Optimal shape design for a frictionless contact problem

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Abstract

In this paper we develop the material derivative method for the optimal shape design of contact problems described by variational inequalities. Since the method can be used for solving the optimal shape problem for systems described by partial differential equations, here we shall use it to solve it for differential inequalities by taking limits of the equations resulting from a penalized approximation. The computations are done by the finite-element method; the gradient of the criteria as a function of coordinates moving nodes is computed, and the performance criterion is then minimized by a material derivative (or speed) method (Zolesio (1981)).

Keywords: Optimal shapes, contact problem, convexity, velocity method, finite-element method, approximation, Sobolev spaces, variational formulation.

1. Introduction

Many problems in mechanical system design involve elastic bodies that come into contact under applied load. Contact problems are nonclassical, in the sense that one does not initially know the contact region or the contact stress. Considerable research has been pursued in recent years to develop constructive methods of determining the contact region and contact stress distribution when two bodies come into contact, a force distribution over the surfaces arises and high contact stresses may occur over the subsets of the contact region. This is undesirable, since plastic deformation of the bodies may occur or high normal forces may lead
to wear of machine parts that move relative to each other. A technique is needed for adjusting
the contour of one or both of the bodies, in order to achieve a minimum contact stress between
them.

In our problem, we consider the linear elastic bodies in frictionless contact, the nonpenetration
condition leading to a unilateral constraint on the displacement field. Consider a smooth
domain $\Omega$ in $\mathbb{R}^n$, $n \geq 2$, with boundary $\Gamma$. From $\Omega$ and given data one constructs a solution
$\varphi_\Omega$, the state function, of the state equation $F(\Omega, \varphi_\Omega)$; from $\Omega$ to $\varphi_\Omega$ one constructs a cost
function $E(\Omega)$; and finally $\Theta$ given a family of domains $\Omega$, one wishes to minimize $E(\Omega)$ over
$\Theta$.

Indeed, consider a membrane in possible contact with a rigid obstacle. The system is
governed by a variational inequality with the constraint on admissible displacements

$$\phi \geq \psi,$$

almost everywhere in $\Omega$,

where $\phi$ is the normal displacement of the membrane and $\psi$ is a given function which
describes the shape of the obstacle. For any solution of the analysis problem, we may define the
contact region $\mathcal{Z}$ as the subdomain of $\Omega$ where $\phi = \psi$. Since $S$, the boundary of this
subdomain, is unknown before the contact problem is solved, it is called a free boundary.

Consider the Sobolev space

$$H_0^1(\Omega) = \{ v | v \in H^1(\Omega), v = 0 \text{ on } \Gamma \}. \quad (1.1)$$

Define the inner products $(\cdot, \cdot)$ and $a(\cdot, \cdot)$ on $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively, by

$$(u, v) = \int_\Omega uv \, dx, \quad \forall u, v \in L^2(\Omega),$$

and

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad \forall u, v \in H_0^1(\Omega),$$

with the associated norms being denoted by $\| f \|^2 = (f, f)$ and $\| u \|^2 = a(u, u)$. We note that
the bilinear form $a(\cdot, \cdot)$ is elliptic, i.e.,

$$a(v, v) \geq \alpha \| v \|^2, \quad \alpha > 0, \quad \forall v \in H_0^1(\Omega). \quad (1.2)$$

Let us define the closed convex subset $K$:

$$K = \{ v | v \in H_0^1(\Omega), v \geq \psi, \psi \text{ given in } \Omega \}. \quad (1.3)$$

Let us assume that $\psi \in H^2(\Omega)$, $\psi \leq 0$ on $\Gamma$ and $A\psi \leq 0$. It is known [3] that under these
conditions the set $K$ is nonempty. The problem we want to consider consists in finding $\phi$ so that

$$\phi \in K, \quad A\phi - f \geq 0, \quad \phi - \psi \geq 0, \quad (A\phi - f)(\phi - \psi) = 0, \quad \text{in } \Omega, \quad (1.4)$$

and

$$\phi = 0, \quad \text{on } \Gamma, \quad \phi = \psi, \quad \text{on } S, \quad \frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial n}, \quad \text{on } S. \quad (1.5)$$
Fig. 1. Physical set-up of the problem, here $Z$ is the contact region containing the subdomain $\Omega_0$ of domain $\Omega$, $S$ is the free boundary and $\Gamma$ is boundary of the domain.

Consequently, there are two sets in $\Omega$:

$$ I = \{ x \mid x \in \Omega, \phi(x) = \psi(x) \}, \quad (1.6) $$

called the \textit{coincidence set}, and

$$ \Omega - I = \{ x \mid x \in \Omega, \phi(x) > \psi(x) \}, \quad (1.7) $$

called the \textit{equilibrium set}. In the two-dimensional case, one can think of this problem therefore as giving the displacement $\phi$ of a membrane subjected to forces $f$ and required to stay above obstacle $\psi$. The membrane touches the obstacle on the coincidence set and the two regions are separated by a surface $S$, which is a free surface, and on which one has two boundary conditions: if $\psi \in H^1(\Omega)$, one has

$$ \phi = \psi \quad \text{and} \quad \frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial n}, \quad \text{on } S. \quad (1.8) $$

Our optimization problem consists of the minimization of the area of $\Omega$ within the constraint that the contact region contains some specified subdomain $\Omega_0$ of $\Omega$. The physical set-up is depicted schematically in Fig. 1. That is, our problem consists of

$$ \begin{align*}
\text{minimize} \quad & |\Omega|, \\
\text{subject to} \quad & Z \supset \Omega_0,
\end{align*} \quad (1.9) $$

where $Z$ is the contact region containing the subdomain $\Omega_0$ of $\Omega$.

To enforce the constraint we introduced a penalty functional, so that, for $\alpha > 0$ the cost functional becomes

$$ E(\Omega) = |\Omega| - \alpha \int_{\Omega_0} (\phi(\Omega) - \psi) \, dx, \quad (1.10) $$

where $\alpha$ is penalty multiplier, $\psi$ defines the shape of the obstacle and $\phi$, as the displacement of the membrane, is the solution of the variational inequality (1.4).

We may express the cost function (1.10) as follows:

$$ E(\Omega) = \int_{\Omega} dx + \alpha \int_{\Omega_0} R_0(\phi(\Omega) - \psi) \, dx, \quad (1.11) $$
with
\[ R_0 = \begin{cases} 1, & \text{on } \Omega_0, \\ 0, & \text{out of } \Omega_0. \end{cases} \]  
(1.12)

also note that we choose \( \Omega_0 \) so that \( \overline{\Omega}_0 \subset \Omega \), but \( \overline{\Omega}_0 \neq \Omega \), because \( \Omega \) does not intersect the subdomain \( \Omega_0 \).

Since we already know that the speed method can be used for solving optimal shape problems for systems described by differential equations [2,6], to solve the optimal shape problem for the systems described by the differential inequality (9.4) we shall introduce a penalized differential equation as follows:
\[ A(\phi_\varepsilon - \psi) + \frac{1}{\varepsilon} (\phi_\varepsilon - \psi) = f, \quad \phi_\varepsilon \in H_0^1(\Omega), \]  
(1.13)

where \( u^- = -\sup(-u, 0) \) and \( A : U = H_0^1 \rightarrow U' \) is linear continuous and a symmetric operator satisfying the coercivity condition; i.e., \((A\phi, \phi) = a(\phi, \phi), A = -\nabla \cdot \nabla \), whose solution \( \phi \) tends to the solution of (1.4) as \( \varepsilon \rightarrow 0 \), i.e., \( \phi \in K \). To verify this, let us write (1.13) in the form
\[ A(\phi_\varepsilon - \psi) + \frac{1}{\varepsilon} (\phi_\varepsilon - \psi) + A\psi = f, \quad \phi_\varepsilon \in H_0^1(\Omega). \]

By taking the scalar product with \((\phi_\varepsilon - \psi)\) in the above equation, assume that \( \psi \in H^2(\Omega), \psi \equiv 0 \) on \( \Gamma \) and \( A\psi \equiv 0 \); also \( a(\psi, \psi^-) = -a(\psi, \psi^-), (\psi, \phi^-) = (\phi^-, \phi^-) \), we obtain
\[ a(\phi_\varepsilon - \psi, \phi_\varepsilon - \psi) + \frac{1}{\varepsilon} \| (\phi_\varepsilon - \psi)^- \|^2 - (f, \phi_\varepsilon - \psi) = (A\psi, \psi - \phi_\varepsilon), \]

since \((A\psi, \psi - \phi_\varepsilon) \leq 0\); then
\[ a(\phi_\varepsilon - \psi, \phi_\varepsilon - \psi) - (f, \phi_\varepsilon - \psi) \leq 0, \quad (\because \| (\phi_\varepsilon - \psi)^- \|^2 \geq 0). \]

Then, by using the elliptic hypothesis, we have
\[ \alpha \| \phi_\varepsilon - \psi \|^2 \leq \| f \| \| \phi_\varepsilon - \psi \| \quad \text{or} \quad \alpha \| \phi_\varepsilon - \psi \| \leq C, \quad (\because \| f \| = C), \]

where \( C \) is constant, and independent of \( \varepsilon \). Also \( \frac{1}{\varepsilon} \| (\phi_\varepsilon - \psi)^- \|^2 \leq C \), but, by taking \( \varepsilon \rightarrow 0 \), the above equation shows that \((\phi - \psi)^- \rightarrow 0 \) in \( L^2(\Omega) \), so that \((\phi - \psi)^- = 0 \) or \( \phi \geq \psi \), which implies that \( \phi \in K \), i.e., \( \phi \geq \psi \). For the existence and uniqueness of the solution of (1.13) see [5]. We shall solve the optimal shape problem for the differential equation (1.13), and then take limits \( \varepsilon \rightarrow 0 \) of the relevant functions, so as to solve the problem associated with differential inequalities.

2. Deformations of domains and optimization

The speed method of shape sensitivity analysis [6] is used here for solving the optimal shape problem for our differential inequality (1.4) by taking limits, as \( \varepsilon \) tends to zero, of the equations resulting from the penalized approximation introduced in (1.13), as explained above. Here we shall explain how this method works for the system described by the differential equation, which will be helpful in solving our problem, i.e., the optimal shape problem for the systems described by a differential inequality.
The main idea of the speed method is to make the derivative of the cost function as negative as possible, by selecting a suitable value of a vector field $V$ to be defined below. This vector field is used for the perturbation of the domain $\Omega$ into $\Omega_t$ at time $t$ (see Fig. 2), and the value of the derivative of the cost function depends only on the value of this vector field at the boundary. To minimize the cost function $E(\Omega)$, we take the vector field $V$, at $t = 0$, in the opposite direction to a vector $G$, the gradient of the cost function. By using this method, various formulae have been obtained [2,6] for the derivative with respect to shape.

Let $\Omega$ be a smooth bounded open set in $\mathbb{R}^n$, $n \geq 2$, and $V$ be a regular, $n$-dimensional vector field defined on $[0, 1] \times U_\epsilon$, where $U_\epsilon$ is an open neighbourhood of $\Omega$. Suppose that the mapping $x \rightarrow V(t, x)$ has continuous space derivatives for any $t \in [0, 1]$, and the mapping $t \rightarrow V(t, x)$ is continuous for the topology given by uniform convergence of these derivatives on any compact subset of $U_\epsilon$. In this approach the deformation of the set $\Omega$ is related to the vector field $V$; the position of a point $x$ in the deformed domain $\Omega_t$ is given by the solution of the ordinary differential equation

$$\frac{dx(t)}{dt} = V(t, x(t)),$$

with the initial condition $x(0) = X \in \Omega$. Let $F_t$ be the transformation, depending on $V$, which is defined by the differential equation (2.1); then

$$X \rightarrow x = x(t, X) \equiv F_t(V)X,$$

since $X$ does not depend on $t$. The domain $\Omega_t$ and its boundary $\Gamma_t$ can now be defined as

$$\Omega_t = F_t(V)(\Omega) = \{x \in \mathbb{R}^n: \text{there exists } X \in \Omega \text{ such that } x = x(t),$$

$$\text{with } \dot{x}(s) = V(x(s), s), 0 < s < t, x(0) = X\},$$

$$\Gamma_t = F_t(V)(\Gamma); \text{ of course, } \Omega_0 = \Omega \text{ and } \Gamma_0 = \Gamma.$$

The cost function can therefore be considered as a function of $\Omega_t$ as

$$E(\Omega_t) = \int_{\Omega_t} \phi \, dx + \alpha \int_{\Omega_t} R_0(\phi_{x,t} - \psi) \, dx,$$

where $\phi_{x,t}$ is the solution of the differential equation (1.13) on $\Omega_t$. It can also be considered as a function of $t$, by the mapping

$$t \rightarrow \Omega_t \rightarrow \phi_{x,t} \rightarrow E(\Omega_t).$$

We wish to define the derivative $\dot{E}(\Omega_t)$ of the cost function $E(\Omega_t)$ at $\Omega_t$, which depends on the vector field $V$; this will be chosen so as to obtain the value of the derivative to be as negative as
possible. Let \( \eta \) be a smooth function of \( x \), which may depend also smoothly on \( t \). Then we know from [6] that

\[
\frac{d}{dt} \int_{\Omega_t} \eta(x, t) \, dx = \int_{\Omega_t} \frac{\partial \eta}{\partial t} \, dx + \int_{\Gamma_t} \eta(x, t) \langle V, n \rangle \, d\Gamma,
\]

(2.6)

where \( n \) is the unit exterior normal to \( \Gamma_t \), \( V \) is the speed, and \( \langle \cdot, \cdot \rangle \) is the scalar product in \( \mathbb{R}^n \).

Now we again return to our main problem, and write (1.13) in variational form:

\[
\int_{\Omega_t} \left( \nabla \phi_{\varepsilon,t} \cdot \nabla \omega + \frac{1}{\varepsilon} (\phi_{\varepsilon,t} - \psi)^- \omega - f \omega \right) \, dx = 0, \quad \omega \in H^1(\Omega_t);
\]

(2.7)

the unknown is denoted by \( \phi_{\varepsilon,t} \in H^1(\Omega_t) \). In order to eliminate certain problems related to the definition of \( \omega \) in \( \Omega_t \), which is variable with \( t \), we suppose that \( \omega \) is the restriction to \( \Omega_t \) of a function \( \omega \in H^1(\mathbb{R}^n) \).

We denote by \( \phi_{\varepsilon,t}' \) the partial derivative of \( \phi_{\varepsilon,t} \) with respect to \( t \); one can in fact prove [6] that \( \phi_{\varepsilon,t} \) exists and is in \( H^1 \). Taking the derivative of the left-hand side of (2.7), as (2.6), we obtain:

\[
\int_{\Omega_t} \left( \nabla \phi_{\varepsilon,t}' \cdot \nabla \omega + \frac{1}{\varepsilon} \frac{d}{d\phi} (\phi_{\varepsilon,t} - \psi)^- \phi_{\varepsilon,t} \omega \right) \, dx
\]

\[
= \int_{\Gamma_t} \left( f \omega - \nabla \phi_{\varepsilon,t} \cdot \nabla \omega - \frac{1}{\varepsilon} (\phi_{\varepsilon,t} - \psi)^- \omega \right) \langle V, n \rangle \, d\Gamma, \quad \omega \in H^1(\Omega_t).
\]

(2.8)

In a similar way, the derivative of the cost function \( E \) is

\[
\dot{E}(\Omega_t) = \int_{\Gamma_t} \langle V, n \rangle \, d\Gamma + \alpha \int_{\Omega_t} R_0 \phi_{\varepsilon,t}' \, dx + \alpha \int_{\Gamma_t} R_0 (\phi_{\varepsilon,t} - \psi) \langle V, n \rangle \, d\Gamma.
\]

(2.9)

Since we have chosen \( \Omega_0 \) so that \( \overline{\Omega}_0 \subset \Omega_{t=0} \), but \( \overline{\Omega}_0 \neq \Omega_{t=0} \), \( \Gamma_t \subset \Omega_{t=0} \setminus \Omega_0 \), and since the value of \( R_0 \) is zero outside of \( \Omega_0 \), the value of \( R_0 \) is zero on \( \Gamma_t \). We can write this equation as

\[
\dot{E}(\Omega_t) = \int_{\Gamma_t} \langle V, n \rangle \, d\Gamma + \alpha \int_{\Omega_t} R_0 \phi_{\varepsilon,t}' \, dx.
\]

(2.10)

We introduce now the adjoint state \( P_{\varepsilon,t} \in H^1(\Omega_t) \), defined as the solution of the variational equation

\[
\int_{\Omega_t} (\nabla P_{\varepsilon,t} \cdot \nabla \omega + F'(\phi_{\varepsilon,t}) P_{\varepsilon,t} \omega - \alpha R_0 \omega) \, dx = 0,
\]

(2.11)

where \( F' (\phi_{\varepsilon,t}) = 1/\varepsilon (d/d\phi (\phi_{\varepsilon,t} - \psi)^-) \), for all \( \omega \in H^1(\Omega_t) \).

We note that the function \( \phi_{\varepsilon,t} \rightarrow \phi_{\varepsilon,t}' \) is not differentiable at \( \phi_{\varepsilon,t} = 0 \), we have defined \( F'(0) = 0 \). This choice turns out to be unimportant because \( \phi_{\varepsilon,t} > 0 \), on \( \Omega_t \), with the exception of a (zero-measure) subset of \( \Gamma_t \); for more details, see [1], where an approximation scheme is introduced for proving this. Choosing \( \omega = P_{\varepsilon,t} \) in (2.8) and \( \omega = \phi_{\varepsilon,t} \) in (2.11), we obtain

\[
\alpha \int_{\Omega_t} R_0 \phi_{\varepsilon,t}' \, dx = - \int_{\Gamma_t} (\nabla P_{\varepsilon,t} \cdot \nabla \phi_{\varepsilon,t}) \langle V, n \rangle \, d\Gamma - \frac{1}{\varepsilon} \int_{\Gamma_t} ((\phi_{\varepsilon,t} - \psi)^- P_{\varepsilon,t}) \langle V, n \rangle \, d\Gamma
\]

\[
+ \int_{\Gamma_t} f P_{\varepsilon,t} \langle V, n \rangle \, d\Gamma.
\]
Putting this value in (2.10), we obtain:

$$
\dot{E}(\Omega_t) = \int_{\Gamma_r} \left( 1 - \nabla P_{e,t} \cdot \nabla \phi_{e,t} - \frac{1}{\epsilon} (\phi_{e,t} - \psi) P_{e,t} + fP_{e,t} \right) \langle V, n \rangle \, d\Gamma
$$

or

$$
\dot{E}(\Omega_t) = \int_{\Gamma_r} (1 - \nabla P_{e,t} \cdot \nabla \phi_{e,t}) \langle V, n \rangle \, d\Gamma, 
$$

(2.12)
since the last two terms of the above equation become zero since the vector $P_{e,t}$ is zero on $\Gamma_r$. So the above expression is linear in $V$ and the gradient can be explicitly computed. We can also write this equation in the form

$$
\dot{E}(\Omega_t) = \int_{\Gamma_r} C_{e,t} \langle V, n \rangle \, d\Gamma, 
$$

(2.13)
where

$$
C_{e,t} = (1 - \nabla P_{e,t} \cdot \nabla \phi_{e,t}),
$$

(2.14)
and $n$ is the normal field on $\Gamma_r$ ($n$ is taken going out of $\Omega_t$). We shall choose $t = 0$ throughout, and write (2.13) as $\dot{E}(\Omega_t) = \langle G, V \rangle_{L^2}$; of course, $G = (C_{e,t} = 0, n)$, the gradient of $E$ at $t = 0$; it is a distribution with support on the boundary $\Gamma_r$. Several different treatments of the gradient have been developed; see [2,6]. The derivative of the cost function $E$ depends only on the value of the vector field $V$ at the boundary $\Gamma_r$; we can then choose the value of $V$ so as to make the derivative of the cost function as negative as possible. The corresponding value of the vector field $V$, to be considered unitary, i.e., $\|V\|_{L^2} = 1$, as explained below, is of course determined by the following relations:

$$
|\langle G, V \rangle| \leq \|G\| \|V\| = \|G\|; 
$$

(2.15)
the value of $V$ must be

$$
V = -\frac{G}{\|G\|}. 
$$

(2.16)
Note that the choice $\|V\|_{L^2} = 1$ is justified, since $V$ is simply a direction of optimum descent; eventually we shall consider changes of the type $\rho V$, $\rho > 0$.

3. Discretization and a first optimization algorithm

We briefly review the method of finite elements. To illustrate the method, let (1.13) be discretized by triangulation elements of degree 1. In variational form (1.13) becomes for all $\phi_{e,t} \in H^1(\Omega_t)$:

$$
\int_{\Omega_t} (\nabla \phi_{e,t} \cdot \nabla \omega + F(\phi_{e,t}) \omega - f \omega) \, dx = 0, \quad \omega \in H^1(\Omega_t),
$$

(3.1)
where $F(\phi_{e,t}) = 1/\epsilon (\phi_{e,t} - \psi)$.
Let $\tau_h$ be the triangulation of $\Omega_t$ and $T_k$ is called the triangle, $\bigcup T_k = \Omega_t$. The parameter $h$ is the size of the largest side or edge, and we assume that we have a family of triangulations of $\tau_h$. Let $P^m$ be the space of polynomials of degree $m$ on $\Omega_h$, and denote by

$$H^m_h(\Omega_h) = \{ \omega_h \in C^0(\Omega_h) : \omega_h \big|_{T_k} \in P^m, \forall T_k \in \tau_h \}$$

(3.2)

the space of continuous piecewise polynomial functions on $\Omega_h$. We know that $H^m_h$ is of finite dimension; then

$$\int_{\Omega_h} \left( \nabla \phi_{h,e,t} \cdot \nabla \omega_h + \frac{1}{\varepsilon} (\phi_{h,e,t} - \psi)^{-1} \omega_h - f \omega_h \right) \, dx = 0. \tag{3.3}$$

If $\{ \omega_i \}_{i=1}^N$ is a basis for $H^m_h$, (3.3) is equivalent to

$$\hat{\Phi} = F, \tag{3.4}$$

where

$$\hat{\Phi}_{ij} = \int_{\Omega_h} \left( \nabla \omega_i \cdot \nabla \omega_j + \frac{1}{\varepsilon} (\phi_e - \psi)^{-1} \omega_j \right) \, dx, \quad F_i = \int_{\Omega_h} f \omega_i \, dx,$$

$$\phi_{h,e} = \sum_{i=1}^N \phi_i \omega_i. \tag{3.5}$$

The $\{ \omega_i \}$ are polynomials of degree $\leq m$ on $T_k$, so $\hat{\Phi}_{ij}$ can be computed exactly. In the case $m = 1$ (conforming finite-element method of degree 1), if $\{ \omega_i \}_{i=1}^N$ denotes the vertices of $\tau_h$, $\{ \omega_i \}$ are uniquely determined by

$$\omega_i(q) = \delta_{ij}, \quad \forall i, j = 1, \ldots, N.$$

It is possible to consider our optimization problem in this new setting. The optimal shape will be found by successive approximations starting with an initial guess $\Omega^0_h$, the algorithm is then developed by means of a material derivative method. We note that the problem has been discretized, so that the shape $\Omega_h$ is defined by the coordinates of the nodes; then, the expression for the cost function $E$ is

$$E(\Omega_h) = \int_{\Omega_h} dx + \alpha \int_{\Omega_h} R(\phi_{h,e,t} - \psi) \, dx, \tag{3.6}$$

where $\phi_{h,e,t}$ is a solution of the differential equation (1.13) on $\Omega_h$. Now we find the derivative $\dot{E}(\Omega_h)$ of the cost function $E(\Omega_h)$ at $\Omega_h$. In variational form,

$$\int_{\Omega_h} \left( \nabla \phi_{h,e,t} \cdot \nabla \omega_h + \frac{1}{\varepsilon} (\phi_{h,e,t} - \psi)^{-1} \omega_h \right) \, dx = \int_{\Omega_h} f \omega_h \, dx, \tag{3.7}$$

for all $\omega_h \in H^1(\Omega_h)$. Taking the derivative of (3.7) with respect to $t$, we get

$$\int_{\Omega_h} \left( \nabla \phi_{h,e,t} \cdot \nabla \omega_h + \frac{1}{\varepsilon} \frac{d}{d\phi} (\phi_{h,e,t} - \psi)^{-1} \phi_{h,e,t} \omega_h \right) \, dx$$

$$= - \int_{\Gamma_h} \left( \nabla \phi_{h,e,t} \cdot \nabla \omega_h + \frac{1}{\varepsilon} (\phi_{h,e,t} - \psi)^{-1} \omega_h - f \omega_h \right) \langle V, n \rangle \, d\Gamma. \tag{3.8}$$
Then,

$$\hat{E}(\Omega_{h,t}) = \int_{\Omega_{h,t}} \langle V, n \rangle \, d\Gamma + \alpha \int_{\Omega_{h,t}} R_0 \phi_{h,\varepsilon,t} \, dx,$$

(3.9)

because $R_0$ is zero on $\Gamma_{h,t}$. Now we shall find the value of the second integral in (3.9). We define the adjoint state $P_{h,\varepsilon,t} \in H^1_0(\Omega_{h,t})$; to that end we introduce our second differential equation as follows:

$$AP_{h,\varepsilon,t} + \frac{1}{\varepsilon} \frac{d}{d\phi} (\phi_{h,\varepsilon,t} - \psi)^{-1} P_{h,\varepsilon,t} = \chi'$$

(3.10)

where $\chi' = \alpha R_0$, and $R_0$ is given by (1.12) and $A = -\nabla \cdot \nabla$; $P_{h,\varepsilon,t} = 0$ on $S_0 \cup \Gamma_t$ ($S_t$ is the free boundary at $t$); the state $P_{h,\varepsilon,t}$ is needed to compute the gradient of the cost function $E(\Omega_{h,t})$. The variational form of (3.10) is

$$\int_{\Omega_{h,t}} \left( \nabla P_{h,\varepsilon,t} \cdot \nabla \omega_h + \frac{1}{\varepsilon} \frac{d}{d\phi} (\phi_{h,\varepsilon,t} - \psi)^{-1} P_{h,\varepsilon,t} \omega_h \right) \, dx$$

$$= \alpha \int_{\Omega_{h,t}} R_0 \omega_h \, dx, \quad \text{for all } \omega_h \in H^1(\Omega_{h,t}).$$

(3.11)

As in the previous section, we can compute

$$\hat{E}(\Omega_{h,t}) = \int_{\Gamma_{h,t}} (1 - \nabla P_{h,\varepsilon,t} \cdot \nabla \phi_{h,\varepsilon,t}) \langle V, n \rangle \, d\Gamma.$$

(3.12)

We could now obtain, as in the previous section, the vector field $V$ so as to minimize this derivative. We consider now the problem associated with the variational inequality.

4. The optimal shape design for a variational inequality

Now we come to the implementation of the main idea of our treatment, that is, to take the limit of these quantities as $\varepsilon$ tends to zero. First we shall find the value of the limit of the cost function, as $\varepsilon$ tends to zero. Since we know that [4]

$$\phi_{h,e,t} \to \phi_{h,t}, \quad \text{in } H^1 \text{ weakly, as } \varepsilon \to 0,$$

and also

$$\phi_{h,e,t} \to \phi_{h,t}, \quad \text{in } L^2(\Omega_{h,t}) \text{ strongly, as } \varepsilon \to 0,$$

by taking the limit (as $\varepsilon \to 0$), on both sides of (3.6), we obtain

$$\lim_{\varepsilon \to 0} \int_{\Omega_{h,t}} R_0 (\phi_{h,\varepsilon,t} - \psi) \, dx, \quad \text{as } \phi_{h,e,t} \to \phi_{h,t} \text{ in } L^2(\Omega_{h,t}) \text{ strongly, so}$$

$$\alpha \lim_{\varepsilon \to 0} \int_{\Omega_{h,t}} R_0 (\phi_{h,\varepsilon,t} - \psi) \, dx \to \alpha \int_{\Omega_{h,t}} R_0 (\phi_{h,t} - \psi) \, dx.$$
since the functional
\[ \phi_{h,t} \to \alpha \int_{\Omega_{h,t}} R_0(\phi_{h,t} - \psi) \, dx \]
is continuous in \( L^2(\Omega_{h,t}) \); thus,
\[ E(\Omega_{h,t}) = \int_{\Omega_{h,t}} dx + \alpha \int_{\Omega_{h,t}} R_0(\phi_{h,t} - \psi) \, dx, \]  
which is the required value of the cost function \( E \) as \( \epsilon \) tends to zero. Now we shall find the value of the derivative of the cost function as \( \epsilon \) tends to zero:
\[ \lim_{\epsilon \to 0} \dot{E}(\Omega_{h,t}) = \lim_{\epsilon \to 0} \int_{\Omega_{h,t}} (1 - \nabla P_{h,e,t} \cdot \nabla \phi_{h,e,t}) (V, n) \, d\Gamma. \]  

Now we will need to find the limit of the vector \( P_{h,e,t} \), as \( \epsilon \to 0 \); in the Appendix we prove the following theorem, which shows that this limit, \( P_{h,t} \), is itself the solution of a variational inequality.

**Theorem 1.** As \( \epsilon \to 0 \), \( P_{h,e,t} \to P_{h,t} \) in \( K \), \( P_{h,t} \) being the solution of the variational inequality
\[ a(P_{h,t}, \omega_h - P_{h,t}) \geq (\chi', \omega_h - P_{h,t}), \quad \omega_h \in K, \]  
where \( \chi' = \alpha R_0 \).

Now we shall find the value of the derivative of the cost function when \( \epsilon \) tends to zero, by using (4.3). Since
\[ \int_{\Gamma_{h,t}} (\nabla P_{h,e,t} \cdot \nabla \phi_{h,e,t}) (V, n) \, d\Gamma = -\alpha \int_{\Omega_{h,t}} R_0 \phi_{h,e,t}' \, dx, \]
then
\[ \lim_{\epsilon \to 0} \int_{\Gamma_{h,t}} (\nabla P_{h,e,t} \cdot \nabla \phi_{h,e,t}) (V, n) \, d\Gamma = -\lim_{\epsilon \to 0} \alpha \int_{\Omega_{h,t}} R_0 \phi_{h,e,t}' \, dx. \]
Since
\[ \phi_{h,e,t} \to \phi_{h,t}, \quad \text{in} \ L^2(\Omega_{h,t}) \ \text{weakly, as} \ \epsilon \to 0, \]
then,
\[ \phi_{h,e,t} \to \phi_{h,t}, \quad \text{in} \ L^2(\Omega_{h,t}) \ \text{strongly, as} \ \epsilon \to 0, \]
so
\[ \alpha \int_{\Omega_{h,t}} R_0 \phi_{h,e,t}' \, dx \to \alpha \int_{\Omega_{h,t}} R_0 \phi_{h,t}' \, dx, \]
since the functional
\[ \phi_{h,t} \to \alpha \int_{\Omega_{h,t}} R_0 \phi_{h,t}' \, dx \]
is continuous in \(L^2(\Omega_{h,t})\), so the above equation becomes
\[
\int_{\Gamma_{h,t}} (\nabla P_{h,t} \cdot \nabla \phi_{h,t}) \langle V, n \rangle \, d\Gamma = -\alpha \int_{\Omega_{h,t}} R_0 \phi_{h,t} \, dx,
\]
and (4.3) gives rise to
\[
\dot{E}(\Omega_{h,t}) = \int_{\Gamma_{h,t}} (1 - \nabla P_{h,t} \cdot \nabla \phi_{h,t}) \langle V, n \rangle \, d\Gamma.
\] (4.5)

As before, the gradient can be explicitly computed and used to minimize \(E\):
\[
\dot{E}(\Omega_{h,t}) = \int_{\Gamma_{h,t}} C_{h,t} \langle V, n \rangle \, d\Gamma,
\] (4.6)
where \(C_{h,t} = (1 - \nabla P_{h,t} \cdot \nabla \phi_{h,t})\); also, we can write (4.6) in this way:
\[
\dot{E}(\Omega_{h,t}) = \langle G, V \rangle_{L_2}, \text{ where } G = (C_{h,t} = 0, n).
\] (4.7)

As before, we choose \(V = -G/\|G\|\).

We can now define an algorithm to solve the optimal shape problem for the differential inequality.

**Algorithm 2.**

0. Choose \(\Omega_{h,t}^0\), i.e., \(\{q^{k,0}\}\).
1. Compute \(\phi_{h,t}^{m,0}\) (with \(m = 1\)).
2. Compute \(P_{h,t}^{m,0}\).
3. Compute \(G\).
4. Compute vector field \(V\).
5. Let \(q^{k,m}(\rho) = q^{k,m-1} + \rho V\); compute \(\rho^m\), an approximation of
\[
\arg \min_{0 < \rho < \rho_{\max}} E(\{q^{k,m}(\rho)\}).
\]

This step involves a one-dimensional optimization in the direction of the gradient; hence \(\rho_{\max}\) is an appropriate value.
6. Set \(q^{k,m+1} = q^{k,m}(\rho)\).
7. Perform a terminal check, i.e., find out whether the domain \(\Omega_{h,t}\) intersects the fixed domain \(\Omega_{h,0}\); if so, stop. Otherwise, change \(m' = m + 1\), and go back to step 1.

5. Description of the program and algorithm used

The optimum design program is composed of the following modules.
1. A module for solving the direct problem (or state problem). We take \(t = 0\) throughout. Find \(\phi_{h,t} \in K\) such that
\[
\int_{\Omega_{h,t}} (\nabla \omega_{h,t} - f_{\omega_h}) \, dx \geq 0, \; \forall \omega_h \in K,
\] (5.1)
or, find $\phi_{h,t} \in K$ such that

$$I(\phi_{h,t}) \leq I(\omega_h), \quad \forall \omega_h \in K,$$

where $I(\phi_{h,t})$ is defined as follows:

$$I(\phi_{h,t}) = \frac{1}{2} \int_{\Omega_{h,t}} \left( |\nabla \phi_{h,t}|^2 - 2f \phi_{h,t} \right) \, dx,$$

minimized over the convex set $K$. The method used for the minimization of this function will be explained briefly. The function $I(\phi_{h,t})$ may be written $I(\phi_1, \phi_2, \ldots, \phi_{p(nh)})$ to emphasize the dependence of $\phi_{h,t}$ on the coefficients in $\phi_{h,t} = \sum_i^N \phi_i \omega^i$. Then the problem can be solved by the relaxation method; with

$$\phi^{n+1}_{h,t} = (\phi^n_1, \ldots, \phi^n_{Nh}), \quad \text{given in } K,$$

with $\phi^n_{h,t}$ known, $\phi^{n+1}_{h,t}$ is determined coordinate by coordinate, further iterations in the algorithm being given by

$$\phi^{n+1} = \phi^n + \omega (\phi^{n+1/2} - \phi^n);$$

$\omega$ is the relaxation parameter $0 < \omega < 2$. The process described above is stopped when

$$\sum_{i=1}^{Nh} |\phi^n_{i+1} - \phi^n_i| / \sum_{i=1}^{Nh} |\phi^n_{i+1}| \leq \epsilon_r.$$

(In our computational experiments we took $\epsilon_r = 10^{-4}$.)

(2) A module for solving the adjoint-state problem, whose solution is needed to compute the derivative of the cost function $E$. The adjoint state $P_{h,t} \in K$ is given by the solution of the variational inequality

$$\int_{\Omega_{h,t}} (\nabla P_{h,t} \cdot \nabla \omega_h - (\chi', \omega_h)) \, dx \geq 0, \quad \text{for every } \omega_h \in K.$$

In the Appendix, we show that this variational inequality has a solution which minimizes the following functional:

$$I(P_{h,t}) = \frac{1}{2} \int_{\Omega_{h,t}} \left( |\nabla P_{h,t}|^2 - 2\chi' P_{h,t} \right) \, dx,$$

over the convex set $K$. For this problem, we use the same method as we used in the case of the state problem.

(3) A module for the computation of the derivative of the cost function $E$ when we know the solution $\phi_{h,t}$ of the state problem and the solution $P_{h,t}$ of the adjoint state problem. In the formula we must account for the variability of the criterion domain.

(4) A module for the computation of the vector field $V$ when we know the vector $G$ from the derivative of the cost function $E$.

(5) A module minimizing the criterion functional when we know a vector field $V$. We used the material (or speed) method with optimal choice of step length $\rho$ and eventually projection.

(6) A drawing module for the plotting (characteristics) related with a given geometry. This is convenient for quickly analyzing computational results.

The finite-element method (on triangles, using first-order polynomials) was used to solve (1.4), (4.4) and (4.7) with $f = -1, \alpha = 100$. The triangulation is composed of 400 nodes at 722
triangles. Since the main idea of our problem is to find the contact region and the free boundary of the contact region at given value of $\psi$ (which is shape of the obstacle), in our example we took the initial shape of the problem as shown in Fig. 3 and we can also see in Fig. 3 the domain $\Omega_h$ where the criterion

$$E(\Omega_h) = \int_{\Omega_h} \text{d}x + \alpha \int_{\Omega_h} R_0 (\phi_h - \psi) \, \text{d}x$$

is evaluated; we take $\alpha = 100$. The starting value of the criterion is $E(\Omega_0) = 5.29865$. We can see the new shapes in Figs. 4 and 5 at iterations 5 and 15 with criterions $E(\Omega_5) = 3.43472$ and $E(\Omega_{15}) = 1.768721$, respectively. Figure 6 shows the final shape of the problem with final criterion $E(\Omega_{21}) = 1.375427$ after 21 iterations; Fig. 7 gives the relation between the performance criterion $E$ and the number of iterations.
Appendix

The main purpose of this appendix is to sketch the proof of Theorem 1. In this, the functions $P_{h,e,e}$ are known to be nonnegative, because the function $\chi' = \alpha R_0$ is nonnegative on $\Omega_{h,e}$, and $P_{h,e,e}$ results from minimizing a functional much like $I_\varepsilon$ (see [1]), which, together with $\chi' \geq 0$ on $\Omega_{h,e}$, implies $P_{h,e,e} \geq 0$ on $\Omega_{h,e}$.

We shall make the following assumption, with respect to the situation concerning Theorem 1.

**Assumption 3.** The functions $P_{h,e}$ are nonnegative on $\Omega_h$, for sufficiently small $\varepsilon > 0$.

In the last resort, this assumption (which, we repeat, is only needed in respect of Theorem 1) shows its validity because it gives rise to a function $P_h$ which itself gives rise to the sought-after gradient of the function $E$.

Before proving Theorem 1 we prove the following lemma.

**Lemma 4.** Let $a(P_{h,e}, \phi_{h,e})$ be a bilinear, continuous form on $H^2_0(\Omega_h) \times H^2(\Omega_h)$ such that

$$a(P_{h,e}, P_{h,e}) \geq 0, \quad \forall P_{h,e} \in H^2_0(\Omega_h);$$

then the function $P_{h,e} \rightarrow a(P_{h,e}, \phi_{h,e})$ is lower semicontinuous with respect to the weak topology.

**Proof.** From the bilinearity, we have for all $P_{h,e} \in H^2_0(\Omega_h), \phi_{h,e} \in H^2(\Omega_h)$,

$$a(\phi_{h,e}, \phi_{h,e}) - a(P_{h,e}, P_{h,e}) + [a(P_{h,e}, \phi_{h,e} - P_{h,e}) + a(\phi_{h,e} - P_{h,e}, P_{h,e}) + a(P_{h,e} - \phi_{h,e}, P_{h,e} - \phi_{h,e})].$$

(A.2)
Now we use the condition of ellipticity, i.e.,
\[ a(P_{h,e}, P_{h,e}) \geq 0; \]
this implies
\[ a(\phi_{h,e}, \phi_{h,e}) \geq a(P_{h,e}, P_{h,e}) + [a(P_{h,e}, \phi_{h,e} - P_{h,e}) + a(\phi_{h,e} - P_{h,e}, P_{h,e})]. \]
Now, let \( \phi_{h,e} \to P_h \) in \( H^2_h(\Omega_h) \) weakly; from the continuity of \( a \) and the fact that
\[ a(P_h, \phi_{h,e} - P_h) \to 0, \quad a(\phi_{h,e} - P_h, P_h) \to 0, \]
it follows that
\[ \liminf_{P_{h,e} \to P_h} a(\phi_{h,e}, \phi_{h,e}) \geq a(P_h, P_h), \quad (A.3) \]
hence, the map \( \phi_{h,e} \to a(\phi_{h,e}, \phi_{h,e}) \) is weakly lower semicontinuous. \( \square \)

In connection with the behaviour of the subsequence \( P_{h,e} \) as \( \epsilon \to 0 \), we have the next theorem.

**Theorem 1.** As \( \epsilon \to 0 \), \( P_{h,e} \to P_h \) in \( K \), \( P_h \) being the solution of the variational inequality
\[ a(P_h, \omega_h - P_h) \geq (\chi', \omega_h - P_h), \quad \omega_h \in K, \quad (A.4) \]
where \( \chi' = \alpha R_0 \), and \( K = \{ \psi | \psi \in H^2(\Omega_h), \psi \geq 0, \text{almost everywhere on } \Omega_h \} \).

**Proof.** Consider the second penalized equation
\[ AP_{h,e} + \frac{1}{\epsilon} d/d\phi(\phi_{h,e})P_{h,e} = \chi', \quad (A.5) \]
or, in variational form, for \( P_{h,e} = \omega_h \), we have
\[ \int_{\Omega_h} \nabla P_{h,e} \cdot \nabla P_{h,e} + (F'(\phi_{h,e})P_{h,e}, P_{h,e}) \, dx = \int_{\Omega_h} (f_1, P_{h,e}) \, dx, \quad (A.6) \]
where \( F'(\phi_{h,e}) = 1/\epsilon d/d\phi(\phi_{h,e}) \geq 0 \); then,
\[ \int_{\Omega_h} d/d\phi(\phi_{h,e}) P_{h,e}^2 \, dx > 0. \]

Then, by (1.2) and (A.6), we have
\[ 0 \leq \alpha \| P_{h,e} \|^2 \leq \int_{\Omega_h} \nabla P_{h,e} \cdot \nabla P_{h,e} + F'(\phi_{h,e}) P_{h,e}^2 \, dx \]
\[ = \int_{\Omega_h} (\chi', P_{h,e}) \, dx, \quad (A.7) \]
or.

\[ \alpha \| P_{h,\varepsilon} \|^2 \leq \int_{\Omega_h} (\chi', P_{h,\varepsilon}) \, dx \leq \| \chi' \| \| P_{h,\varepsilon} \|, \]

\[ \alpha \| P_{h,\varepsilon} \|^2 \leq C_1 \| P_{h,\varepsilon} \|, \quad C_1 = 1, \]

\[ \| P_{h,\varepsilon} \| \leq C_{11}, \quad C_{11} = \frac{C_1}{\alpha} = \text{constant, independent of } \varepsilon. \]  

(A.8)

A subsequence, also denoted by \( P_{h,\varepsilon} \), can then be extracted from the sequence \( P_{h,\varepsilon} \), such that \( P_{h,\varepsilon} \to P_h \) weakly in \( H^2_h(\Omega_h) \).

Since we have assumed that \( P_{h,\varepsilon} \geq 0 \) on \( \Omega_h \), \( P_h \geq 0 \) on \( \Omega_h \). By writing equation (A.8) in the following form:

\[ a(P_{h,\varepsilon}, \omega_h - P_{h,\varepsilon}) - (\chi', \omega_h - P_{h,\varepsilon}) = -(F'(\phi_{h,\varepsilon}) P_{h,\varepsilon}, \omega_h - P_{h,\varepsilon}), \]  

(A.9)

\[ a(P_{h,\varepsilon}, \omega_h - P_{h,\varepsilon}) - (\chi', \omega_h - P_{h,\varepsilon}) = \frac{1}{\varepsilon} \left[ (\hat{H} P_{h,\varepsilon}, P_{h,\varepsilon}) - (\hat{H} P_{h,\varepsilon}, \omega_h) \right], \]  

(A.10)

where \( \hat{H} = d/d\phi(\phi_{h,\varepsilon}) \). Consider now (A.10) only for those \( \omega_h = W_h \in B \subset K \), with \( B \) the subset of the convex set \( K \) composed of the basis elements for \( H^2_h(\Omega_h) \). Now we shall prove that the right-hand side of (A.10) is positive, that is

\[ (\hat{H} P_{h,\varepsilon}, P_{h,\varepsilon}) > (\hat{H} P_{h,\varepsilon}, W_h), \]  

(A.11)

provided \( h \) is sufficiently small.

Since \( \hat{H} \) is a positive operator, \( (\hat{H} P_{h,\varepsilon}, P_{h,\varepsilon}) > 0 \); we can assume that \( (\hat{H} P_{h,\varepsilon}, W_h) \geq 0 \); otherwise (A.10) is automatically true. We can make the right-hand side of the inequality (A.11) as small as possible; note that \( P_{h,\varepsilon} \) does not depend much on \( h \) (from (A.4)), but that the support of \( W_h \) can be made as small as possible by taking \( h \) small enough, the maximum value of \( W_h \) is of course 1. Therefore from (A.11) we can see that, under these conditions,

\[ a(P_{h,\varepsilon}, W_h - P_{h,\varepsilon}) - (\chi', W_h - P_{h,\varepsilon}) = \frac{1}{\varepsilon} \left[ (\hat{H} P_{h,\varepsilon}, P_{h,\varepsilon}) - (\hat{H} P_{h,\varepsilon}, W_h) \right] \geq 0, \]  

(A.12)

\[ W_h \in B \subset K; \]

hence (A.12) can be written as

\[ a(P_{h,\varepsilon}, W_h - P_{h,\varepsilon}) - (\chi', W_h - P_{h,\varepsilon}) \geq 0, \quad W_h \in B \subset K, \]

or,

\[ a(P_{h,\varepsilon}, W_h) - (\chi', W_h - P_{h,\varepsilon}) \geq a(P_{h,\varepsilon}, P_{h,\varepsilon}); \]  

(A.13)

letting \( \varepsilon \to 0 \) in (A.13), we obtain

\[ a(P_h, W_h) - (\chi', W_h - P_h) \geq \inf_{\varepsilon \to 0} a(P_{h,\varepsilon}, P_{h,\varepsilon}). \]

By applying Lemma 4 we obtain now

\[ \lim_{\varepsilon \to 0} \inf a(P_{h,\varepsilon}, P_{h,\varepsilon}) \geq a(P_h, P_h) > 0, \]
which implies that
\[ a(P_h, W_h - P_h) \geq (\chi', W_h - P_h), \quad W_h \in B \subset K. \]  \hspace{1cm} (A.14)

Now we shall show that (A.14) holds for all \( \omega_h \in K \). Indeed,
\[ \omega_h = \sum_i \alpha_i \omega_i, \quad \alpha_i \geq 0, \quad \omega_i \in B, \]
so that
\[ a(P_h, \omega_h - P_h) = a\left(P_h, \left( \sum_i \alpha_i \omega_i \right) - P_h \right) = \sum_i \alpha_i a(P_h, \omega_i - P_h) \geq \sum_i \alpha_i (\chi', \omega_i - P_h), \]
since the \( \alpha_i \)'s are positive, and equation (A.14) is valid for all \( \omega_i \)'s. Thus,
\[ a(P_h, \omega_h - P_h) \geq (\chi', \omega_h - P_h), \quad \omega_h \in K, \]  \hspace{1cm} (A.15)
which shows that \( P_h \) is a solution of the inequality (A.4).

Let the function \( P_{h,\varepsilon} \) minimize the function
\[ I_\varepsilon(P_{h,\varepsilon}) = \frac{1}{2} \int_{\Omega_h} |\nabla P_{h,\varepsilon}|^2 \, dx - \int_{\Omega_h} (\chi', P_{h,\varepsilon}) \, dx, \] \hspace{1cm} (A.16)
over the convex set \( K \); the function \( \chi' \) equals \( \alpha R_0 \). Then it is the unique solution of (A.6). As \( \varepsilon \to 0 \), \( P_{h,\varepsilon} \to P_h \), a function which minimizes the functional
\[ I(P_h) = \frac{1}{2} \int_{\Omega_h} |\nabla P_h|^2 \, dx - \int_{\Omega_h} (\chi', P_h) \, dx, \] \hspace{1cm} (A.17)
over the convex set \( K \). We note that such a minimizer is a unique solution of the variational inequality of (A.15). By taking the limits as \( \varepsilon \to 0 \) of both sides of (A.16), we can show (see [1]) that \( I_\varepsilon \) tends to \( I \). Theorem 1 follows from the fact that corresponding minimizers of these functionals are unique. \( \Box \)

References