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A Comparison Theorem for Permanents and a Proof of a Conjecture on (t, m)-Families*

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A comparison theorem for permanents is established and it is used to prove a conjecture on (t, m)-families. © 1992 Academic Press, Inc.

1. INTRODUCTION

Let $F_1, F_2, ..., F_m$ be m sets and $S = \bigcup_{i=1}^{m} F_i$. A sequence $(f_1, f_2, ..., f_m)$ of m distinct elements of S is said to form a system of distinct representatives (SDR) of the family $F = (F_1, F_2, ..., F_m)$ if $f_i \in F_i$ for each $1 \leqslant i \leqslant m$. Two SDRs are distinct if they are different as sequences. Let $N(F)$ denote the number of distinct SDRs of the family F. The problem of finding the value of and the bounds for $N(F)$ has been investigated extensively in the literature. For details, see, for example, [1, 3–6].

Recently, G. J. Chang [2] considered the following problem. Let t be a nonnegative integer. A family $F = (F_1, F_2, ..., F_m)$ is called a (t, m)-family if

$$\left|\bigcup_{i \in I} F_i\right| \geqslant |I| + t \qquad \text{for any nonempty subset } I \subseteq [1, m].$$

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The problem proposed by Chang [2] is: What is the value of

$$M(t, m) = \min\{N(F) \mid F \text{ is a } (t, m)\text{-family}\}?$$

It is easy to see that for the family $F^*_{t,m} = (F^*_1, F^*_2, ..., F^*_m)$ with

$$F^*_i = \{i, m+1, m+2, ..., m+t\}, \qquad 1 \leqslant i \leqslant m,$$

the value of $N(F^*_{t,m})$ is

$$U(t, m) = \sum_{j=0}^{\min(t,m)} j! \binom{t}{j} \binom{m}{j}.$$

He [2] proved that

$$M(t, m) = U(t, m) \qquad \text{for} \quad t = 0, 1, \text{ and } 2,$$

and that $F^*_{t,m}$ is the only (t, m)-family with

$$N(F) = M(t, m) \qquad \text{for} \quad t = 2.$$

He [2] also determined all the (t, m)-families with $N(F) = M(t, m)$ for $t = 0$ and 1. Based on these results, he conjectured that

$$M(t, m) = U(t, m) \qquad \text{for} \quad t \geqslant 3, \tag{1.1}$$

and that $F^*_{t,m}$ is the only (t, m)-family with

$$N(F) = M(t, m) \qquad \text{for all} \quad t \geqslant 3. \tag{1.2}$$

In this paper we first prove a comparison theorem for permanents of matrices in Section 2, and then use the result to prove the conjecture in Section 3.

2. A Comparison Theorem for Permanents

Let $B = (b_{i,j})$ be an $m \times n$ matrix over a ring R. We define the permanent of B as

$$\operatorname{per}(B) = \sum_{j_1 j_2 \cdots j_m} \prod_{i=1}^{m} b_{i,j_i}, \tag{2.1}$$

where $j_1 j_2 \cdots j_m$ is an m-permutation of $[1, n]$. When $m > n$, the sum on the right-hand side of (2.1) is 0.

Let $S=\bigcup_{i=1}^{m} F_i=\{s_1, s_2, ..., s_n\}$. Let $A=(a_{i,j})$ be the incidence matrix of the family $F=(F_1, F_2, ..., F_m)$; i.e.,

$$a_{i,j}=\begin{cases}1 & \text{if } \ s_j\in F_i\\ 0 & \text{otherwise.}\end{cases}$$

Then $N(F)=\operatorname{per}(A)$.

We write A as

$$A=\begin{pmatrix}A_1\\ A_2\\ \vdots\\ A_m\end{pmatrix},$$

where A_i is the ith row vector of A. For a vector V, we define

$$\|V\|=\text{the number of nonzero components in } V.$$

Then a family F is a (t, m)-family iff its incidence matrix A satisfies

$$\left\|\sum_{i\in I} A_i\right\|\geqslant |I|+t \qquad \text{for any nonempty subset } I\subseteq[1, m]. \tag{2.2}$$

A (0, 1)-matrix with this property will be called a (t, m)-matrix. Then we have

$$M(t, m)=\min\{\operatorname{per}(A)\mid A \text{ is a } (t, m)\text{-matrix}\}, \tag{2.3}$$

and the conjecture becomes

Conjecture M.

$$M(t, m)=U(t, m) \qquad \text{for all} \quad t\geqslant 3, \tag{2.4}$$

and the matrices that achieve the minimal value in (2.3) are only those that can be obtained by row and column permutations of the following matrix A^*:

$$A^*=\left(\underbrace{\begin{array}{ccccc}1 & & & \bigcirc & \\ & 1 & & & \\ & & \ddots & & \\ & & & & \\ & \bigcirc & & 1 & \\ & & & & 1\end{array}}_{m}\ \left|\ \underbrace{\begin{array}{c}1\cdots 1\\ 1\cdots 1\\ \cdots\\ \cdots\\ 1\cdots 1\\ 1\cdots 1\end{array}}_{t}\ \right|\ \underbrace{\begin{array}{c} \\ \\ \bigcirc \\ \\ \\ \end{array}}_{n-m-t}\right).$$

For convenience, we adopt some notation from Minc [6]. Let

$$\Gamma_{r,n} = \{w = (w_1, w_2, ..., w_r) \mid 1 \leqslant w_i \leqslant n \text{ for all } 1 \leqslant i \leqslant r\},$$

$$Q_{r,n} = \{(w_1, w_2, ..., w_r) \in \Gamma_{r,n} \mid 1 \leqslant w_1 < w_2 < \cdots < w_r \leqslant n\}.$$

Let B be an $m \times n$ matrix, $\alpha \in Q_{h,m}$ and $\beta \in Q_{k,n}$. Let

$B[\alpha \mid \beta]$ denote the $h \times k$ submatrix of B formed by the rows α and the columns β,

$B(\alpha \mid \beta)$ denote the $(m-h) \times (n-k)$ submatrix of B complimentary to $B[\alpha \mid \beta]$,

$B[\alpha \mid \beta)$ denote the $h \times (n-k)$ submatrix of B formed by the rows α and the columns that are not in β,

$B(\alpha \mid \beta]$ denote the $(m-h) \times k$ submatrix of B formed by the columns β and the rows that are not in α,

$B(- \mid \beta] = B[\,[1, m] \mid \beta]$, $B[\alpha \mid -\,) = B[\alpha \mid [1, n]\,]$,

$B(- \mid \beta) = B[\,[1, m] \mid \beta)$, $B(\alpha \mid -\,) = B(\alpha \mid [1, n]\,]$.

The following lemma is needed for the proof of Theorem 1; its proof can be found in [6].

LEMMA 1. *If B is an $m \times m$ matrix, $m \geqslant 2$ and $\beta \in Q_{r,m}$, then*

$$\operatorname{per}(B) = \sum_{\alpha \in Q_{r,m}} \operatorname{per}(B[\alpha \mid \beta]) \times \operatorname{per}(B(\alpha \mid \beta)).$$

In particular, for any $j \in [1, m]$,

$$\operatorname{per}(B) = \sum_{i=1}^{m} b_{i,j} \times \operatorname{per}(B(i \mid j)).$$

Our comparison theorem for permanents can be stated as follows.

THEOREM 1. *Let $B = (b_{i,j})$ be an $m \times n$ $(0, 1)$-matrix, $m \leqslant n$, and p and q be given, $1 \leqslant p < q \leqslant n$. Suppose $\hat{B} = (\hat{b}_{i,j})$ is obtained from B by changing the pth and qth columns as follows:*

$$(\hat{b}_{i,p}, \hat{b}_{i,q}) = \begin{cases} (1, 0) & \text{if } (b_{i,p}, b_{i,q}) = (0, 1), \\ (b_{i,p}, b_{i,q}) & \text{otherwise.} \end{cases} \tag{2.6}$$

Then

$$\operatorname{per}(B) \geqslant \operatorname{per}(\hat{B}). \tag{2.7}$$

And the strict inequality in (2.7) *holds iff there are two indices i and j such that*

$$\begin{pmatrix} b_{i,p} & b_{i,q} \\ b_{j,p} & b_{j,q} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad or \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.8}$$

and

$$\operatorname{per}(B(i, j \mid p, q)) \neq 0. \tag{2.9}$$

Note. This theorem is similar to the lemma proved by Brualdi *et al.* [1]. However, their lemma requires the following condition to hold: Column q of matrix B dominates column p, i.e.,

$$b_{iq} \geqslant b_{ip} \qquad (1 \leqslant i \leqslant m)$$

and

$$\sum_{i=1}^{m} (b_{iq} - b_{ip}) \geqslant 2.$$

Our theorem does not require the above condition to hold.

Proof. Since the permanent of a matrix is invariant under row and column permutations, we may assume, without loss of generality, that $(p, q) = (1, 2)$. Let $I \subseteq [1, m]$ be the subset of indices such that $(b_{i,1}, b_{i,2}) = (0, 1)$ for each $i \in I$. If $I = \varnothing$, the conclusion is trivial. Thus, we may assume that $|I| \geqslant 1$.

By definition, we have

$$\operatorname{per}(\hat{B}) = \sum_{w \in Q_{m,n}} \operatorname{per}(\hat{B}(- \mid w]). \tag{2.10}$$

All the w's in $Q_{m,n}$ can be divided into four types and so can all the submatrices in $\{\hat{B}(- \mid w] \mid w \in Q_{m,n}\}$:

$$\begin{aligned} U_1 &= \{\hat{B}(- \mid w] \mid w \in Q_{m,n} \text{ and } w_1 \geqslant 3\}, \\ U_2 &= \{\hat{B}(- \mid w] \mid w \in Q_{m,n} \text{ and } w_1 = 2\}, \\ U_3 &= \{\hat{B}(- \mid w] \mid w \in Q_{m,n},\ w_1 = 1 \text{ and } w_2 \geqslant 3\}, \\ U_4 &= \{\hat{B}(- \mid w] \mid w \in Q_{m,n},\ w_1 = 1 \text{ and } w_2 = 2\}. \end{aligned}$$

Clearly, if $\hat{B}(- \mid w] \in U_1$, then

$$\operatorname{per}(\hat{B}(- \mid w]) = \operatorname{per}(B(- \mid w]). \tag{2.11}$$

If $\hat{B}(- \mid w] \in U_2$, then by Lemma 1,

$$\operatorname{per}(\hat{B}(-\mid w]) = \sum_{i=1}^{m} \hat{b}_{i,2} \times \operatorname{per}(\hat{B}(i \mid w_2, ..., w_m]) = \sum_{i \in \ddot{I}} b_{i,2} \times \operatorname{per}(B(i \mid w_2, ..., w_m]),$$

where $\ddot{I} = [1, m] - I$. For $\hat{B}(- \mid w] \in U_3$, we have

$$\operatorname{per}(\hat{B}(-\mid w]) = \sum_{i=1}^{m} \hat{b}_{i,1} \times \operatorname{per}(\hat{B}(i \mid w_2, ..., w_m]) = \sum_{i \in I} b_{i,2} \times \operatorname{per}(B(i \mid w_2, ..., w_m]) + \sum_{i \in \ddot{I}} b_{i,1} \times \operatorname{per}(B(i \mid w_2, ..., w_m]). \tag{2.13}$$

For $\hat{B}(- \mid w] \in U_4$, we have

$$\begin{aligned}\operatorname{per}(\hat{B}(-\mid w]) &= \sum_{i<j} \operatorname{per}\left(\begin{pmatrix} \hat{b}_{i,1} & \hat{b}_{i,2} \\ \hat{b}_{j,1} & \hat{b}_{j,2} \end{pmatrix}\right) \times \operatorname{per}(\hat{B}(i, j \mid w_3, ..., w_m]) \\ &= \sum_{i<j; i,j \in I} \operatorname{per}\left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\right) \times \operatorname{per}(B(i, j \mid w_3, ..., w_m]) \\ &\quad + \sum_{i<j; i \in I, j \in \ddot{I}} \operatorname{per}\left(\begin{pmatrix} 1 & 0 \\ b_{j,1} & b_{j,2} \end{pmatrix}\right) \times \operatorname{per}(B(i, j \mid w_3, ..., w_m]) \\ &\quad + \sum_{i<j; i \in \ddot{I}, j \in I} \operatorname{per}\left(\begin{pmatrix} b_{i,1} & b_{i,2} \\ 1 & 0 \end{pmatrix}\right) \times \operatorname{per}(B(i, j \mid w_3, ..., w_m]) \\ &\quad + \sum_{i<j; i,j \in \ddot{I}} \operatorname{per}\left(\begin{pmatrix} b_{i,1} & b_{i,2} \\ b_{j,1} & b_{j,2} \end{pmatrix}\right) \times \operatorname{per}(B(i, j \mid w_3, ..., w_m]) \\ &= 0 \\ &\quad + \sum_{i<j; i \in I, j \in \ddot{I}} b_{j,2} \times \operatorname{per}(B(i, j \mid w_3, ..., w_m]) \\ &\quad + \sum_{i<j; i \in \ddot{I}, j \in I} b_{i,2} \times \operatorname{per}(B(i, j \mid w_3, ..., w_m]) \\ &\quad + \sum_{i<j; i,j \in \ddot{I}} \operatorname{per}\left(\begin{pmatrix} b_{i,1} & b_{i,2} \\ b_{j,1} & b_{j,2} \end{pmatrix}\right) \times \operatorname{per}(B(i, j \mid w_3, ..., w_m]) \\ &= \sum_{j \in I, i \in \ddot{I}} b_{i,2} \times \operatorname{per}(B(i, j \mid w_3, ..., w_m]) \\ &\quad + \sum_{i<j; i,j \in \ddot{I}} \operatorname{per}\left(\begin{pmatrix} b_{i,1} & b_{i,2} \\ b_{j,1} & b_{j,2} \end{pmatrix}\right) \times \operatorname{per}(B(i, j \mid w_3, ..., w_m]).\end{aligned} \tag{2.14}$$

Thus, (2.10)–(2.14) give

$$\begin{aligned}\operatorname{per}(\hat{B}) = &\sum_{3 \leqslant w_1 < \cdots < w_m \leqslant n} \operatorname{per}(B(- \mid w_1, ..., w_m])\\
&+ \sum_{3 \leqslant w_2 < \cdots < w_m \leqslant n} \Big(\sum_{i \in I} \operatorname{per}(B(i \mid w_2, ..., w_m])\\
&+ \sum_{i \in \bar{I}} (b_{i,1} + b_{i,2}) \times \operatorname{per}(B(i \mid w_2, ..., w_m]) \Big)\\
&+ \sum_{3 \leqslant w_3 < \cdots < w_m \leqslant n} \Big(\sum_{j \in I, i \in \bar{I}} b_{i,2} \times \operatorname{per}(B(i, j \mid w_3, ..., w_m])\\
&+ \sum_{i<j; i,j \in \bar{I}} \operatorname{per}\left(\begin{pmatrix} b_{i,1} & b_{i,2} \\ b_{j,1} & b_{j,2} \end{pmatrix}\right) \times \operatorname{per}(B(i, j \mid w_3, ..., w_m]) \Big). \qquad (2.15)\end{aligned}$$

On the other hand,

$$\begin{aligned}\operatorname{per}(B) &= \sum_{w \in Q_{m,n}} \operatorname{per}(B(- \mid w])\\
&= \sum_{3 \leqslant w_1 < \cdots < w_m \leqslant n} \operatorname{per}(B(- \mid w_1, ..., w_m])\\
&\quad + \sum_{3 \leqslant w_2 < \cdots < w_m \leqslant n} \sum_{i=1}^{m} b_{i,2} \times \operatorname{per}(B(i \mid w_2, ..., w_m])\\
&\quad + \sum_{3 \leqslant w_2 < \cdots < w_m \leqslant n} \sum_{i=1}^{m} b_{i,1} \times \operatorname{per}(B(i \mid w_2, ..., w_m])\\
&\quad + \sum_{3 \leqslant w_3 < \cdots < w_m \leqslant n} \sum_{i<j} \operatorname{per}\left(\begin{pmatrix} b_{i,1} & b_{i,2} \\ b_{j,1} & b_{j,2} \end{pmatrix}\right) \times \operatorname{per}(B(i, j \mid w_3, ..., w_m]).\end{aligned}$$

Since

$$\begin{aligned}&\sum_{i=1}^{m} b_{i,2} \times \operatorname{per}(B(i \mid w_2, ..., w_m])\\
&\qquad = \sum_{i \in I} 1 \times \operatorname{per}(B(i \mid w_2, ..., w_m]) + \sum_{i \in \bar{I}} b_{i,2} \times \operatorname{per}(B(i \mid w_2, ..., w_m]),\\
&\sum_{i=1}^{m} b_{i,1} \times \operatorname{per}(B(i \mid w_2, ..., w_m])\\
&\qquad = \sum_{i \in I} 0 \times \operatorname{per}(B(i \mid w_2, ..., w_m]) + \sum_{i \in \bar{I}} b_{i,1} \times \operatorname{per}(B(i \mid w_2, ..., w_m])\end{aligned}$$

and

$$
\begin{aligned}
&\sum_{i<j} \operatorname{per}\left(\begin{pmatrix} b_{i,1} & b_{i,2} \\ b_{j,1} & b_{j,2} \end{pmatrix}\right) \times \operatorname{per}(B(i, j \mid w_3, \ldots, w_m]) \\
&\quad = \sum_{i<j;\, i,j \in I} \operatorname{per}\left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right) \times \operatorname{per}(B(i, j \mid w_3, \ldots, w_m]) \\
&\qquad + \sum_{i<j;\, i \in I,\, j \in \bar{I}} \operatorname{per}\left(\begin{pmatrix} 0 & 1 \\ b_{j,1} & b_{j,2} \end{pmatrix}\right) \times \operatorname{per}(B(i, j \mid w_3, \ldots, w_m]) \\
&\qquad + \sum_{i<j;\, i \in \bar{I},\, j \in I} \operatorname{per}\left(\begin{pmatrix} b_{i,1} & b_{i,2} \\ 0 & 1 \end{pmatrix}\right) \times \operatorname{per}(B(i, j \mid w_3, \ldots, w_m]) \\
&\qquad + \sum_{i<j;\, i,j \in \bar{I}} \operatorname{per}\left(\begin{pmatrix} b_{i,1} & b_{i,2} \\ b_{j,1} & b_{j,2} \end{pmatrix}\right) \times \operatorname{per}(B(i, j \mid w_3, \ldots, w_m]) \\
&\quad = \sum_{i<j;\, i \in I,\, j \in \bar{I}} b_{j,1} \times \operatorname{per}(B(i, j \mid w_3, \ldots, w_m]) \\
&\qquad + \sum_{i<j;\, i \in \bar{I},\, j \in I} b_{i,1} \times \operatorname{per}(B(i, j \mid w_3, \ldots, w_m]) \\
&\qquad + \sum_{i<j;\, i,j \in \bar{I}} \operatorname{per}\left(\begin{pmatrix} b_{i,1} & b_{i,2} \\ b_{j,1} & b_{j,2} \end{pmatrix}\right) \times \operatorname{per}(B(i, j \mid w_3, \ldots, w_m]) \\
&\quad = \sum_{j \in I,\, i \in \bar{I}} b_{i,1} \times \operatorname{per}(B(i, j \mid w_3, \ldots, w_m]) \\
&\qquad + \sum_{i<j;\, i,j \in \bar{I}} \operatorname{per}\left(\begin{pmatrix} b_{i,1} & b_{i,2} \\ b_{j,1} & b_{j,2} \end{pmatrix}\right) \times \operatorname{per}(B(i, j \mid w_3, \ldots, w_m]),
\end{aligned}
$$

we have

$$
\begin{aligned}
\operatorname{per}(B) = &\sum_{3 \leqslant w_1 < \cdots < w_m \leqslant n} \operatorname{per}(B(- \mid w_1, \ldots, w_m]) \\
&+ \sum_{3 \leqslant w_2 < \cdots < w_m \leqslant n} \left(\sum_{i \in I} \operatorname{per}(B(i \mid w_2, \ldots, w_m]) \right. \\
&\left. + \sum_{i \in \bar{I}} (b_{i,1} + b_{i,2}) \times \operatorname{per}(B(i \mid w_2, \ldots, w_m]) \right) \\
&+ \sum_{3 \leqslant w_3 < \cdots < w_m \leqslant n} \left(\sum_{j \in I,\, i \in \bar{I}} b_{i,1} \times \operatorname{per}(B(i, j \mid w_3, \ldots, w_m]) \right. \\
&\left. + \sum_{i<j;\, i,j \in \bar{I}} \operatorname{per}\left(\begin{pmatrix} b_{i,1} & b_{i,2} \\ b_{j,1} & b_{j,2} \end{pmatrix}\right) \times \operatorname{per}(B(i, j \mid w_3, \ldots, w_m]) \right). \qquad (2.16)
\end{aligned}
$$

Note that when $i \in \bar{I}$, $b_{i,2} = 1$ implies $b_{i,1} = 1$. Thus, we have

$$\sum_{3 \leqslant w_3 < \cdots < w_m \leqslant n} \sum_{j \in I, i \in \ddot{I}} b_{i,2} \times \operatorname{per}(B(i, j \mid w_3, ..., w_m])$$

$$\leqslant \sum_{3 \leqslant w_3 < \cdots < w_m \leqslant n} \sum_{j \in I, i \in \ddot{I}} b_{i,1} \times \operatorname{per}(B(i, j \mid w_3, ..., w_m]). \quad (2.17)$$

Combining (2.15)–(2.17), we obtain

$$\operatorname{per}(\hat{B}) \leqslant \operatorname{per}(B), \quad (2.18)$$

and the strict inequality in (2.18) holds iff the strict inequality in (2.17) holds. The latter is equivalent to the existence of indices $w_3, ..., w_m$, i and j such that $i \in \ddot{I}$, $j \in I$, $(b_{i,1}, b_{i,2}) = (1, 0)$, $3 \leqslant w_3 < \cdots < w_m \leqslant n$, and $\operatorname{per}(B(i, j \mid w_3, ..., w_m]) \neq 0$. This is just the same as (2.8) and (2.9) with $p = 1$ and $q = 2$. This completes the proof. ▌

3. Proof of Conjecture M

In this section we establish the validity of Conjecture M. We first prove the following lemma.

Lemma 2. *Let* $t \geqslant 1$,

$M'(t, m)$
$= \min\{\operatorname{per}(A) \mid A$ *is a* (t, m)*-matrix with* $\|A_i\| = t + 1$ *for all* $1 \leqslant i \leqslant m\}$,

$M''(t, m)$
$= \min\{\operatorname{per}(A) \mid A$ *is a* (t, m)*-matrix with* $\|A_i\| \geqslant t + 2$ *for some* $1 \leqslant i \leqslant m\}$.

Then we have

$$M''(t, m) > M'(t, m). \quad (3.1)$$

Proof. Let A be a (t, m)-matrix with $\|A_i\| \geqslant t + 2$ for some $1 \leqslant i \leqslant m$. Since A is a (t, m)-matrix, A must satisfy (2.2). Thus, by permuting rows and columns if necessary, A can be reduced to the following form $\hat{A}$ with $\|\hat{A}_1\| \geqslant t + 2$:

$$\hat{A} = \left(\begin{array}{c|cccc|c} 1\ 1 \cdots 1 & & B & & & C \\ \hline & 1 & * & & * & \\ & * & 1 & & * & \\ D & & & \ddots & & E \\ & * & * & & 1 & \end{array}\right) \begin{array}{l} \}\,1 \\ \\ \left.\vphantom{\begin{array}{c} 1\\1\\1\\1 \end{array}}\right\} m-1, \\ \\ \end{array}$$
$$\underbrace{\qquad}_{t+1} \quad \underbrace{\qquad}_{m-1} \quad \underbrace{\qquad}_{n-m-t}$$

where $*$ stands for 0 or 1.

Let $c = \|\hat{A}_1\| - (t + 1)$. Since $\|\hat{A}_1\| \geqslant t + 2$, we have $c \geqslant 1$. Furthermore, there are exactly c ones in the submatrix (B, C) of $\hat{A}$. We have two cases to consider, depending on whether there is a one in C or not.

Case I. There is a one in C. In this case let

$$X = \left(\begin{array}{c|c|c} 1 \cdots 1 & 0 \cdots 0 & 0 \cdots 0 \\ \hline \underbrace{D}_{t+1} & \underbrace{\begin{matrix} 1 & & * \\ & \ddots & \\ * & & 1 \end{matrix}}_{m-1} & \underbrace{E}_{n-m-t} \end{array}\right) \begin{matrix} \}1 \\ \\ \}m-1 \end{matrix}$$

and

$$Y = \left(\begin{array}{c|c|c} \underbrace{D}_{t+1} & \underbrace{\begin{matrix} 1 & * & \cdots & * \\ * & 1 & & * \\ & & \ddots & \\ * & * & \cdots & 1 \end{matrix}}_{m-1} & \underbrace{0}_{n-m-t-1} \end{array}\right) \Big\} m-1.$$

If we expand $\operatorname{per}(\hat{A})$ by its first row, we see that

$$\begin{aligned} \operatorname{per}(\hat{A}) &\geqslant \operatorname{per}(X) + \operatorname{per}(Y) \\ &\geqslant \operatorname{per}(X) + 1 \\ &\geqslant \operatorname{per}(X). \end{aligned}$$

Note that X is also a (t, m)-matrix with $\|X_1\| = t + 1$ and $\|X_i\| = \|\hat{A}_i\|$ for all $2 \leqslant i \leqslant m$.

Case II. There is no one in C. In this case there are exactly c ones in B. By permuting rows and columns if necessary, we may assume that $\hat{A}$ is of the form

$$\hat{A} = \left(\begin{array}{c|c|c|c} 1 \cdots 1 & 1 \cdots 1 & 0 \cdots 0 & 0 \cdots 0 \\ \hline F & \begin{matrix} 1 & & * \\ & \ddots & \\ * & & 1 \end{matrix} & G & H \\ \hline \underbrace{*}_{t+1} & \underbrace{*}_{c} & \underbrace{\begin{matrix} 1 & & * \\ & \ddots & \\ * & & 1 \end{matrix}}_{m-1-c} & \underbrace{*}_{n-m-t} \end{array}\right) \begin{matrix} \}1 \\ \}c \\ \\ \}m-1-c, \end{matrix}$$

where $*$ stands for 0 or 1. Since

$$\|\hat{A}_2 + \cdots + \hat{A}_{c+1}\| \geqslant t + c \geqslant 1 + c,$$

there is at least one 1 in F, G, or H. Let there be a one in the ith row of F, G, or H. We have $1 \leqslant i \leqslant c$. Now, let

$$W=\left(\begin{array}{c|c|c|c} 1\cdots 1 & 0\cdots 0 & 0\cdots 0 & 0\cdots 0 \\ \hline F & \begin{matrix}1 & & *\\ & \ddots & \\ * & & 1\end{matrix} & G & H \\ \hline * & * & \begin{matrix}1 & & *\\ & \ddots & \\ * & & 1\end{matrix} & * \\ \underbrace{\qquad}_{t+1} & \underbrace{\qquad}_{c} & \underbrace{\qquad}_{m-1-c} & \underbrace{\qquad}_{n-m-t} \end{array}\right)\begin{matrix}\}\,1\\ \}\,c \\ \}\,m-1-c,\end{matrix}$$

and

$$Z=\left[\begin{array}{c|c|c|c} F & \begin{matrix}1 & & & & *\\ & \ddots & & & \\ & & 1 & & \\ & & * & & \\ & & 1 & & \\ & & & \ddots & \\ * & & & & 1\end{matrix} & G & H \\ \hline * & * & \begin{matrix}1 & & *\\ & \ddots & \\ * & & 1\end{matrix} & * \\ \underbrace{\qquad}_{t+1} & \underbrace{\qquad\qquad}_{c-1} & \underbrace{\qquad}_{m-1-c} & \underbrace{\qquad}_{n-m-t} \end{array}\right]\begin{matrix}\}\,c \\ \}\,m-1-c,\end{matrix}$$

where Z is the matrix obtained by deleting the first row and the $(t+1+i)$th column of $\hat{A}$. If we expand $\hat{A}$ by its first row, we see that

$$\begin{aligned}\operatorname{per}(\hat{A}) &\geqslant \operatorname{per}(W) + \operatorname{per}(Z)\\ &\geqslant \operatorname{per}(W) + 1\\ &> \operatorname{per}(W).\end{aligned}$$

Note that W is also a (t, m)-matrix with $\|W_1\| = t+1$ and $\|W_i\| = \|\hat{A}_i\|$ for all $2 \leqslant i \leqslant m$.

Repeating the above argument, we obtain (3.1). This completes the proof. ∎

We are now ready to prove the following theorem that covers both Theorem 3 of Chang [2] and Conjecture M.

THEOREM 2. *$M(t, m) = U(t, m)$ for all $t > 1$, and the matrices that achieve the minimal value in* (2.3) *are only those that can be obtained by row and column permutations of the matrix A^* shown in* (2.5).

Proof. By Lemma 2, it suffices to prove that $M'(t, m) = U(t, m)$. Let $A = (a_{i,j})$ be a (t, m)-matrix with exactly $t+1$ ones in each row. Since A is a (t, m)-matrix, A must satisfy (2.2). Thus, by permuting rows and columns if necessary, A can be reduced to the form

$$A = \left(\begin{array}{c|c|c} \underbrace{\begin{matrix} 1 & * & & a_{1,m} \\ & 1 & & a_{2,m} \\ * & & \cdots & \\ & & 1 & a_{m-1,m} \\ 0 \cdots 0 & & & 1 \end{matrix}}_{m} & \underbrace{\begin{matrix} \\ \\ * \\ \\ 1 \cdots 1 \end{matrix}}_{t} & \underbrace{\begin{matrix} \\ \\ * \\ \\ \\ \end{matrix}}_{n-m-t} \end{array}\right),$$

where * stands for 0 or 1.

By induction on m we now prove the following claim: By applying the procedure stated in Theorem 1 and by permuting rows and columns, we can transform A into A^*,

$$A^* = (I_m J_{m \times t} O_{m \times (n-m-t)}), \tag{3.2}$$

where I_m is the identity matrix of order m, $J_{m \times t}$ is the $m \times t$ matrix of ones, and $O_{m \times (n-m-t)}$ is the $m \times (n-m-t)$ zero matrix. Note that A^* is exactly the same matrix as in (2.5).

If $a_{i,m} = 1$ for some $1 \leqslant i \leqslant m-1$, then there must be a j, $m+1 \leqslant j \leqslant m+t$, such that $a_{i,j} = 0$. By applying the procedure in Theorem 1, A can be transformed into

$$\left(\begin{array}{c|c|c} \underbrace{\begin{matrix} 1 & * & & 0 \\ & \cdot & & \cdot \\ * & \cdot & & \cdot \\ & & \cdot & \cdot \\ & & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{matrix}} & \underbrace{\begin{matrix} \\ \\ * \\ \\ \\ 1 \cdots 1 \end{matrix}}_{t} & \underbrace{\begin{matrix} \\ \\ * \\ \\ \\ \\ \end{matrix}}_{n-m-t} \end{array}\right).$$

By permuting its columns, the above matrix can be transformed into

$$A' = \left(\begin{array}{ccc|c|cc} 1 & * & & & & 0 \\ & & & & & \cdot \\ * & \ddots & & * & * & \cdot \\ & & & & & \cdot \\ & & 1 & & & 0 \\ \multicolumn{3}{c|}{\underbrace{0 \cdots 0}_{m-1}} & \underbrace{1 \cdots 1}_{t} & \underbrace{\quad}_{n-m-t} & \underbrace{1}_{1} \end{array}\right).$$

Let $B = A'(m \mid n)$. We have

$$\|B_i\| = t + 1 \qquad \text{for each } 1 \leqslant i \leqslant m-1,$$

$$\left\| \sum_{i \in I} B_i \right\| \geqslant \|I\| + t \qquad \text{for each nonempty subset } I \subseteq [1, m-1].$$

By the induction hypothesis, B can be transformed into

$$B' = (I_{m-1} J_{(m-1)\times t} O_{(m-1)\times(n-m-t)}).$$

Corresponding, A' is transformed into

$$\hat{A} = \left(\begin{array}{ccc|ccc|ccc|c} 1 & 0 & & 1 & \cdot & 1 & & & & 0 \\ 0 & \cdot & & & & & & 0 & & \cdot \\ & \cdot & 1 & 1 & \cdot & 1 & & & & 0 \\ \multicolumn{3}{c|}{\underbrace{\hat{a}_{m,1} \cdot \hat{a}_{m,m-1}}_{m-1}} & \multicolumn{3}{c|}{\underbrace{\hat{a}_{m,m} \cdot \hat{a}_{m,m-1+t}}_{t}} & \multicolumn{3}{c|}{\underbrace{\hat{a}_{m,m+t} \cdot \hat{a}_{n-1}}_{n-m-1}} & \underbrace{1}_{1} \end{array}\right).$$

Since $\|\hat{A}_m\| = t + 1$, the number of ones in $\{\hat{a}_{m,1}, \ldots, \hat{a}_{m,m-1}, \hat{a}_{m,m+t}, \ldots, \hat{a}_{m,n-1}\}$ is exactly the same as the number of zeros in $\{\hat{a}_{m,n}, \ldots, \hat{a}_{m,m-1+t}\}$. Thus, by applying the procedure in Theorem 1, $\hat{A}$ can be transformed into

$$\left(\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1 \cdots 1 & & 0 \\ & & & & & \cdot \\ 0 & \ddots & & & 0 & \cdot \\ & & & & & \cdot \\ & & 1 & 1 \cdots 1 & & 0 \\ \multicolumn{3}{c|}{\underbrace{0 \cdots \cdots 0}_{m-1}} & \underbrace{1 \cdots 1}_{t} & \underbrace{\quad}_{n-m-t} & \underbrace{1}_{1} \end{array}\right).$$

By permuting its columns, the above matrix can finally be transformed into A^*. Hence we have $M'(t, m) = U(t, m)$.

To prove the second part of the theorem, we will show that if A cannot be transformed into A^* by row and column permutations only, then $\mathrm{per}(A) > \mathrm{per}(A^*)$. Suppose that the procedure in Theorem 1 must be used in the above transformation from A to A^*. The last time the procedure in Theorem 1 is used, A must have been transformed into one of the following two forms (up to an isomorphism of row and column permutations):

$$A_0 = \left(\begin{array}{ccc|c|c|ccc|c} 1 & & 0 & & & 1 & \cdots & 1 & \\ & \ddots & & D & E & & & & 0 \\ 0 & & 1 & & & 1 & \cdots & 1 & \\ \hline 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1 & \end{array}\right) \tag{3.3}$$

$$\underbrace{\qquad}_{m-1} \quad \underbrace{\quad}_{1} \quad \underbrace{\quad}_{1} \quad \underbrace{\qquad}_{t-1} \quad \underbrace{\qquad}_{n-m-t}$$

or

$$A_0' = \left(\begin{array}{ccc|ccc|c|c|c} 1 & & 0 & 1 & \cdots & 1 & & & \\ & \ddots & & & & & D' & E' & 0 \\ 0 & & 1 & 1 & \cdots & 1 & & & \end{array}\right), \tag{3.4}$$

$$\underbrace{\qquad}_{m} \quad \underbrace{\qquad}_{t-1} \quad \underbrace{\quad}_{1} \quad \underbrace{\quad}_{1} \quad \underbrace{\qquad}_{n-m-t-1}$$

where D and E are two $(m-1)$-dimensional $(0, 1)$-vectors, and D' and E' are two m-dimensional $(0, 1)$-vectors, with the following properties: none of D, E, D', and E' is a 0-vector, while $D + E$ and $D' + E'$ are 1-vectors.

For case (3.3), we transfer all the ones of D to the corresponding positions of E. Since D and E are not 0-vectors, there must be a submatrix of order 2 of (D, E), say $\left(\begin{smallmatrix} d_i & e_i \\ d_j & e_j \end{smallmatrix}\right)$, having the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. For case (3.4), we transfer all the ones of E' to the corresponding positions of D'. Since D' and E' are not 0-vectors, there must be a submatrix of order 2 of (D', E'), say $\left(\begin{smallmatrix} d_k' & e_k' \\ d_l' & e_l' \end{smallmatrix}\right)$, having the form $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. It is also clear that

$$\mathrm{per}(A_0(i, j \mid m, m+1)) \neq 0$$

and

$$\mathrm{per}(A_0'(i, j \mid m+t, m+t+1)) \neq 0.$$

Therefore, we have

$$\mathrm{per}(A_0) > \mathrm{per}(A^*)$$

and

$$\mathrm{per}(A_0') > \mathrm{per}(A^*).$$

This completes the proof of the theorem. ▮

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