \wedge \cdots \wedge (\xi|y)^0).
\]

Proof. We have that \( (\xi|y)^{l-1} \wedge \cdots \wedge (\xi|y)^0 = \xi^{l-1} \wedge \cdots \wedge \xi^0 \) and \( \sigma^{\text{fact}}(\xi^{l-1} \wedge \cdots \wedge \xi^0) = 1 \). Hence assertion (1) immediately follows from Theorem 2.3(2). Moreover, the elements \( f_1(\xi) \wedge \cdots \wedge f_n(\xi) \) and \( \text{Res}(f_1/q, \ldots, f_l/q)((\xi|y)^{l-1} \wedge \cdots \wedge (\xi|y)^0) \) in \( \bigwedge^l A[\xi] \) map to the same element in \( \text{Fact}^l_A(\rho) \) by \( \sigma^{\text{fact}} \). Hence assertion (2) also follows from Theorem 2.3(2).

Assertion (3) is a special case of assertion (2). \( \square \)

3. Properties of factorial Schur functions

We interpret the theory of factorial Schur functions in terms of the combinatorial formalism of Knutson and Tao [13] used to describe the equivariant cohomology of grassmannians. To facilitate the understanding of the correspondence between the combinatorial formalism of Knutson and Tao and our approach we have followed the notation of Knutson and Tao as closely as possible.

3.1. Notation. Denote by \( \{k\} \) all strings \( \lambda: \lambda_1 \ldots \lambda_n \) consisting of zeros and ones, with exactly \( k \) ones. We consider \( \{n\} \) as a lattice with inequality \( \lambda' \geq \lambda \) when \( \sum_{i=1}^{j} \lambda'_i \geq \sum_{i=1}^{j} \lambda_i \) for \( j = 1, \ldots, n \).

An inversion in \( \lambda \) is a pair \( (i, j) \) with \( i < j \) such that \( 1 = \lambda_i > \lambda_j = 0 \). Denote by \( \text{inv}(\lambda) \) the inversions in \( \lambda \) and write \( l(\lambda) = |\text{inv}(\lambda)| \).

We introduce a similar terminology and notation for partitions. Recall that \( l+k=n \). Let \( \{k\}_P \) consist of all partitions \( b: k \geq b_1 \geq \cdots \geq b_l \). We consider \( \{n\}_P \) as a lattice with inequality \( b' \geq b \) when \( b'_i \geq b_i \) for \( i = 1, \ldots, l \).

An inversion in \( b \) is a pair \( (i, b_j + l - j + 1) \) such that \( i < b_j + l - j + 1 \) and \( i \notin \{b_{j+1} + l - j, b_j + 1\} \). We denote the inversions in \( b \) by \( \text{inv}(b) \) and write \( l(b) = |\text{inv}(b)| \). Then \( l(b) = \sum_{i=1}^{l} b_i \).

Let \( \lambda: \lambda_1 \ldots \lambda_n \) be a string in \( \{n\}_P \). Denote by \( b(\lambda)_1 + 1 < \cdots < b(\lambda)_l + l \) the positions where the zeros appear in \( \lambda_1 \ldots \lambda_n \), that is, \( b(\lambda)_{i+1} + l - i = 0 \) for \( i = 1, \ldots, l \). We obtain a partition \( b(\lambda) \): \( b(\lambda)_1 \geq \cdots \geq b(\lambda)_l \) in \( \{n\}_P \) and

\[
b(\lambda)_i = \{ \text{the number of ones to the left of zero number } l-i+1 \text{ in the string } \lambda \}.
\]
It is clear that \( \text{inv}(\lambda) = \text{inv} b(\lambda) \) and thus \( l(\lambda) = \sum_{i=1}^{l} b(\lambda)_i = l(b(\lambda)) \).

From a partition \( b: b_1 \geq \cdots \geq b_l \in \{n\}_P \) we obtain a string \( \lambda(b) \): \( \lambda(b)_1 \ldots \lambda(b)_n \) in \( \{n\}_k \) defined by \( \lambda(b)_i = \lambda(b)_1 \) for \( i = 1, \ldots, l \), and the remaining \( \lambda(b)_i \) being one.

**3.2. Lemma.** There is a length preserving bijection of lattices between partitions \( \{n\}_k \) and \( \{n\}_P \) that maps \( b \) to \( \lambda(b) \), and \( \lambda \) to \( b(\lambda) \).

**Proof.** It is obvious from the definitions that the map described in the lemma gives a bijection between \( \{n\}_k \) and \( \{n\}_P \), and we observed above that the map preserves length.

That we obtain a homomorphism of lattices follows since \( b(\lambda)_i \) and \( b(\lambda')_i \) is the number of ones to the left of zero number \( l - i + 1 \) in \( \lambda \), respectively \( \lambda' \). \( \square \)

**3.3. Lemma.** Let \( \lambda \) and \( \lambda' \) be lists in \( \{n\}_k \), and let \( b \) and \( b' \) be partitions in \( \{n\}_P \).

(1) When \( \lambda \) and \( \lambda' \) differ in the \( i \)th and \( j \)th position only and \( (i,j) \in \text{inv}(\lambda) \), then \( j = b(\lambda)_p + l - p + 1 \) and \( i = b(\lambda')_q + l - q + 1 \) for some \( p \) and \( q \), and the remaining elements in the sequences \( b(\lambda)_1 + l > \cdots > b(\lambda)_i + 1 \) and \( b(\lambda')_1 + l > \cdots > b(\lambda')_j + 1 \) are the same.

(2) When \( b \) and \( b' \) are such that the sequences \( b_1 + 1 < \cdots < b_1 + l \) and \( b'_1 + 1 < \cdots < b'_1 + l \) differ only where \( j = b_1 + l - p + 1 \) and \( i = b'_q + l - q + 1 \) with \( i < j \), then the lists \( \lambda(b) \) and \( \lambda(b') \) differ only in the positions \( i \) and \( j \), and \( (i,j) \in \text{inv}(\lambda(b)) \).

**Proof.** Assertion (1) follows since the zeros in \( \lambda \) are in the same positions as the zeros in \( \lambda' \) except in positions \( i \) and \( j \) where \( \lambda_j = 0 \) and \( \lambda'_j = 0 \).

Similarly assertion (2) follows since the zeros in \( \lambda(b) \) and \( \lambda(b') \) are in the same positions except zero number \( j \) in \( \lambda(b) \) and zero number \( i \) in \( \lambda(b') \). \( \square \)

We next give the main properties of factorial Schur functions.

**3.4. Theorem** (Vanishing theorem, see [22] and [16]). Let \( y_1, \ldots, y_n \) be elements in \( A \). For every partition \( b: b_1 \geq \cdots \geq b_l \) in \( \{n\}_P \) we have:

(1) \( s_b(y_{b_1 + 1}, \ldots, y_{b_l + 1}) = \prod_{(i,j) \in \text{inv}(\lambda(b))} (y_j - y_i) \).

(2) Let \( h: h_1 \geq \cdots \geq h_l \) be a partition in \( \{n\}_P \) such that \( h \) is not greater than or equal to \( b \). Then \( s_b(y_{h_1 + 1}, \ldots, y_{h_l + 1}, y) = 0 \).

**Proof.** This is a special case of Theorem 2.4 in [16]. \( \square \)

**3.5. Theorem.** Let \( A = \mathbb{Z}[y_1, \ldots, y_n] \) be the polynomial ring in the independent variables \( y_1, \ldots, y_n \) over \( \mathbb{Z} \), and let \( g \in A\{T_1, \ldots, T_l\}_{\text{sym}} \).

(1) \( g \) satisfies the GKM (Goresky–Kottwitz–MacPherson) condition. That is:

When \( b \) and \( b' \) are partitions in \( \{n\}_P \) such that the sequences \( b_1 + 1 < \cdots < b_1 + l \) and \( b'_1 + 1 < \cdots < b'_1 + l \) differ only where an element in the first sequence is equal to \( j \) and an element in the second is equal to \( i \) with \( i \neq j \), then

\[
g(y_{b_1 + 1}, \ldots, y_{b_l + 1}) - g(y_{b'_1 + 1}, \ldots, y_{b'_l + 1})
\]

is divisible by \( y_i - y_j \).
(2) Let \( g = \sum_{h \in I} Y_h s_h(T | y) \) with \( Y_h \in A \) and with \( I \subseteq \{ n \} \), and assume that \( g(y_{h_1+1}, \ldots, y_{h_l+1}) = 0 \) for all partitions \( h \in \{ n \} \). Then \( Y_h = 0 \) for all \( h \in I \).

(3) For a given partition \( b \) the conditions (1) and (2) of Theorem 3.4 characterize the homogeneous symmetric functions. More precisely: We have that \( g(y_{c_1+1}, \ldots, y_{c_l+1}) = s_b(y_{c_1+1}, \ldots, y_{c_l+1} | y) \) for all \( c \in \{ n \} \) when the following three conditions are fulfilled:

(i) \( g(y_{b_1+1}, \ldots, y_{b_l+1}) = \prod_{1 \leq i < j \leq \text{inv}(b)} (y_i - y_j) \) for \( b \in \{ n \} \).

(ii) \( g(y_{h_1+1}, \ldots, y_{h_l+1}) = 0 \) for all partitions \( h \in \{ n \} \) that are not greater than or equal to \( b \).

(iii) \( g(y_{c_1+1}, \ldots, y_{c_l+1}) \) is homogeneous in \( y_1, \ldots, y_n \) of degree \( l(b) \) for all \( c \in \{ n \} \).

Proof. All three assertions are special cases of the corresponding assertions of Theorem 2.5 in [16]. \( \square \)

4. Factorial Schur functions and Schubert classes

We give the precise correspondence between the combinatorial formalism of Knutson and Tao [13], used to describe the equivariant cohomology of grassmannians, and our description in Section 2 of the general Schubert calculus interpreted via factorial Schur functions.

4.1. Notation. Let \( A := Z[y_1, \ldots, y_n] \) be the polynomial ring in the independent variables \( y_1, \ldots, y_n \) over \( Z \). We denote by \( \bigoplus_{\lambda \in \{ n \}} A \) all lists \( \alpha = (\alpha | \lambda) \) of elements in \( A \). This is a ring with component-wise multiplication, and the unit is the list with \( 1|\lambda = 1 \) for all \( \lambda \). We consider \( \bigoplus_{\lambda \in \{ n \}} A \) as an \( A \)-algebra mapping \( a \in A \) to the list \( a_{\alpha} \) with \( \alpha_{\lambda} = a \) for all \( \lambda \).

The support of a list \( \alpha \) is the set of \( \lambda \in \{ n \} \) such that \( \alpha | \lambda \neq 0 \).

4.2. Theorem. There is an \( A \)-algebra homomorphism

\[
\sigma^\text{equ} : \text{Fact}^l_A((T | y)^n) \rightarrow \bigoplus_{\lambda \in \{ n \}} A
\]
determined by \( \sigma^\text{equ}(\rho(g)|\lambda) = g(y_{b(\lambda)_1+1}, \ldots, y_{b(\lambda)_l+1}) \) for all \( g \in A[T_1, \ldots, T_l]^\text{sym} \) and all \( \lambda \in \{ n \} \), where \( \rho : A[T_1, \ldots, T_l]^\text{sym} \rightarrow \text{Fact}^l_A((T | y)^n) \) is the residue homomorphism of Section 2.1.

Proof. It is clear that we have an \( A \)-algebra homomorphism \( \sigma : A[T_1, \ldots, T_l]^\text{sym} \rightarrow \bigoplus_{\lambda \in \{ n \}} A \) determined by \( \sigma(g)|\lambda = g(y_{b(\lambda)_1+1}, \ldots, y_{b(\lambda)_l+1}) \). In order to prove the proposition it suffices to show that the kernel of the residue homomorphism \( \rho : A[T_1, \ldots, T_l]^\text{sym} \rightarrow \text{Fact}^l_A((T | y)^n) \) is mapped to zero by \( \sigma \). It follows from the exact sequence \( 0 \rightarrow ((T | y)^n) \rightarrow A[T] \rightarrow A[\xi] \rightarrow 0 \) that the kernel of the homomorphism \( \bigwedge^l \rho((T | y)^n) : \bigwedge^l A[T] \rightarrow \bigwedge^l A[\xi] \) is generated, as an \( A \)-
module, by elements $f_1 \wedge \cdots \wedge f_l$ with $f_i(T) = (T|y)^n$ for some $i$. Hence, since $\sigma^{\text{sym}}$ and $\sigma^{\text{fact}}$ are isomorphisms by 2.1 and Theorem 2.3(2), it follows from the commutative diagram

\[
\begin{array}{ccc}
\bigwedge^l A[T] & \xrightarrow{\sigma^{\text{sym}}} & A[T_1, \ldots, T_l]^{\text{sym}} \\
\bigwedge^l \rho(T|y)^n & \downarrow & \bigwedge^l A[\xi] \\
\bigwedge^l A[\xi] & \xrightarrow{\sigma^{\text{fact}}} & \text{Fact}_A^l((T|y)^n)
\end{array}
\]

that we must show that the image of the elements $\text{Res}(f_1/Q, \ldots, f_l/Q)$ by the homomorphism $\sigma$ are zero when $f_i(T) = (T|y)^n$ for some $i$. When $f_i(T) = (T|y)^n$ the $i$th row in the matrix that defines $\text{Res}(f_1/Q, \ldots, f_l/Q)$ consists of the coefficients of $T^{-1}, T^{-2}, \ldots, T^{-l}$ in the Laurent expansion of the rational function $(T|y)^n/Q(T) = ((T - y_1)(T - y_2)\cdots(T - y_0))/((T - T_1)(T - T_2)\cdots(T - T_l)).$ However, the Laurent series of $((T|y)^n/Q(y_{b(\lambda)}+1,\ldots,y_{b(\lambda)+1}) = ((T - y_1)(T - y_2)\cdots(T - y_n))/((T - y_{b(\lambda)+1}))$ is a polynomial in $T$ since $k \geq b(\lambda_1) \geq \cdots \geq b(\lambda_l) \geq 0$. It follows that

$$\sigma(\text{Res}(f_1/Q, \ldots, f_l/Q)|\lambda = \text{Res}(f_1/Q, \ldots, f_l/Q)(y_{b(\lambda)+1}, \ldots, y_{b(\lambda)+1}) = 0$$

when $f_i(T) = (T|y)^n$ for some $i$, as we wanted to prove. \qed

**4.3. Definition.** An element $\alpha \in \bigoplus_{\lambda \in [n]} A$ is called a class if it satisfies the GKM (Goresky–Kottwitz–MacPherson) condition, that is:

If $\lambda, \lambda'$ in $[n]$ differ in the positions $i$ and $j$ only, the element $\alpha|\lambda - \alpha|\lambda'$ is divisible by $y_i - y_j$.

A class $\alpha \in \bigoplus_{\lambda \in [n]} A$ is a Schubert class corresponding to $\lambda \in [n]$ if it satisfies the following three conditions:

1. $\alpha|\lambda = \prod_{(i,j) \in \text{inv}(\lambda)} (y_j - y_i)$.
2. If $\alpha|\lambda' \neq 0$ then $\lambda' \geq \lambda$.
3. For all $\mu \in [n]$ the element $\alpha|\mu$ is homogeneous of degree $l(\lambda)$ in $y_1, \ldots, y_n$.

**4.4. Proposition.** The $A$-algebra of classes in $\bigoplus_{\lambda \in [n]} A$ is the image of the algebra $\text{Fact}_A^l((T|y)^n)$ by the homomorphism $\sigma^{\text{equ}} : \text{Fact}_A^l((T|y)^n) \rightarrow \bigoplus_{\lambda \in [n]} A$.

**Proof.** It follows from Theorem 3.5(1) that the elements in the image of $\sigma^{\text{equ}}$ satisfy the GKM condition.

Conversely, let $\alpha$ be a class. It follows from [16] Proposition 3.3 that $\alpha$ is a linear combination, with coefficients in $A$, of the images of the elements $s_b(T_1, \ldots, T_l|y)$ by the composite homomorphism $A[T_1, \ldots, T_l]^{\text{sym}} \xrightarrow{\rho} \text{Fact}_A^l((T|y)^n) \xrightarrow{\sigma^{\text{equ}}} \bigoplus_{\lambda \in [n]} A$. Hence $\alpha$ is in the image of $\sigma^{\text{equ}}$. \qed

The correspondence between the equivariant Schubert calculus of Section 3 and the formalism of Knutson and Tao [13] is given by the following result and its corollary.

**4.5. Theorem.** For $\lambda \in [n]$ we let $\tilde{S}_\lambda := \sigma^{\text{equ}} \rho(s_{b(\lambda)}(T_1, \ldots, T_l|y))$.
(1) The homomorphism $\sigma^{\text{equiv}} : \text{Fact}^l_A((T|y)^n) \rightarrow \bigoplus_{\lambda \in \{n\}^l} A$ induces an isomorphism between the $A$-algebra $\text{Fact}^l_A((T|y)^n)$ and the $A$-algebra of classes.

(2) $\tilde{S}_\lambda$ is the unique Schubert class belonging to $\lambda$.

(3) The Schubert classes $\tilde{S}_\lambda$ for $\lambda \in \{n\}^l$ form a basis for the $A$-module of classes.

**Proof.** It follows from Proposition 4.4 that the image of $\sigma^{\text{equiv}}$ is the $A$-algebra of classes. To prove assertion (1) it therefore remains to prove that $\sigma^{\text{equiv}}$ is injective.

From Theorem 2.3(2) it follows that every element $f$ in $\text{Fact}^l_A((T|y)^n)$ can be written as $f = \sum_{h \in \mathcal{I}} Y_h s_b(\xi_1, \ldots, \xi_l|y)$ with $Y_h \in A$ and $\mathcal{I} \subseteq \{l\}^n_\mathcal{P}$. Let $g = \sum_{h \in \mathcal{I}} Y_h s_b(T_1, \ldots, T_l|y)$, and let $\rho : A[T_1, \ldots, T_l]^{\text{sym}} \rightarrow \text{Fact}^l_A((T|y)^n)$ be the residue homomorphism. Then $\rho(g) = f$ and thus $\sigma^{\text{equiv}}(f) = 0$ if and only if $\sigma^{\text{equiv}}(\rho(g))|\lambda(b) = g(y_b(\mu_1+l-1), \ldots, y_b(\mu_l+1)) = 0$ for all $\lambda \in \{n\}^l$. However, from Theorem 3.5(2) we obtain that the condition $g(y_b(\mu_1+l-1), \ldots, y_b(\mu_l+1)) = 0$ for all $\lambda \in \{n\}^l$ implies that $Y_h = 0$ for all $h \in \mathcal{I}$ and thus that $f = 0$. Consequently $\sigma^{\text{equiv}}$ is injective.

To prove assertion (2). We note that the class $\tilde{S}_\lambda = \sigma^{\text{equiv}}(s_b(\lambda)(T_1, \ldots, T_l|y))$ is a Schubert class corresponding to $\lambda$ because the conditions (1) and (2) of Definition 4.3 are satisfied by Theorem 3.4(1) and (2), and it is clear that the polynomials $\tilde{S}_\lambda|\mu = s_b(\lambda)(y_{b(\mu_1+l-1)}, \ldots, y_{b(\mu_l+1)})$ are homogeneous of degree $l(\lambda)$ in $y_1, \ldots, y_n$. That $\tilde{S}_\lambda$ is the unique Schubert class belonging to $\lambda$ follows from assertion (1) and Theorem 3.5(3).

Assertion (3) follows from assertion (1) and Theorem 2.3(2). □

**4.6. Corollary.** When we consider the classes in $\bigoplus_{\lambda \in \{n\}^l} A$ as a $\text{Fact}^l_A((T|y)^n)$-module via $\sigma^{\text{equiv}}$ we obtain an isomorphism of $\text{Fact}^l_A((T|y)^n)$-modules

$$\bigwedge^l A[\xi] \rightarrow \text{classes}$$

determined by mapping $(\xi|y)^{b_1+\cdots+i-1} \wedge \cdots \wedge (\xi|y)^{b_l} = s_b(\xi_1, \ldots, \xi_l|y)((\xi|y)^{i-1} \wedge \cdots \wedge (\xi|y)^{0})$ to $\tilde{S}_{\lambda(b)}$, for each partition $b: b_1 \geq \cdots \geq b_l$ in $\{n\}^l_\mathcal{P}$.

**Proof.** We define the homomorphism as the composite map of the isomorphism $\text{Fact}^l_A((T|y)^n) \rightarrow \bigoplus A[\xi] \rightarrow \text{classes}$ induced by $\sigma^{\text{equiv}} : \text{Fact}^l_A((T|y)^n) \rightarrow \bigoplus_{\lambda \in \{n\}^l} A$ and the isomorphism $\sigma^{\text{fact}} : \bigwedge^l A[\xi] \rightarrow \text{Fact}^l_A((T|y)^n)$. □

**4.7. Remark.** As we observed in [16] the proofs of Theorems 3.4 and 3.5 are strongly influenced by the methods used by Knutson and Tao [13]. We could therefore have proceeded in the opposite direction and used the results of [13] to prove parts of Theorem 4.5. It is however more in the spirit of our points of view to put the emphasis on factorial Schur functions and the general form of Schubert calculus.

5. Pieri’s formula

The equivariant Pieri formula is given by the multiplication on $\bigwedge^l A[\xi]$ of the elements in $\text{Fact}^l_A((T|y)^n)$ corresponding to Segre classes or Chern classes. The explicit expression for this multiplication is given by the definitions of Section 2 and 3. The cancellations that put the expressions in their usual form is performed in [16, Section 5] (see also [8] and [26]). It gives
the following result with $s_i = s_i(T_1, \ldots, T_l)$ denoting the $i$th complete symmetric function in $T_1, \ldots, T_l$ for $i = 0, 1, \ldots$:

5.1. Pieri’s formula. Let $h_1 \geq \cdots \geq h_l$ be a partition and let $h$ be a non-negative integer. Then

$$s_h(T_1, \ldots, T_l) \left( (\xi|y)^{h_1} \wedge \cdots \wedge (\xi|y)^{h_l} \right)$$

$$= \sum_{i=0}^{h} \sum_{(j_1, \ldots, j_l) \in J_{h-i}} s_i(y_{h_1+1}, \ldots, y_{h_1+j_1+1}, \ldots, y_{h_l+1}, \ldots, y_{h_l+j_l+1})$$

$$\times \left( (\xi|y)^{h_1+j_1} \wedge \cdots \wedge (\xi|y)^{h_l+j_l} \right),$$

where $J_{h-i}$ is the collection of all $l$-tuples $(j_1, \ldots, j_l)$ such that $j_1 + \cdots + j_l = h - i$ and $j_1 + h_1 \geq h_1 > j_2 + h_2 \geq h_2 > \cdots > j_l + h_l \geq h_l$. We also have

$$s_{h0\cdots0}(T_1, \ldots, T_l|y) \left( (\xi|y)^{h_1} \wedge \cdots \wedge (\xi|y)^{h_l} \right)$$

$$= \sum_{i=0}^{h} \sum_{j=0}^{h} (-1)^j c_j(y_1, \ldots, y_{h+j-1})$$

$$\times s_i(y_{h_1+1}, \ldots, y_{h_1+j_1+1}, \ldots, y_{h_l+1}, \ldots, y_{h_l+j_l+1}) \left( (\xi|y)^{h_1+j_1} \wedge \cdots \wedge (\xi|y)^{h_l+j_l} \right).$$

6. Relations to geometry

We recall in this section the connection between the combinatorial formalism of Knutson and Tao and the geometry of grassmannians. The notation is that of [13]. The connection between the general cohomology of grassmannians and geometry is given in a much more general setting in [17].

6.1. Notation. Let $K$ be a field. For every string $\lambda \in \{1^n\}$ we denote by $K^\lambda$ the corresponding $k$-plane in $K^n$. The Schubert variety corresponding to $\lambda$ in the grassmannian $\text{Grass}_k(K^n)$ of $k$-dimensional sub-vector-spaces of $K^n$ is defined by

$$X_\lambda = \{ V_k \in \text{Grass}_k(K^n) : \dim(V_k \cap F_i) \geq \dim(K^\lambda \cap F_i), \text{ for all } i \in [1, n] \},$$

where $F_i := K^{[n-i,1]}$ is the anti-standard $i$-plane. Denote by $S_\lambda \in A_k(\text{Grass}_k(K^n))$ the Poincaré dual of the class $X_\lambda$. That is, $S_\lambda$ is the class of the Schubert variety

$$\{ V_k \in \text{Grass}_k(K^n) : \dim(V_k \cap F'_i) \geq \dim(K^\lambda \cap F'_i), \text{ for all } i \in [1, n] \},$$

where $F'_i = \bigoplus_{j=1}^{n} K^{[1,0,n-j]}$ is the standard $i$-plane.

Let $a(\lambda)_1 < \cdots < a(\lambda)_k$ be the indices such that $\lambda_{a(\lambda)_i} = 1$. Then $n - a(\lambda)_k + 1 < n - a(\lambda)_{k-1} + 1 < \cdots < n - a(\lambda)_1 + 1$ are the indices $i$ where the sequence of numbers

$$\dim(K^\lambda \cap F_1) \leq \cdots \leq \dim(K^\lambda \cap F_{i-1}) \leq (K^\lambda \cap F_i) \leq \cdots \leq \dim(K^\lambda \cap F_n)$$
In other words, the partitions increase strictly. Hence

\[ X_\lambda = \{ V_k \in \text{Grass}_k(K^n) : \dim(V_k \cap F_{n-a(\lambda)_{k-i+1}}) \geq i, \text{ for all } i \in [1, k] \}. \]

We define \( \Omega(F_{n-a(\lambda)_{k+1}}, \ldots, F_{n-a(\lambda)_{1+1}}) := X_\lambda. \)

In a similar way \( 1 \leq a(\lambda)_1 < \cdots < a(\lambda)_k \leq n \) are the indices \( i \) where the numbers in the sequence

\[ \dim(K^\lambda \cap F_i^\prime) \leq \cdots \leq \dim(K^\lambda \cap F_{i-1}^\prime) \leq \dim(K^\lambda \cap F_i^1) \leq \cdots \leq \dim(K^\lambda \cap F_k^1) \]

increase strictly. Consequently the variety that gives the class \( S_\lambda \) is

\[ \Omega(F_{a(\lambda)_1}, \ldots, F_{a(\lambda)_k}) := \{ V_k \in \text{Grass}_k(K^n) : \dim(V_k \cap F_i^\prime) \geq \dim(K^\lambda \cap F_i^1), \text{ for all } i \in [1, n] \} \]

\[ = \{ V_k \in \text{Grass}_k(K^n) : \dim(V_k \cap F_{a(\lambda)_i}) \geq i, \text{ for all } i \in [1, k] \}. \]

We write \( S_\lambda := \Omega(a(\lambda)_1, \ldots, a(\lambda)_k). \)

6.2. Relation to partitions. With the notation of Section 4 the zeros in \( \lambda : \lambda_1 \lambda_2 \ldots \) are in the places \( b(\lambda)_l + 1 < b(\lambda)_{l-1} + 2 < \cdots < b(\lambda)_1 + l \) and have \( b(\lambda)_l, b(\lambda)_{l-1}, \ldots, b(\lambda)_1 \) ones to their left. Hence

\[ \Omega(F'_{a(\lambda)_1}, \ldots, F'_{a(\lambda)_k}) = \{ V_k \in \text{Grass}_k(K^n) : \dim(V_k \cap F_{b_l+i+1+l-1}) \geq b_{l-i+1}, \text{ for all } i \in [1, n] \} \]

\[ = \{ V_k \in \text{Grass}_k(K^n) : \dim(V_k \cap F_{b_l+l-i}) \geq b_l, \text{ for all } i \in [1, l] \}. \]

Note that \( \dim \Omega(F'_{a(\lambda)_1}, \ldots, F'_{a(\lambda)_k}) = \sum_{i=1}^k (a(\lambda)_i - i) \), and thus,

\[ \text{codim } S_\lambda = \text{codim } \Omega(a(\lambda)_1, \ldots, a(\lambda)_k) = \sum_{i=1}^k l - a(\lambda)_i + i = l(\lambda). \]

We have a partition \( l - a(\lambda)_1 + 1 \geq l - a(\lambda)_2 + 2 \geq \cdots \geq l - a(\lambda)_k + k. \) It is clear that

\[ l - a(\lambda)_i + i = \{ \text{the number of zeros to the right of one number } i \text{ in } \lambda_1 \ldots \lambda_n \}. \]

In particular \( l(\lambda) = \sum_{i=1}^k (l - a(\lambda)_i + i) \). We earlier defined

\[ b_l(\lambda) = \{ \text{the number of ones to the left of zero number } l - i + 1 \text{ in } \lambda_1 \ldots \lambda_n \}. \]

Since zero number \( l - i + 1 \) in \( \lambda_1 \ldots \lambda_n \) has \( l - (l - i + 1) + 1 = i \) zeros to its right, including itself, we have that \( b_l \) is the number of ones that has at least as many as \( i \) zeros to its right. On the other hand there are \( l - a(\lambda)_j + j \) zeros to the right of one number \( j \) in \( \lambda_1 \lambda_2 \ldots \). Hence

\[ b_l(\lambda) = \left| \left\{ j : l - a(\lambda)_j + j \geq i \right\} \right|. \]

In other words, the partitions \( l - a(\lambda)_1 + 1 \geq l - a(\lambda)_2 + 2 \geq \cdots \geq l - a(\lambda)_k + k \) and \( b(\lambda)_1 \geq b(\lambda)_2 \geq \cdots \geq b(\lambda)_l \) are conjugate. In particular \( l(\lambda) = \sum_{i=1}^k (l - a(\lambda)_i + 1) = \sum_{i=1}^l b_l(\lambda). \)
6.3. Equivariant Schubert classes. In [13] the Schubert class $\tilde{S}_\lambda$ is defined to be the equivariant class of $\Omega(a(\lambda)_1, \ldots, a(\lambda)_k)$ and in [17] we show that the class $\Omega(a(\lambda)_1, \ldots, a(\lambda)_k)$ is represented by $T^{b_1} \cdot T^{b_2} \cdots T^{b_l} \neq 0 \wedge \cdots \wedge T^0$ where $s_{b_1, \ldots, b_l} = \det(s_{b_j} + j - i)$ is the Schur function with $s_i$ the $i$th complete symmetric function in $T_1, \ldots, T_l$. The equivariant version is $(T|y)^{b_1} \wedge \cdots \wedge (T|y)^{b_l} = s_b(T_1, \ldots, T_l|y)((T|y)^{l-1} \wedge \cdots \wedge (T|y)^0)$.

7. Cohomology theories and presentations of factorization algebras

This section is included to show the connection between the general Schubert calculus that we developed in Section 2 and the description of various cohomology theories for Grassmann schemes. We show that Fact $^A$ becomes the cohomology ring for various cohomology theories when we vary the polynomial $p$. In [17] we saw how the action of Fact $^A$ on $\wedge^l A[\xi]$ gives a generalization to the Grassmann schemes $\tilde{S}_\lambda(E)$ for arbitrary locally free $O_S$-modules $E$ over a scheme $S$, where a bivariant theory for $S$ exists (see [5] and [4]). Here $p(T)$ corresponds to the Chern polynomial $T^n - c_1(E)T^{n-1} + \cdots + (-1)^{n}c_n(E)$ of $E$.

From the explicit presentation of Fact $^A$ by generators and relations in Proposition 1.3(2) we see that when $p(T) = (T - y_1)(T - y_2) \cdots (T - y_n)$ the $A$-algebra Fact $^A$ is the equivariant quantum cohomology of A. Givental and B. Kim [9,11,12] in the case of a Grassmannian manifold. Here the $y_1, \ldots, y_n$ serve as equivariant parameters.

We shall below give another presentation of Fact $^A$ by generators and relations that shows that when $p(T) = (T - y_1)(T - y_2) \cdots (T - y_n) + (-1)^n q$ we obtain the equivariant quantum cohomology in a form similar to the one in the main result of Mihalcea [21], but in an unshifted version, that is, using only generalized factorial powers $(T|y)^j$ and not their shifted counterparts $(T|\tau^s y)^j$ defined in [21]. Further specialization of the polynomial to $p(T) = T^n + (-1)^n q$ gives the small quantum cohomology of E. Witten [27] (see also A. Bertram [2], or [7], where the Giambelli formula is proved. Bertram also gives a Pieri formula). Moreover, when $p(T) = (T|y)^n$ we obtain, as we have seen in Sections 2 and 4, the equivariant cohomology.

More presentations of factorization algebras are given in [17].

7.1. Notation. Let $a: a_1, a_2, \ldots$ and $c: c_1, c_2, \ldots$ be elements in the ring $A$. For each non-negative number $h$ we write

$$b_h(a : c) = \det \begin{pmatrix} 1 & c_1 & c_2 & \cdots & c_{h-1} & c_h \\ 1 & a_1 & a_2 & \cdots & a_{h-1} & a_h \\ 0 & 1 & a_1 & \cdots & a_{h-2} & a_{h-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_1 \end{pmatrix},$$

and we let $b_h(a) := b_h(a : 0)$ where $0: 0, 0, \ldots$.

7.2. Lemma. We have

$$b_h(a : c) = b_h(a) - c_1 b_{h-1}(a) + \cdots + (-1)^h c_h = \sum_{i=0}^{h} (-1)^i c_i b_{h-i}(a).$$
Proof. The proof consists in developing the matrix whose determinant gives $b_h(a : c)$ along the first row.

7.3. Proposition. We have an equality

$$1 - c_1 T + c_2 T^2 - \cdots = (1 - a_1 T + a_2 T^2 - \cdots)(1 + b_1(a : c)T + b_2(a : c)T^2 + \cdots),$$

and this equality determines the elements $b_h(a : c)$ uniquely.

Proof. Expansion of the matrix whose determinant gives $b_h(a : c)$ along the last column gives $b_h(a : c) = a_1 b_{h-1}(a : c) - a_2 b_{h-2}(a : c) + \cdots + (-1)^{h-1} a_h + (-1)^h c_h$, that is, $c_h = a_h - a_{h-1} b_1(a : c) + \cdots + (-1)^{h-1} a_{h-1} b_1(a : c) + (-1)^h b_h(a : c)$. For $h = 0, 1, \ldots$ we obtain the equation of the proposition.

7.4. Example. With the notation of Proposition 1.3 the equations defining the ideal $\mathfrak{I}$ in the algebra $A[X_1, \ldots, X_l, Y_1, \ldots, Y_k]$ are obtained by setting the coefficients of $T, T^2, \ldots, T^n$ equal in the equation

$$1 - c_1(p) T + \cdots + (-1)^n c_n(p) T^n = (1 - X_1 T + \cdots + (-1)^l X_l T^l)(1 - Y_1 T + \cdots + (-1)^k Y_k T^k).$$

By Proposition 7.3 we obtain equations

$$b_h(X : c(p)) = \det \begin{pmatrix} 1 & c_1(p) & c_2(p) & \cdots & c_{h-1}(p) & c_h(p) \\ 1 & X_1 & X_2 & \cdots & X_{h-1} & X_h \\ 0 & 1 & X_1 & \cdots & X_{h-2} & X_{h-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & X_1 \end{pmatrix},$$

for $h = 0, \ldots, n$ with $X: X_1, X_2, \ldots, X_l, 0, 0, \ldots$ and $c(p): c_1(p), c_2(p), \ldots$, and $Y_h = (-1)^h b_h(X : c(p))$ for $h = 1, \ldots, k$. We use the latter $k$ equations to eliminate the residue classes in $\text{Fact}_A^l(p) = A[X_1, \ldots, X_l, Y_1, \ldots, Y_k]/\mathfrak{J}$ of the variables $Y_1, \ldots, Y_k$. After the elimination the ideal in $A[X_1, \ldots, X_l]$ corresponding to $\mathfrak{J}$ is generated by the elements $b_h(X : c(p))$, for $h = k + 1, \ldots, n$. Thus we have an isomorphism

$$\text{Fact}_A^l(p) \cong A[X_1, \ldots, X_l]/(b_{k+1}(X : c(p)), \ldots, b_n(X : c(p))).$$

Let $\mathfrak{J}_1 = \langle \sum_{i=0}^{k+1} (-1)^i c_i(p)b_{k+1-i}(X), \ldots, \sum_{i=0}^n (-1)^i c_i(p)b_{n-i}(X) \rangle$. We obtain from Lemma 7.2 an isomorphism

$$\text{Fact}_A^l(p) \cong A[X_1, \ldots, X_l]/\mathfrak{J}_1.$$ 

The example is symmetric in $X_1, \ldots, X_l$ and $Y_1, \ldots, Y_k$ so that we obtain a corresponding presentation of $\text{Fact}_A^l(p)$ expressed in the generators $Y_1, \ldots, Y_k$. 


7.5. Example. Let $X_i = c_i(T_1, \ldots, T_l)$ and $Y_i = c_i(T_{l+1}, \ldots, T_n)$ as in Proposition 1.3(2). Then the generators of $J$ are obtained by setting the coefficients of $T, T^2, \ldots, T^n$ equal in the equations

$$T^n p(1/T) = 1 - c_1(p) T + \cdots + (-1)^n c_n(p) T^n = \prod_{i=0}^{l-1} (1 - T T_i) \prod_{i=0}^{k} (1 - T T_{i+1}).$$

Define the sequence $s(p): s_1(p), s_2(p), \ldots$

by

$$\left(1 - c_1(p) T + \cdots + (-1)^n c_n(p) T^n\right) \left(1 + s_1(p) T + s_2(p) T^2 + \cdots\right) = 1.$$

Then the equations of $J$ are obtained by setting the coefficients of $T, T^2, \ldots$ equal in the equation

$$1 + s_1(p) T + s_2(p) T^2 + \cdots = \left(1 + s_1(T_1, \ldots, T_l) T + s_2(T_1, \ldots, T_l) T^2 + \cdots\right) \times \left(1 + s_1(T_{l+1}, \ldots, T_n) T + s_2(T_1, \ldots, T_l) T^2 + \cdots\right),$$

where $s_i(T_1, \ldots, T_l)$ and $s_i(T_{l+1}, \ldots, T_n)$ for $i = 1, 2, \ldots$ are the complete symmetric functions in $T_1, \ldots, T_l$, respectively $T_{l+1}, \ldots, T_n$. Substituting $-T$ for $T$ we obtain from Proposition 7.3 the equation

$$s_h(T_{l+1}, \ldots, T_n) = (-1)^h \det \begin{pmatrix} 1 & s_1(p) & s_2(p) & \cdots & s_{h-1}(p) & s_h(p) \\ s_1(T_1, \ldots, T_l) & 1 & s_2(T_1, \ldots, T_l) & \cdots & s_{h-1}(T_1, \ldots, T_l) & s_h(T_1, \ldots, T_l) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & s_1(T_1, \ldots, T_l) \end{pmatrix}$$

for $h = 0, 1, \ldots$. From Lemma 7.2 we obtain

$$s_h(T_{l+1}, \ldots, T_n) = (-1)^h s_1^h(T_1, \ldots, T_l) + (-1)^{h+1} s_1(p)s_1^{h-1}(T_1, \ldots, T_l) + \cdots + s_h(p)$$

where $s_1^i(T_1, \ldots, T_l)$ is the usual Schur function in the variables $T_1, \ldots, T_l$ obtained from the partition $1^i: 1 \geq 1 \geq \cdots \geq 1$ with $i$ ones. With

$$\mathcal{J}_2 = \left(\sum_{i=0}^{k+1} (-1)^i s_{k+1-i}^i(p)s_i^i(T_1, \ldots, T_l), \ldots, \sum_{i=0}^{n} (-1)^i s_{n-i}^i(p)s_i^i(T_1, \ldots, T_l)\right),$$

this gives, as in Example 7.4, that

$$\text{Fact}_A(p) = A[c_1(T_1, \ldots, T_l) \ldots c_l(T_1, \ldots, T_l)]/\mathcal{J}_2,$$

or, as we saw in Proposition 1.5, that

$$\text{Fact}_A(p) = A[s_1(T_1, \ldots, T_l) \ldots s_l(T_1, \ldots, T_l)]/\mathcal{J}_2.$$
7.6. Example. Let \( p(T) = T^n - c_1(p)T^{n-1} + \cdots + (-1)^n c_n(p) = (T - y_1) \cdots (T - y_n) + (-1)^n q \). Then we have, with the notation of Example 7.5, that \( s_i(p) = s_i(y_1, \ldots, y_n) \) for \( i = 0, 1, \ldots, n - 1 \), and \( s_n(p) = s_n(y_1, \ldots, y_n) + (-1)^{n+1} q \). With

\[
\exists_3 = \left( \sum_{i=0}^{k+1} s_{k+1-i}(y_1, \ldots, y_n)s_1^i(T_1, \ldots, T_l), \ldots, \right.
\]

\[
\sum_{i=0}^{n-1} (-1)^i s_{n-1-i}(y_1, \ldots, y_n)s_1^i(T_1, \ldots, T_l),
\]

\[
\sum_{i=0}^{n} (-1)^i s_{n-i}(y_1, \ldots, y_n)c_i(T_1, \ldots, T_l) + (-1)^{n+1} q
\]

\[
= \left( \sum_{i=0}^{k+1} s_{k+1-i}(y_1, \ldots, y_n)c_i(T_1, \ldots, T_l), \ldots, \right.
\]

\[
\sum_{i=0}^{n-1} (-1)^i s_{n-1-i}(y_1, \ldots, y_n)c_i(T_1, \ldots, T_l),
\]

\[
\sum_{i=0}^{n} (-1)^i s_{n-i}(y_1, \ldots, y_n)c_i(T_1, \ldots, T_l) + (-1)^{n+1} q
\]

we therefore obtain from Example 7.5 that

\[
\text{Fact}_A(p) = A[s_1(T_1, \ldots, T_l), \ldots, s_l(T_1, \ldots, T_l)] / \exists_3.
\]

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