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Equilibria of Noncompact Generalized Games with $\mathcal{U}$-Majorized Preference Correspondences

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Abstract—In this paper, some existence theorems of equilibria for qualitative games and generalized games with an infinite number of agents with noncompact strategy sets and with $\mathcal{U}$-majorized preference correspondences are proved. Our theorems improve some recent results in the literatures.

Keywords—Generalized game, $\mathcal{U}$-majorized, Equilibrium.

1. INTRODUCTION

Ding and Tan [1–3], Ding *et al.* [4,5], Im *et al.* [6], Tan and Yuan [7], and Tian [8,9] have proved some existence theorems of equilibria for generalized games with an infinite number of agents, with noncompact strategy sets and with the preference correspondences which have open lower sections or are majorized by the correspondences with open lower sections. Tan and Yuan [10] proved some new existence theorems for qualitative games and generalized games with compact strategy sets and with the preference correspondences which are majorized by upper semicontinuous correspondences.

In this paper, some existence theorems of equilibria for noncompact qualitative games and for noncompact generalized games with an infinite number of agents, with noncompact strategy sets and with the preference correspondences which are majorized by upper semicontinuous correspondences, are proved. These theorems generalize the corresponding results of Tan and Yuan [10].

2. PRELIMINARIES

Let A be a nonempty subset of a topological space X. We shall denote by 2^A the family of all subsets of A and by $\mathrm{cl}_X A$ the closure of A in X. If A is a nonempty subset of a vector space E, we shall denote by $\mathrm{co}A$ the convex hull of A. If $S, T : A \to 2^E$ are correspondences, then $(S \cap T) : A \to 2^E$ is a correspondence defined by $(S \cap T)(x) = T(x) \cap S(x)$ for each $x \in A$. If X and Y are topological spaces and $T : X \to 2^Y$ is a correspondence, then

(1) T is said to be upper semicontinuous on X if for each $x \in X$ and for each open set U of Y containing $T(x)$, the set $\{z \in X : T(z) \subset U\}$ is an open neighborhood of x in X;
(2) the graph of T, denoted by $\mathrm{Gr}\,(T)$, is the set $\{(x, y) \in X \times Y : y \in T(x)\}$;
(3) the correspondence $\overline{T} : X \to 2^Y$ is defined by $\overline{T}(x) = \{y \in Y : (x, y) \in \mathrm{cl}_{X \times Y} \mathrm{G}\,(T)\}$.

Let X be a topological space, Y be a nonempty subset of a vector space E, $\theta : X \to E$ be a map, and $\phi : X \to 2^Y$ be a correspondence. Then,

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(1) ϕ is said to be of class $\mathcal{U}_\theta$ if
   (a) for each $x \in X$, $\theta(x) \notin \phi(x)$, and
   (b) ϕ is upper semicontinuous with closed convex values in Y;
(2) ϕ_x is a $\mathcal{U}_\theta$-majorant of ϕ at x if there exist an open neighborhood $N(x)$ of x in X and $\phi_x : N(x) \to 2^Y$ such that
   (a) for each $z \in N(x)$, $\phi(z) \subset \phi_x(z)$, $\theta(z) \notin \phi_x(z)$, and
   (b) ϕ_x is upper semicontinuous with closed convex values;
(3) ϕ is said to be $\mathcal{U}_\theta$-majorized if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists a $\mathcal{U}_\theta$-majorant ϕ_x of ϕ at x.

In this paper, we shall deal with either the case
(A) $X = Y$ is a nonempty convex subset of a topological vector space E and $\theta = I_X$, the identity map on X, or the case
(B) $X = \prod_{i\in I} X_i$ and $\theta = \pi_j : X \to X_j$ is the projection of X onto X_j and $Y = X_j$ is a nonempty convex subset of a topological vector space.

In both cases (A) and (B), we shall write $\mathcal{U}$ in place of $\mathcal{U}_\theta$.

Let I be a (finite or infinite) set of agents. A generalized game is a family of quadruples $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$, where X_i is a nonempty subset of a topological vector space, $A_i, B_i : X = \prod_{j\in I} X_j \to 2^{X_i}$ are constraint correspondences, and $P_i : X \to 2^{X_i}$ is a preference correspondence. An equilibrium of Γ is a point $\hat{x} \in X$ such that for each $i \in I$, $\hat{x}_i \in \overline{B}_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$. Following [11], a qualitative game is a collection $\Gamma = (X_i, P_i)_{i\in I}$, where $P_i : X \to 2^{X_i}$ is a preference correspondence. A point $\hat{x} \in X$ is said to be an equilibrium of Γ if $P_i(\hat{x}) = \emptyset$ for all $i \in I$.

The following result is Theorem 2.2 of [10].

LEMMA 2.1. *Let X be a nonempty subset of a locally convex Hausdorff topological vector space and D be a nonempty compact subset of X. Let $P : X \to 2^D$ be $\mathcal{U}$-majorized. Then there exists a point $\hat{x} \in \mathrm{co}D$ such that $P(\hat{x}) = \emptyset$.*

3. EXISTENCE OF EQUILIBRIA

Now we shall prove some new existence theorems of equilibria for noncompact qualitative games and noncompact generalized games with any (countable or uncountable) number of agents and with noncompact strategy sets of agents in locally convex Hausdorff topological vector spaces in which either the preference correspondences are $\mathcal{U}$-majorized or the intersection of constraint and preference correspondences are $\mathcal{U}$-majorized.

THEOREM 3.1. *Let $\Gamma = (X_i, P_i)_{i\in I}$ be a qualitative game such that for each $i \in I$,*
(a) *X_i is a nonempty convex subset of a locally convex Hausdorff topological vector space and D_i is a nonempty compact subset of X_i,*
(b) *$P_i : X = \prod_{j\in I} X_j \to 2^{D_i}$ is $\mathcal{U}$-majorized,*
(c) *the set $E^i = \{x \in X : P_i(x) \neq \emptyset\}$ is open in X,*
(d) *there exists a nonempty closed convex subset F_i of D_i such that $F_i \cap P_i(x) \neq \emptyset$ for all $x \in E^i$.*

Then there exists a point $\hat{x} \in \mathrm{co}D = \mathrm{co}\,(\prod_{j\in I} D_i)$ such that $P_i(x) = \emptyset$ for all $i \in I$.

PROOF. Let $D = \prod_{j\in I} D_j$. Then D is a compact subset of $X = \prod_{j\in I} X_j$. For each $x \in X$, let $I(x) = \{i \in I : P_i(x) \neq \emptyset\}$. Define a correspondence $P : X \to 2^D$ by

$$P(x) = \begin{cases} \bigcap_{i\in I(x)} P'_i(x), & \text{if } I(x) \neq \emptyset, \\ \emptyset, & \text{if } I(x) = \emptyset, \end{cases}$$

where $P'_i(x) = \prod_{j\neq i} F_j \times P_i(x)$ for each $x \in X$. Then for each $x \in X$ with $I(x) \neq \emptyset$, $P(x) \neq \emptyset$ and hence $P'_i(x) \neq \emptyset$ for each $i \in I(x)$. For any fixed $i \in I(x)$, we have $P_i(x) \neq \emptyset$. By (b), there exist an open neighborhood $N(x)$ of x in X and an $\mathcal{U}$-majorant ϕ_i of P_i at x such that

(1) for each $z \in N(x), P_i(z) \subset \phi_i(z)$ and $z_i \notin \phi_i(z)$;

(2) ϕ_i is upper semicontinuous with closed convex values in D_i.

By the assumption, we may assume $N(x) \subset E^i$ so that $P_i(z) \neq \emptyset$ for each $z \in N(x)$. Now define the correspondence $\Phi_x : N(x) \to 2^D$ by

$$\Phi_x(z) = \prod_{j \neq i} F_j \times \phi_i(z)$$

for each z in $N(x)$. We claim that Φ_x is an $\mathcal{U}$-majorant of P at x. Indeed, for each $z \in N(x)$, by (1),

$$P(z) = \prod_{j \in I(z)} P_j'(z) \subset P_i'(z) = \prod_{j \neq i} F_j \times P_i(z) \subset \prod_{j \neq i} F_j \times \phi_i(z) = \Phi_x(z)$$

and $z \notin \Phi_x(z)$, since $z_i \notin \phi_i(z)$. By (b), (2), and Lemma 3 of [12], $\Phi_x : N(x) \to 2^D$ is upper semicontinuous with nonempty closed convex values in D. Therefore Φ_x is a $\mathcal{U}$-majorant of P at x. This shows that P is $\mathcal{U}$-majorized. By Lemma 2.1, there exists a point $\hat{x} \in \mathrm{co}D$ such that $P(\hat{x}) = \emptyset$ so that $I(\hat{x}) = \emptyset$. Hence, $P_i(\hat{x}) = \emptyset$ for all $i \in I$.

REMARK 3.1. If for each $i \in I$, X_i is compact convex, the condition (d) is satisfied trivially. Hence Theorem 3.1 generalizes Theorem 3.2 of [10] in the following ways:

(1) X_i may not be compact,

(2) the set $E^i = \{x \in X : P_i(x) \neq \emptyset\}$ may not be paracompact.

COROLLARY 3.1. *Let $\Gamma = (X_i, P_i)_{i \in I}$ be a qualitative game such that for each $i \in I$,*

(a) *X_i is a nonempty compact convex subset of a locally convex Hausdorff topological vector space,*

(b) *$P_i : X = \prod_{i \in I} X_i \to 2^{X_i}$ is $\mathcal{U}$-majorized,*

(c) *the set $E^i = \{x \in X : P_i(x) \neq \emptyset\}$ is open in X.*

Then Γ has an equilibrium point $\hat{x} \in X$.

REMARK 3.2. Corollary 3.1 improves Theorem 3.2 of [10], the set E^i may not be paracompact.

THEOREM 3.2. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be a generalized game (= abstract economy) such that for each $i \in I$,*

(a) *X_i is a nonempty convex subset of a locally convex Hausdorff topological vector space E_i and D_i is a nonempty compact subset of X_i,*

(b) *for each $x \in X = \prod_{i \in I} X_i$, $A_i(x)$ is nonempty, $A_i(x) \subset B_i(x) \subset D_i$, and $B_i(x)$ is convex,*

(c) *the set $E^i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X,*

(d) *$A_i \cap P_i : X \to 2^{D_i}$ is $\mathcal{U}$-majorized,*

(e) *there exists a nonempty closed convex set $K_i \subset D_i$ such that for each $x \in E^i \cap F_i$, $(A_i \cap P_i)(x) \cap K_i \neq \emptyset$ and for each $x \in X \setminus F_i$, $\overline{B}_i(x) \cap K_i \neq \emptyset$, where $F_i = \{x \in X : x \in \overline{B}_i(x)\}$.*

Then Γ has an equilibrium point $\hat{x} \in \mathrm{co}D = \mathrm{co}\left(\prod_{i \in I} D_i\right)$, i.e., $\hat{x}_i \in \overline{B}_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ for each $i \in I$.

PROOF. For each $i \in I$, by the definition of $\overline{B}_i$, $\overline{B}_i$ has closed graph and hence F_i is closed in X. Define a correspondence $Q_i : X \to 2^{D_i}$ by

$$Q_i(x) = \begin{cases} A_i(x) \cap P_i(x), & \text{if } x \in F_i, \\ \overline{B}_i(x), & \text{if } x \notin F_i. \end{cases}$$

We shall prove that the qualitative game $\Gamma = (X_i, Q_i)_{i \in I}$ satisfies all hypotheses of Theorem 3.1.

For each $i \in I$, we have

$$\begin{aligned}\{x \in X : Q_i(x) \neq \emptyset\} &= \{X \setminus F_i : Q_i(x) \neq \emptyset\} \cup \{x \in F_i : Q_i(x) \neq \emptyset\} \\ &= (X \setminus F_i) \cup \{x \in F_i : (A_i \cap P_i)(x) \neq \emptyset\} \\ &= (X \setminus F_i) \cup (F_i \cap E^i) \\ &= (X \setminus F_i) \cup E^i.\end{aligned}$$

Hence the set $\{x \in X : Q_i(x) \neq \emptyset\}$ is open in X. For any fixed $i \in I$, let $x \in X$ be such that $Q_i(x) \neq \emptyset$. If $x \notin F_i$, then $x \in N(x) = X \setminus F_i$ and $N(x)$ is an open neighborhood of x. Define a correspondence $\phi_x : N(x) \to 2^D$ by

$$\phi_x(z) = \overline{B}_i(z), \qquad \text{for each } z \in N(x).$$

Then we have $Q_i(z) = \overline{B}_i(z)$ and $z_i \notin Q_i(z)$ for each $z \in N(x)$ by the definition of F_i. By Proposition 3.1.9 of [13], $\phi_x = \overline{B}_i$ is upper semicontinuous with closed convex values. Therefore $\phi_x = \overline{B}_i$ is a $\mathcal{U}$-majorant of Q_i at x. If $x \in F_i$, then $Q_i(x) = A_i(x) \cap P_i(x) \neq \emptyset$. By condition (d), there exist an open neighborhood $N_1(x)$ of x in X and a $\mathcal{U}$-majorant $\psi_x : N_1(x) \to 2^D$ such that $Q_i(z) = A_i(z) \cap P_i(z) \subset \psi_x(z)$ and $z_i \notin \psi_x(z)$ for each $z \in N_1(x)$, and $\psi_x : N_1(x) \to 2^D$ is upper semicontinuous with closed convex values. Hence ψ_x is a $\mathcal{U}$-majorant of Q_i at x. Therefore Q_i is a $\mathcal{U}$-majorized correspondence. Now let $x \in X$ with $Q_i(x) \neq \emptyset$, then $x \in (X \setminus F_i) \cup \{z \in F_i : (A_i \cap P_i)(z) \neq \emptyset\}$. It follows from assumption (e) that $Q_i(x) \cap K_i \neq \emptyset$. Hence all hypotheses of Theorem 3.1 are satisfied so that there exists a point $\hat{x} \in \mathrm{co}D$ such that $Q_i(\hat{x}) = \emptyset$ for all $i \in I$. By condition (b), this implies that for each $i \in I$, $\hat{x}_i \in \overline{B}_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

REMARK 3.3. If X_i is compact convex for each $i \in I$, then condition (e) of Theorem 3.2 is satisfied by letting $X_i = D_i = K_i$ for each $i \in I$. Hence Theorem 3.2 generalizes Theorem 3.1 of [10] in the following ways:

(1) X_i may not be compact,

(2) the set $E^i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ may not be paracompact.

COROLLARY 3.2. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be a generalized game such that for each $i \in I$,*

(a) *X_i is a nonempty compact convex subset of a locally convex Hausdorff topological vector space E_i,*

(b) *for each $x \in X = \prod_{i \in I} X_i$, $A_i(x)$ is nonempty, $A_i(x) \subset B_i(x)$, and $B_i(x)$ is convex,*

(c) *$A_i \cap P_i : X \to 2^{X_i}$ is $\mathcal{U}$-majorized.*

Then Γ has an equilibrium point, i.e., there exists a point $\hat{x} \in X$ such that for each $i \in I$, $\hat{x}_i \in \overline{B}_i(\hat{x})$ and $A_i(x) \cap P_i(x) = \emptyset$.

REMARK 3.4. Corollary 3.2 improves Theorem 3.1 of [10].

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