Perturbations of Periodic Competitive Parabolic Systems

Xiao-Qiang Zhao

Department of Mathematics and Statistics, Memorial University of Newfoundland, St. John’s, Newfoundland A1C 5S7, Canada
E-mail: xzhao@math.mun.ca

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It is shown that the competitive exclusion and coexistence in two species periodic competitive parabolic systems are robust under a class of perturbations. This result is also applied to a reaction-diffusion model for the evolution of dispersal rates.

1. INTRODUCTION

Competitive models have received extensive investigations. Due to the discrete monotone dynamical system approach (see [6, 8]), the global dynamics of two species periodic competitive parabolic systems is well understood. For example, if one semitrivial periodic solution is linearly unstable and the other one is linearly stable, and there is no positive periodic solution, then the stable semitrivial periodic solution is globally attractive with respect to positive initial values. One natural question is about the robustness of the global dynamics of two species competitive systems under certain perturbations. Recently, a mutation-selection multi-phenotypic model for the evolution of dispersal rates has been studied in [2]. When the model is restricted to two phenotypes of a species, it becomes a reaction-diffusion system of two species competitive type with perturbations. The global dynamics in the case of temporal homogeneity and small rate of mutation is clear (see [2]): the semitrivial equilibrium

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with the slower dispersal rate is a global attractor. In order to take into account heterogeneous time-varying environments, a time-periodic version of this reaction diffusion system without mutation is analysed nontrivially in [9]. It turns out that the dynamics in the periodic case is much richer than that in the autonomous case. The purpose of this paper is to discuss perturbations of periodic competitive parabolic systems. Our approach is via the Poincaré map associated with the periodic system under consideration. In order to appeal to the abstract results on the perturbation of a globally stable steady state (see [12]) and the upper semi-continuity of global attractors (see [3]), we need to choose a suitable closed subset of the state space and prove that it is repelling for the dynamics on the complementary set uniformly for small perturbations, which can be done by using the recent result on the uniform persistence uniform in parameters (see [7]). Our results show that the competitive exclusion and coexistence are robust under a large class of perturbations. We should point out that a similar idea on perturbations was used for a periodically pulsed bio-reactor model of parabolic type in [13]. Moreover, the techniques in this paper may work for other types of perturbations and multi-species competitive systems.

In Section 2, we recall some basic definitions and a theorem on uniform persistence uniform in parameters. For a general periodic system with parameters, we provide a result on uniform weak repellors in terms of periodic-parabolic principal eigenvalues, and a result on continuity of solution operators with respect to parameters uniformly for initial values. In Section 3, we first show the uniform boundedness and ultimate boundedness of solutions for two species competitive systems with perturbations. Then we prove our main results on the competitive exclusion and coexistence, which are also applied to the afore-mentioned reaction-diffusion model for the evolution of dispersal rates.

2. PRELIMINARIES

Let \((X, d)\) and \((
\Lambda, 
\rho)\) be two metric spaces. Let \(f: X \to X\) be a continuous map and \(X_0 \subset X\) be an open set with \(f(X_0) \subset X_0\). Define \(\partial X_0 = X \setminus X_0\), and \(M_0 = \{x \in \partial X_0: f^n(x) \in \partial X_0, \forall n \geq 0\}\), which may be empty. Recall that \(f: X \to X\) is said to be uniformly persistent with respect to \((X_0, \partial X_0)\) if there exists \(\eta > 0\) such that \(\liminf_{n \to \infty} d(f^n(x), \partial X_0) \geq \eta\) for all \(x \in X_0\). A finite sequence \(\mathcal{M} = \{M_1, \ldots, M_k\}\) of pairwise disjoint, compact, and isolated invariant sets in \(\partial X_0\) is said to be an acyclic covering of \(f\) in \(\partial X_0\) in \(\Omega(M_0) := \bigcup_{x \in M_0} \omega(x) \subset \bigcup_{i=1}^k M_i\) and no subset of \(\mathcal{M}\) forms a cycle in \(\partial X_0\). Let \(\mathcal{A}\) be the maximal compact invariant set of \(f\) in \(\partial X_0\). Then \(\mathcal{M} = \{M_1, \ldots, M_k\}\)
is an acyclic covering of \( f \) in \( \partial X_0 \) if and only if it (after re-ordering) is a Morse decomposition of \( A_\beta \) (see [7, Lemma 4.1]).

**Theorem A** [7, Corollary 4.5] (Uniform Persistence Uniform in Parameters). For each \( \lambda \in \Lambda \), let \( S^\lambda_\cdot : X \to X \) be a continuous map that takes \( X_0 \) into itself, and such that \( S^\lambda_\cdot(x) \) is continuous in \((\lambda, x)\). Assume that every positive orbit for \( S^\lambda_\cdot \) has compact closure in \( X \), that the set \( \bigcup_{\lambda \in \Lambda, x \in X} \omega_0(x) \) has compact closure, where \( \omega_0(x) \) denotes the omega limit of \( x \) for discrete semiflow \( \{S^\lambda_\cdot\} \), and that there exist \( \lambda_0 \in \Lambda \) and \( \delta_0 > 0 \) such that

(1) \( S^\lambda_\cdot : X \to X \) has a global attractor \( A \) and admits an acyclic covering \( \mathcal{M} = \{M_1, \ldots, M_k\} \) in \( \partial X_0 \);

(2) For each \( \lambda \in \Lambda \) with \( \rho(\lambda, \lambda_0) < \delta_0 \) and any \( x \in X_0 \),
\[
\limsup_{n \to \infty} d(S^n_\cdot x, M_i) \geq \delta_0, \quad 1 \leq i \leq k.
\]

Then there exists \( \delta > 0 \) such that \( \liminf_{n \to \infty} d(S^n_\cdot x, \partial X_0) \geq \delta \) for any \( \lambda \in \Lambda \) with \( \rho(\lambda, \lambda_0) < \delta \) and any \( x \in X_0 \).

Let \( \mathbb{R}_+ = [0, \infty) \) and \( \Omega \) be a bounded subset of \( \mathbb{R}^N \) with smooth boundary. We consider the following periodic parabolic systems with parameters

\[
\begin{aligned}
\frac{\partial u_i}{\partial t} &= A_i(t) u_i + F_i(x, t, u, \lambda), & \text{in } \Omega \times (0, \infty) \\
B_i u_i &= 0, & \text{in } \partial \Omega \times (0, \infty),
\end{aligned}
\]  

(2.1)

where \( u = (u_1, \ldots, u_m) \in \mathbb{R}^m \), \( \lambda \in \Lambda \), \( B_i u_i = u_i \) or \( B_i u_i = \partial u_i / \partial n + \alpha_i u_i \), \( \alpha_i \geq 0 \), \( A_i(t) \) are uniform elliptic operators with the coefficients being smooth in \((x, t)\), \( F_i \) are continuous functions, and for some positive number \( T \), \( A_i(t + T) = A_i(t) \), \( F_i(x, t + T, u, \lambda) = F_i(x, t, u, \lambda) \), \( F_i(x, t, 0, \lambda) = 0 \), \( 1 \leq i \leq m \), \( \forall x \in \overline{\Omega}, \quad t \in [0, \infty) \), \( u \in \mathbb{R}^m \), \( \lambda \in \Lambda \).

Let \( X_i^+ = C(\overline{\Omega}, \mathbb{R}_+) \) if \( B_i u_i = \partial u_i / \partial n + \alpha_i u_i \); and \( X_i^+ = \{\phi \in C(\overline{\Omega}, \mathbb{R}_+) \mid \phi(x) = 0, \forall x \in \partial \Omega\} \) if \( B_i u_i = u_i \). Let \( |\cdot| \) be the standard norm in the Euclidean space \( \mathbb{R}^m \). For \( \phi, \psi \in X_i^+ := \prod_{i=1}^m X_i^+ \), we define \( \|\phi\| = \max_{x \in \overline{\Omega}} |\phi(x)| \) and \( d(\phi, \psi) = \|\phi - \psi\| \). Then \((X_i^+, d)\) is a metric space.

We assume that for any \( \phi = (\phi_1, \ldots, \phi_m) \in X_i^+ \), the unique (mild) solution \( u(x, t, \phi, \lambda) \) of (2.1) with \( u(\cdot, 0, \phi, \lambda) = \phi \) exists globally on \([0, \infty)\) and \( u_i(x, t, \phi, \lambda) \geq 0, 1 \leq i \leq m, \forall x \in \overline{\Omega}, \quad t \geq 0, \quad \lambda \in \Lambda \).

For each \( 1 \leq i \leq m \) and any \( m(\cdot) \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R}) \) with \( m(x, t + T) = m(x, t), \forall x \in \overline{\Omega}, \quad t \in \mathbb{R} \), let \( \mu(A_i(-), m(\cdot, \cdot)) \) be the unique principal
eigenvalue of the periodic-parabolic eigenvalue problem (see [5, Chap. II])

\[
\begin{aligned}
\frac{\partial \varphi}{\partial t} &= A_i(t) \varphi + m(x, t) \varphi + \mu \varphi, \quad x \in \overline{\Omega}, \ t \in \mathbb{R} \\
B_i \varphi &= 0, \quad x \in \partial \Omega, \ t \in \mathbb{R} \\
\varphi &= T\text{-periodic in } t.
\end{aligned}
\]  

(2.2)

For various properties and estimates of periodic-parabolic principal eigenvalues, we refer to [5, 9, 10].

**Lemma 2.1.** Let \( \lambda_0 \in \Lambda \) be fixed. Assume that there exists some \( 1 \leq i \leq m \), such that

1. \( F_i(x, t, u, \lambda) \geq u_i G_i(x, t, u, \lambda) \) for a continuous function \( G(\cdot) \);
2. System (2.1) with \( \lambda = \lambda_0 \) admits a componentwise nonnegative \( T \)-periodic solution \( u_0^{\lambda}(x, t) = (u_{01}^{\lambda}(x, t), \ldots, u_{0m}^{\lambda}(x, t)), 0, u_{0i+1}^{\lambda}(x, t), \ldots, u_{0m}^{\lambda}(x, t); \)
3. \( \mu(A_i(\cdot), G_i(x, t, u_0^{\lambda}(x, t), \lambda_0)) < 0 \).

Then there exists \( \delta > 0 \) such that for each \( \rho(\lambda, \lambda_0) < \delta \), and any \( \phi \in X^+ \) with \( \phi(t) \neq 0 \),

\[
\lim_{n \to \infty} \| u(\cdot, nT, \phi, \lambda) - u_0^{\lambda}(\cdot, 0) \| \geq \delta.
\]

**Proof.** The proof is essentially the same as that of [13, Proposition 4.1]. It is only necessary to replace \( C(\overline{\Omega}, \mathbb{R}) \) with \( C^1(\overline{\Omega}, \mathbb{R}) \) in the case where \( B_i u_i = u_i \).

**Lemma 2.2.** Assume that solutions of (2.1) are uniformly bounded uniformly for \( \lambda \in \Lambda \); i.e., for any \( r > 0 \), there exists \( B = B(r) > 0 \) such that for any \( \phi \in X^+ \) with \( \| \phi \| \leq r \), \( \| u(\cdot, t, \phi, \lambda) \| \leq B(r), \) for all \( t \geq 0, \lambda \in \Lambda \). Then for each \( \lambda_0 \in \Lambda \), and any integer \( k \),

\[
\lim_{\lambda \to \lambda_0} \| u(\cdot, t, \phi, \lambda) - u(\cdot, t, \phi, \lambda_0) \| = 0
\]

uniformly for \( t \in [T, kT] \) and \( \phi \) in any bounded subset of \( X^+ \).

**Proof.** Let \( E_i := X_i^{\beta} \), \( \beta \in (0, 1) \), be the fractional power space of \( L^p(\Omega) \), \( p > N \), with respect to \( (A_i(0), B_i) \), and let \( E = \prod_{i=1}^{m} E_i \). Then the smooth effect of parabolic operators implies that each \( u(\cdot, t, \phi, \lambda) \in E \) for \( t \geq T \). By the variation of constant’s formula and a generalized Gronwall’s inequality argument as in the proof of [16, Proposition 3.1], it then follows that Lemma 2.2 holds.
3. GLOBAL DYNAMICS UNDER PERTURBATIONS

Consider two species periodic competitive parabolic systems with perturbations

\[
\begin{aligned}
\frac{\partial u_i}{\partial t} &= A_i(t)u_i + u_ig_i(x,t,u) + \epsilon \sum_{j=1}^{2} M_{ij}(x,t)u_j, & \text{in } \Omega \times (0, \infty) \\
B_iu_i &= 0, & 1 \leq i \leq 2, \text{ on } \partial \Omega \times (0, \infty),
\end{aligned}
\]

(3.1)

where \( u = (u_1, u_2) \in \mathbb{R}^2, \epsilon \in [0, \epsilon_0], \epsilon_0 > 0, \) and \( \Omega, A_i(t), \) and \( B_i \) are as in Section 2. Throughout this section, we assume that

(H1) \( g = (g_1, g_2) \in C^2(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}_+^2, \mathbb{R}^2) \) is \( T \)-periodic in \( t, \) and \( \partial g_i / \partial u_j < 0, \forall 1 \leq i \neq j \leq 2; \)

(H2) \( \partial g_1(x, t, u_1, 0) / \partial u_1 < 0, \partial g_2(x, t, u_2, 0) / \partial u_2 < 0, \) and there exists \( K_0 > 0 \) such that \( g_1(x, t, K_0, 0) < 0 \) and \( g_2(x, t, 0, K_0) < 0; \)

(H3) \( M_{ij} \in C^2(\overline{\Omega} \times \mathbb{R}, \mathbb{R}) \) is \( T \)-periodic in \( t, \) and \( M_{ij}(x, t) < 0, M_{ij}(x, t) \geq 0, \forall 1 \leq i \neq j \leq 2. \)

Let \( X^+ = X_1^+ \times X_2^+ \) be defined as in Section 2 with \( m = 2. \) Then for any \( \phi = (\phi_1, \phi_2) \in X^+, \) the unique (mild) solution \( u(x, t, \phi, \epsilon) \) of (3.1) with \( u(\cdot, 0, \phi, \epsilon) = \phi \) exists on its maximal interval \([0, \sigma_\phi]\) and \( u(x, t, \phi, \epsilon) \geq 0, 1 \leq i \leq 2, x \in \Omega, t \in [0, \sigma_\phi]. \) The following lemma implies that when \( \epsilon \) is sufficiently small, \( \sigma_\phi = \infty \) for each \( \phi \in X^+. \)

**Lemma 3.1.** There exists an \( \epsilon_1 > 0 \) such that solutions of (3.1) are uniformly bounded and ultimately bounded uniformly for \( \epsilon \in [0, \epsilon_1]. \)

**Proof.** Choose a small \( \epsilon_1 > 0 \) such that

\[
\begin{aligned}
g_1(x, t, K_0, 0) + \epsilon_1 M_{12}(x, t) < 0, & \quad g_2(x, t, 0, K_0) + \epsilon_1 M_{21}(x, t) < 0.
\end{aligned}
\]

Let \( \epsilon \in [0, \epsilon_1] \) and \((u_1(x, t), u_2(x, t)) = u(x, t, \phi, \epsilon). \) Then \((u_1(x, t), u_2(x, t))\) satisfies

\[
\begin{aligned}
\frac{\partial u_1}{\partial t} &\leq A_1(t)u_1 + u_1g_1(x, t, u_1, 0) + \epsilon_1 M_{12}(x, t)u_2, & \text{in } \Omega \times (0, \infty) \\
\frac{\partial u_2}{\partial t} &\leq A_2(t)u_2 + u_2g_2(x, t, u_2, 0) + \epsilon_1 M_{21}(x, t)u_1, & \text{in } \Omega \times (0, \infty) \\
B_iu_i &= 0, & 1 \leq i \leq 2, \text{ on } \partial \Omega \times (0, \infty).
\end{aligned}
\]

(3.2)
Clearly, the following periodic parabolic system

\[
\begin{aligned}
\frac{\partial u_1}{\partial t} &= A_1(t)u_1 + u_1 g_1(x,t,u_1,0) + \epsilon_1 M_{12}(x,t)u_2, \quad \text{in } \Omega \times (0,\infty) \\
\frac{\partial u_2}{\partial t} &= A_2(t)u_2 + u_2 g_2(x,t,0,u_2) + \epsilon_1 M_{21}(x,t)u_1, \quad \text{in } \Omega \times (0,\infty) \\
B_i u_i &= 0, \quad 1 \leq i \leq 2, \text{ on } \partial \Omega \times (0,\infty)
\end{aligned}
\]  

(3.3)

is quasimonotone (i.e., cooperative) on \( X^+ \). Moreover, the reaction term

\( h(x,t,u) \)

in the right hand side of (3.3) is strongly sublinear on \( \mathbb{R}_+^2 \) in the sense that

\( h(x,t,\alpha u) \geq \alpha h(x,t,u) \)

for all \( \alpha \in (0,1), x \in \bar{\Omega}, \) and \( u \gg 0; \)

here \( u \gg v \) means \( u - v \in \text{int}(\mathbb{R}_+^2) \). By the choice of \( \epsilon_i \), it easily follows that for each \( K \geq K_0, [0,K]^2 \) is positively invariant with respect to solutions of (3.3). Then solutions of (3.3) are uniformly bounded. Moreover, by applying [15, Theorem 2.3] to the Poincaré map associated with the periodic system (3.3) on the ordered interval \([0,0),(K,K)]_{X^+}\), we get the global attractivity of either the trivial solution or a unique positive periodic solution of (3.3). Thus the standard comparison theorem of quasimonotone parabolic systems completes the proof. \( \blacksquare \)

If \( \mu(A, g(\cdot , 0, 0)) < 0, i = 1, 2 \), by [15, Theorem 3.3], two scalar periodic parabolic equations

\[
\begin{aligned}
\frac{\partial u_1}{\partial t} &= A_1(t)u_1 + u_1 g_1(x,t,u_1,0), \quad \text{in } \Omega \times (0,\infty) \\
B_i u_i &= 0, \quad \text{on } \partial \Omega \times (0,\infty)
\end{aligned}
\]  

(3.4)

and

\[
\begin{aligned}
\frac{\partial u_2}{\partial t} &= A_2(t)u_2 + u_2 g_2(x,t,0,u_2), \quad \text{in } \Omega \times (0,\infty) \\
B_i u_i &= 0, \quad \text{on } \partial \Omega \times (0,\infty)
\end{aligned}
\]  

(3.5)

admit two positive \( T \)-periodic solutions \( u_i^+(x,t) \) and \( u_i^+(x,t) \), respectively, and they are globally attractive with respect to positive initial values.

**Theorem 3.1 (Competitive Exclusion).** Assume that

(A1) \( \mu(A_i(\cdot), g_i(\cdot,0,0)) < 0, i = 1, 2; \)

(A2) \( \mu(A_1(t), g_1(x,t,0,u_2^+(x,t))) < 0, \mu(A_2(t), g_2(x,t,u_1^+(x,t),0)) > 0; \)
(A3) The unperturbed system (3.1) with \( \epsilon = 0 \) admits no positive periodic solution.

Then there exists \( \delta > 0 \) such that for each \( \epsilon \in [0, \delta] \), the perturbed system (3.1) admits a semitrivial \( T \)-periodic solution \((u^\epsilon_i(x, t, \epsilon), 0)\) with \( u^\epsilon_i(x, t, \epsilon) > 0 \), \( \forall x \in \Omega \), \( t \in \mathbb{R}_+ \), and \( u^\epsilon_i(x, t, 0) \equiv u^\epsilon_i(x, t) \), and for each \( \phi = (\phi_1, \phi_2) \in X^+ \) with \( \phi_1(\cdot) \neq 0 \), \( \lim_{t \to -\infty}(u(x, t, \phi, \epsilon) - (u^\epsilon_i(x, t, \epsilon), 0)) = 0 \) uniformly for \( x \in \Omega \).

**Proof.** For any \( \epsilon \in [0, \epsilon_1] \), let \( S_\epsilon = S(\epsilon, \cdot) : X^+ \to X^+ \) be the Poincaré map associated with (3.1), that is, \( S(\epsilon, \phi) = u(\cdot, T, \phi, \epsilon) \), \( \phi \in X^+ \). Then \( S(\cdot, \cdot) : [0, \epsilon_1] \times X^+ \to X^+ \) is continuous. By Lemma 3.1, it follows that for each \( \epsilon \in [0, \epsilon_1] \), \( S_\epsilon : X^+ \to X^+ \) is compact and point dissipative uniformly for \( \epsilon \in [0, \epsilon_1] \); i.e., there exists a bounded and closed subset \( B_0 \) of \( X^+ \), independent of \( \epsilon \in [0, \epsilon_1] \), such that for any \( \phi \in X^+ \), \( \epsilon \in [0, \epsilon_1] \), there exists \( n_0 = n_0(\phi, \epsilon) \) such that \( S_\epsilon^n(\phi) \in B_0 \) for all \( n \geq n_0 \). Then, by [3, Theorem 2.4.7], for each \( \epsilon \in [0, \epsilon_1] \), there exists a global attractor \( A_\epsilon \) for \( S_\epsilon : X^+ \to X^+ \). Clearly, \( A_\epsilon \subset B_0 \). By Lemmas 3.1 and 2.2, \( S(\epsilon, \cdot) : [0, \epsilon_1] \to X^+ \) is continuous uniformly for \( \phi \) in any bounded subset of \( X^+ \). It then easily follows that for any bounded subset \( B \subset X^+ \), \( \bigcup_{\epsilon \in [0, \epsilon_1]} S_\epsilon(B) \) is compact in \( X^+ \). Thus \( \bigcup_{\epsilon \in [0, \epsilon_1]} S_\epsilon(B_0) \) is compact in \( X^+ \). Let

\[ X^+_0 = \left\{ (\phi_1, \phi_2) \in X^+ : \phi_1(\cdot) \neq 0 \right\} \quad \text{and} \quad \partial X^+_0 = X^+ \setminus X^+_0. \]

Then \( S_\epsilon : X^+_0 \to X^+_0 \) and \( S_\epsilon : \partial X^+_0 \to \partial X^+_0 \). Let \( M_0 = (0, 0) \), \( M_1 = (u^\epsilon_1(\cdot, 0), 0) \), and \( M_2 = (0, u^\epsilon_2(\cdot, 0)) \). Then \( S_\epsilon(M_i) = M_i \), \( i = 0, 1, 2 \). Since \( u^\epsilon_i(x, t) \) is a positive \( T \)-periodic solution of (3.4), we get

\[ \mu(A(\cdot), g_j(x, t, u^\epsilon_i(x, t), 0)) = 0. \]

Clearly, \( g_j(x, t, u^\epsilon_i(x, t), 0) > g_j(x, t, u^\epsilon_i(x, t), 0) + u^\epsilon_i(x, t) \times (\partial g_j(x, t, u^\epsilon_i(x, t), 0) / \partial u_i) \). By [5, Lemma 15.5], we have

\[ \mu \left( A(\cdot), g_j(x, t, u^\epsilon_i(x, t), 0) + u^\epsilon_i(x, t) \frac{\partial g_j(x, t, u^\epsilon_i(x, t), 0)}{\partial u_i} \right) > 0, \]

which, together with the assumption \( \mu(A_j(t), g_j(x, t, u^\epsilon_i(x, t), 0)) > 0 \), implies that \( (u^\epsilon_i(x, t), 0) \) is linearly stable in the sense that the spectral radius of the Poincaré map associated with the linearized periodic system of the unperturbed system at \( (u^\epsilon_i(x, t), 0) \) is less than 1. Thus \( r(D_k S(0, M_j)) < 1 \). By the standard theory of abstract competitive systems (see [6, 8]), we further have \( \lim_{n \to \infty} S_0^n \phi = M_1 \) for every \( \phi \in X_0 \). It is easy to see that \( \{M_0, M_2\} \) is an acyclic covering of \( S_0 \) in \( \partial X_0 \). By Lemma 2.1, there exists \( \delta_1 > 0 \) such that for each \( \epsilon \in [0, \delta_1] \), and any \( \phi \in X_0 \),

\[ \lim_{n \to \infty} \|S_\epsilon^n(\phi) - M_i\| = \lim_{n \to \infty} \|u(\cdot, nT, \phi, \epsilon) - M_i\| \geq \delta_1, \quad i = 0, 2. \]
By Theorem A in Section 2, it then follows that there exists \( \delta_2 \in (0, \delta_1] \) such that for any \( \epsilon \in [0, \delta_2] \), \( \phi \in X_0 \), \( \liminf_{n \to \infty} d(S^n_{\epsilon} \phi, \partial X_0) \geq \delta_2 \). Thus there exists a global attractor \( A^0_{\epsilon} \) for \( S_{\epsilon}: X_0 \to X_0 \) (see, e.g., [14, Theorem 2.3]). Moreover, there exists a bounded and closed subset \( B^*_0 \) of \( X_0 \) such that \( A^0_{\epsilon} \subset B^*_0 \) for all \( \epsilon \in [0, \delta_2] \). Then there exists such that for any \( \epsilon \in [0, \delta_2] \). Then \( \bigcup_{\epsilon \in [0, \delta_2]} S_{\epsilon}\left(A^0_{\epsilon}\right) \subset \bigcup_{\epsilon \in [0, \delta_2]} S_{\epsilon}\left(B^*_0\right) \) is compact, and \( \bigcup_{\epsilon \in [0, \delta_2]} S_{\epsilon}\left(A^0_{\epsilon}\right) = \bigcup_{\epsilon \in [0, \delta_2]} A^0_{\epsilon} \subset B^*_0 \subset X_0 \). By [12, Theorem 2.1 and Remark 2.1] on the perturbation of a globally stable fixed point, as applied to \( S_{\epsilon}(\cdot): X^+ \to X^+ \) with \( U = X_0 \) and \( B_{\epsilon} = A^0_{\epsilon} \), \( \epsilon \in [0, \delta_2] \), there exists \( \delta > 0 \) such that for each \( \epsilon \in [0, \delta] \), \( S_{\epsilon} \) admits a fixed point \( \phi^*_{\epsilon} \in X_0 \) with \( \phi^*_{\epsilon} = M_{\epsilon} \), and for each \( \phi \in X_0 \), \( \lim_{n \to \infty} S^n_{\epsilon} \phi = \lim_{n \to \infty} u(\cdot, nt, \phi, \epsilon) = \phi^*_{\epsilon} \).

By taking \( \phi = (\phi_1, 0) \in X_0 \), we get \( u(x, t, \phi, \epsilon) = (u_i(x, t, \phi, \epsilon), 0) \), and hence each \( \phi^*_{\epsilon} \) of the form \( (\phi^*_{\epsilon}, 0) \) with \( \phi^*_{\epsilon}(x) > 0, \forall x \in \Omega \).

**Theorem 3.2 (Competitive Coexistence).** Let \( e_0 \in \text{int}(C^1_0(\overline{\Omega}, R_+)) \) be fixed. Assume that (A1) holds and

\[
\mu(A_1(t), g_1(x, t, 0, u^*_2(x, t))) < 0, \quad \mu(A_2(t), g_2(x, t, u^*_2(x, t), 0)) < 0.
\]

Then there exists \( \beta > 0 \) such that for each \( \epsilon \in [0, \beta] \), (3.1) admits at least one positive T-periodic solution, and for any \( \phi = (\phi_1, \phi_2) \in X^+ \) with \( \phi_t(\cdot) \neq 0, i = 1, 2 \), there exists \( t_0 = t_0(\phi, \epsilon) > 0 \) such that

\[
u_i(x, t, \phi, \epsilon) \geq \beta e_i(x), \quad i = 1, 2, \text{ for all } x \in \Omega, \ t \geq t_0,
\]

where \( e_i = 1 \) if \( B_i u_i = \partial u_i / \partial n + a_i u_i \), and \( e_i = e_0 \) if \( B_i u_i = u_i \).

**Proof:** Let \( S_{\epsilon}: X^+ \to X^+, A_{\epsilon}, B_{\epsilon}, \) and \( M_{\epsilon} \) be as in the proof of Theorem 3.1. Let

\[
Y_0 = \{(\phi_1, \phi_2) \in X^+: \phi_i \neq 0, i = 1, 2\} \quad \text{and} \quad \partial Y_0 = X^+ \setminus Y_0.
\]

Then \( S_{\epsilon}: Y_0 \to Y_0 \) and \( S_{\epsilon}: \partial Y_0 \to \partial Y_0 \). Clearly, \( \{M_0, M_1, M_2\} \) is an acyclic covering of \( S_{\epsilon} \) on \( \partial Y_0 \). By Lemma 2.1, there exists \( \delta_1 > 0 \) such that for each \( \epsilon \in [0, \delta_1] \), and any \( \phi \in Y_0 \),

\[
\limsup_{n \to \infty} \|S^n_{\epsilon}(\phi) - M_i\| = \limsup_{n \to \infty} \|u(\cdot, nt, \phi, \epsilon) - M_i\| \geq \delta_1, \quad i = 0, 1, 2.
\]

By Theorem A in Section 2, it follows that there exist \( \delta_2 \in (0, \delta_1] \) such that for any \( \epsilon \in [0, \delta_2] \), \( \phi \in Y_0 \), \( \liminf_{n \to \infty} d(S^n_{\epsilon} \phi, \partial Y_0) \geq \delta_2 \). Then, by [14, Theorem 2.3], \( S_{\epsilon} \) admits a fixed point \( S_{\epsilon}(\phi_2) = \phi_2 \in Y_0 \), and hence the parabolic maximum principle implies that (3.1) admits a positive T-periodic solution \( u(x, t, \phi, \epsilon) \). Let \( A^0_{\epsilon}, \epsilon \in [0, \delta_2] \), be the global attractor for \( S_{\epsilon} \).
Theorem 3.1. In the case where \( u \) satisfies \( \lim_{t \to \infty} u(t) = 0 \), it then follows that \( A_\epsilon^0 \) is upper semi-continuous in \( \epsilon \in [0, \delta_2] \) (see, e.g., [3, Theorem 2.5.2]). In the case where \( \epsilon = 0 \), by the assumption (A4) and the standard theory of abstract competitive systems (see [6, 8]), the unperturbed periodic parabolic system admits two positive \( T \)-periodic solutions \( (\bar{u}_{1,0}(x,t), \bar{u}_{2,0}(x,t)) \) and \( (\bar{u}_{1,1}(x,t), \bar{u}_{2,1}(x,t)) \) with \( \bar{u}_{i,j}(x,t) \geq \bar{u}_{1,0}(x,t), i = 1, 2, \forall x \in \Omega, t \in \mathbb{R}_+ \), and for any \( \phi = (\phi_1, \phi_2) \in X^+ \) with \( \phi_i \neq 0, i = 1, 2, u(x,t,0) = (u_1(x,t), u_2(x,t)) \) satisfies

\[
\lim_{t \to \infty} d(u_i(x,t), [\bar{u}_{i,0}(x,t), \bar{u}_{i,1}(x,t)]) = 0, \quad i = 1, 2, \text{ uniformly for } x \in \Omega.
\]

By the upper semi-continuity of \( A_\epsilon^0 \) at \( \epsilon = 0 \) and [14, Theorem 2.1], we then get the order persistence stated in Theorem 3.2.

As applications of Theorems 3.1 and 3.2 and the results in [2, 9], we consider a reaction-diffusion model for two phenotypes of a species in a heterogeneous time-periodic environment

\[
\begin{aligned}
\begin{cases}
\frac{\partial u_i}{\partial t} = d_i \Delta u_i + u_i(a(x,t) - u_1 - u_2) + \epsilon \sum_{j=1}^{2} M_{ij}(x,t) u_j, \\
\frac{\partial u_i}{\partial n} = 0, \quad 1 \leq i \leq 2, \text{ on } \partial \Omega \times (0, \infty)
\end{cases}
\end{aligned}
\]  

(3.6)

where \( 0 < d_1 < d_2 \), \( a(x,t) \), and \( M_{ij}(x,t) \) are \( C^2 \)-smooth and \( T \)-periodic in \( t \). We assume (H3) holds.

**Example 1.** Let \( a(x,t) = a(x), \forall t \in \mathbb{R} \). Assume that either

1. \( a(x) > 0, \forall x \in \Omega, \) or
2. for some \( x_0 \in \Omega, a(x_0) > 0 \) and \( 0 < d_1 < d_2 \) are sufficiently small.

Then for sufficiently small \( \epsilon > 0 \), system (3.6) admits a semitrivial \( T \)-periodic solution \( (u^\epsilon_1(x,t), 0, 0) \) with \( u^\epsilon_1(x,t) > 0, \forall x \in \Omega, t \in \mathbb{R}_+ \), and for each \( \phi = (\phi_1, \phi_2) \in X^+ \) with \( \phi_i(\cdot) \neq 0 \), the solution \( u(x,t, \phi, \epsilon) \) of (3.6) satisfies \( \lim_{t \to \infty} (u(x,t, \phi, \epsilon) - (u^\epsilon_1(x,t, \epsilon), 0)) = 0 \) uniformly for \( x \in \Omega \).

**Proof.** This is the consequence of [2, Theorem 3.2; 9, Lemma 2.4] and Theorem 3.1.
EXAMPLE 2. Let \( b(x,t) \) be \( C^2 \)-smooth and 1-periodic in \( t \), and \( a(x,t) := b(x, \omega t), \omega > 0 \). Assume that \( \int_0^1 b(x,t) \, dt > 0 \) and \( b(x) := \int_0^1 b(x,t) \, dt \) is not constant. Then the conclusion of Example 1 with \( T = \frac{1}{\omega} \) holds in each of the following cases:

1. \( d_2 \) is large enough;
2. \( \omega \) is large enough.

Proof. This is the consequence of [9, Lemmas 3.3 and 3.6 and Theorem 5.3] and Theorem 3.1.

EXAMPLE 3. There exist a smooth function \( b(x,t) \), 1-periodic in \( t \), and \( \beta > 0 \) such that for \( d > d_1 > 0 \) sufficiently close to \( d \), and sufficiently small \( \varepsilon > 0 \), system (3.6) with \( a(x,t) := b(x, \omega t) \) and \( T = \frac{1}{\omega}, \omega > 0 \), admits a semitrivial \( \frac{1}{\omega} \)-periodic solution \((0, u_0^\varepsilon(x,t,\varepsilon)) \) with \( u_0^\varepsilon(x,t,\varepsilon) > 0 \), \( \forall x \in \Omega, t \in \mathbb{R}_+ \), and for each \( \phi = (\phi_1, \phi_2) \in X^+ \) with \( \phi_i(\cdot) \neq 0 \), the solution \( u(x,t,\phi,\varepsilon) \) of (3.6) satisfies \( \lim_{t \to \infty} (u(x,t,\phi,\varepsilon) - (0, u_0^\varepsilon(x,t,\varepsilon))) = 0 \) uniformly for \( x \in \Omega \).

Proof. This is the consequence of [9, Lemma 3.9 and Theorem 5.2] and Theorem 3.1 (after exchanging \( u_1 \) and \( u_2 \)).

EXAMPLE 4. There exists a class \( \Gamma \) of smooth, 1-periodic in \( t \), functions such that for each \( b(\cdot, \cdot) \in \Gamma \), there are \( \omega > 0, 0 < d < d_1 < d_2, \beta > 0 \), such that for sufficiently small \( \varepsilon > 0 \), (3.6) with \( a(x,t) := b(x, \omega t) \) and \( T = \frac{1}{\omega} \) admits at least one positive \( \frac{1}{\omega} \)-periodic solution, and for any \( \phi = (\phi_1, \phi_2) \in X^+ \) with \( \phi_i(\cdot) \neq 0, i = 1, 2 \), there exists \( t_0 = t_0(\phi, \varepsilon) \) such that

\[
u_i(x,t,\phi,\varepsilon) \geq \beta, \quad i = 1, 2, \quad \text{for all } x \in \Omega, t \geq t_0.
\]

Proof. This is the consequence of [9, Lemmas 3.4 and 3.6] and Theorem 3.2.

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