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Examples and Counterexamples for Existence of Categorical Quotients

Annette A'Campo-Neuen and Jürgen Hausen

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We give examples for existence and non-existence of categorical quotients for algebraic group actions in the categories of algebraic varieties and prevarieties. All our examples are subtorus actions on toric varieties.

INTRODUCTION

For group actions in the category of algebraic varieties, various notions of quotients have been introduced. Among these, the *categorical quotient* is a basic concept; here one only requires universality with respect to invariant morphisms. In practice, it is a delicate problem whether or not a given action admits a categorical quotient. A possible way to obtain existence statements is to treat the problem in a suitably modified category. For example, if a finite group acts on a variety, then this action in general admits no algebraic variety as orbit space but in the category of algebraic spaces it has a geometric quotient.

In this note we investigate the effect of allowing non-separated quotient spaces on the existence of categorical quotients. Our aim is to show by means of examples that concerning categorical quotients the separated and the non-separated case behave surprisingly independent from each other. We work with the following terminology. Let G be a complex algebraic group, let $\mathfrak{R}$ denote any subcategory of the category of complex algebraic prevarieties containing G, and assume that G acts $\mathfrak{R}$-morphically on an object X of $\mathfrak{R}$. Then we will call a morphism $p \in \mathrm{Mor}_{\mathfrak{R}}(X, Y)$ a $\mathfrak{R}$-*quotient* for the action of G on X if for every G-invariant morphism $f \in \mathrm{Mor}_{\mathfrak{R}}(X, Z)$ there is a unique morphism $\tilde{f} \in \mathrm{Mor}_{\mathfrak{R}}(Y, Z)$ with $f = \tilde{f} \circ p$.

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We consider actions of subtori on complex toric varieties. In this setting, it makes sense to ask for quotients in the categories AV of complex algebraic varieties, PV of complex algebraic prevarieties, TV of complex toric varieties, and TP of complex toric prevarieties. In [1, 2] we have shown that TV- and TP-quotients always exist. For AV- and PV-quotients it is well known that these notions need not coincide if both exist (see Example 4.5). Concerning existence and non-existence we here give the following results and examples:

(i) If H is a subtorus of the big torus T of a toric variety X with $\dim(T/H) \leq 2$, then the TV-quotient for the action of H on X is also an AV-quotient (see Section 4).

(ii) A $\mathbb{C}^*$-action without AV-quotient but with PV-quotient (see Section 5).

(iii) A $\mathbb{C}^*$-action admitting neither an AV-quotient nor a PV-quotient (see Section 6).

(iv) A $\mathbb{C}^*$-action with AV-quotient and without PV-quotient (see Section 7).

The examples (ii) and (iii) in fact do not even admit a quotient in the categories of algebraic or analytic spaces. The existence result (i) is proved in a slightly more general framework. Let X be a toric prevariety and let H be a subtorus of the acting torus T of X. We call a regular map q: $X \to Y$ an *H-invariant separation* of X, if Y is a variety, q is H-invariant, and every H-invariant regular map from X to a variety Z factors uniquely through q. We prove:

THEOREM. *If* $\dim(T/H) \leq 2$, *then there exists an H-invariant separation of X.*

The present note is organized as follows. Sections 1–3 are devoted to obtaining some general criteria for existence and non-existence of H-invariant separations and categorical quotients. The main result and the examples are presented in Sections 4–7. Throughout the note we make use of the basic concepts introduced in [1, 2, 4].

Notation

We fix some notation. Let N be a lattice, i.e., a free $\mathbb{Z}$-module of finite rank. The dual lattice is $M := \mathrm{Hom}(N, \mathbb{Z})$. The canonical pairing is denoted by

$$M \times N \to \mathbb{Z}, \qquad (u, v) \mapsto u(v) =: \langle u, v \rangle.$$

Let $N_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} N$ denote the real vector space associated to N. Moreover, for a homomorphism $F\colon N \to N'$ of lattices, denote by $F_{\mathbb{R}}$ its extension to the real vector spaces associated to N and N'.

When we speak of a cone in N we always think of a convex rational polyhedral cone in $N_{\mathbb{R}}$. For two cones τ, σ in N we write $\tau \prec \sigma$ if τ is a face of σ. The relative interior of a cone $\sigma \subset N_{\mathbb{R}}$ is denoted by σ°. The dual cone of a cone σ in N is the cone

$$\sigma^{\vee} := \{u \in M;\ \forall_{v \in \sigma}\, \langle u, v \rangle \geq 0\}.$$

A *fan* in N is a finite set Δ of strictly convex cones in N such that $\sigma, \sigma' \in \Delta$ implies $\sigma \cap \sigma' \prec \sigma$ and $\sigma \in \Delta$ implies that also every face of σ lies in Δ. For example, for any given cone σ the set $\mathfrak{F}(\sigma)$ of its faces forms a fan in N. For two fans Δ, Δ' we will use the notation $\Delta \prec \Delta'$ if Δ is a subfan of Δ'.

A *system of fans* in N is a finite family $\mathscr{S} := (\Delta_{ij})_{i,j \in I}$ of fans in N such that $\Delta_{ij} = \Delta_{ji}$ and $\Delta_{ij} \cap \Delta_{jk} \prec \Delta_{ik}$ holds for any $i, j, k \in I$. In particular, one has $\Delta_{ij} \prec \Delta_{ii} \cap \Delta_{jj}$ for all $i, j \in I$. A system $(\Delta_{ij})_{i,j \in I}$ of fans is called *affine*, if for every $i \in I$ the fan Δ_{ii} is the fan of faces of a single cone $\sigma(i)$. The set of labelled cones of a system $\mathscr{S} = (\Delta_{ij})_{i,j \in I}$ of fans is

$$\mathfrak{F}(\mathscr{S}) := \{(\sigma, i);\ i \in I,\ \sigma \in \Delta_{ii}\}.$$

For a system $\mathscr{S} = (\Delta_{ij})_{i,j \in I}$ of fans in a lattice, we define its *support* to be the set

$$|\mathscr{S}| := \bigcup_{(\sigma, i) \in \mathfrak{F}(\mathscr{S})} \sigma.$$

In [2] we showed that every affine system $\mathscr{S}$ of fans defines a toric prevariety $X_{\mathscr{S}}$. Moreover, we introduced the concept of a map of systems of fans and proved that the assignment $\mathscr{S} \to X_{\mathscr{S}}$ is an equivalence of categories.

1. FACTORIZATION OF REGULAR MAPS

In this section we prove a criterion for the existence of a factorization of a regular map. The result may be of interest independent from its application in our proof of Theorem 4.1. By a (*pre*-) *variety* we mean throughout this paper an algebraic (pre-) variety over the field $\mathbb{C}$ of complex numbers. Recall that any prevariety carries in a natural manner the structure of a possibly non-Hausdorff complex analytic space.

Let X denote a prevariety. By a *local curve* in $x \in X$ we mean a holomorphic mapping germ $\gamma: \mathbb{C}_0 \to X_x$ arising from an algebraic curve; i.e., there is an algebraic curve X' in X through x with $\gamma(\mathbb{C}_0) \subset X'_x$. Let $p: X \to Y$ be a regular map of prevarieties. We say that a local curve $\tilde{\gamma}$ in $x \in X$ is a *weak p-lifting* of a local curve γ in $y \in Y$ if there is a non-constant holomorphic mapping germ $\alpha: \mathbb{C}_0 \to \mathbb{C}_0$ and a commutative diagram

$$\begin{array}{ccc} \mathbb{C}_0 & \xrightarrow{\tilde{\gamma}} & X_x \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{C}_0 & \xrightarrow{\gamma} & Y_y \end{array}$$

We call the map p *weakly proper*, if any local curve in Y admits a weak p-lifting. Note that a weakly proper map is necessarily surjective. Moreover, every proper regular map is weakly proper. For a related notion in the context of algebraic spaces see [5, Sect. 3].

1.1. PROPOSITION. *Let $p: X \to Y$ be a weakly proper regular map of prevarieties, assume that Y is normal, and let $f: X \to Z$ be a regular map into a variety. If f is constant on the fibres of p, then there is a unique regular map $\tilde{f}: Y \to Z$ such that $f = \tilde{f} \circ p$.*

Proof. By assumption $\tilde{f}$ exists as a uniquely determined map of sets. We have to show that $\tilde{f}$ is regular. Since Y was assumed to be normal, it suffices to show that the graph Γ of $\tilde{f}$ is closed with respect to the Zariski topology in $Y \times Z$. Note that

$$\Gamma = \{(p(x), f(x));\ x \in X\}.$$

Hence Γ is a constructible subset of $Y \times Z$. In particular the Zariski closure $\overline{\Gamma}$ of Γ in $Y \times Z$ coincides with the metric closure of Γ and there is a dense subset $\Gamma^0 \subset \Gamma$ which is Zariski open in $\overline{\Gamma}$. Now, let (y, z) be a point of $\overline{\Gamma} \subset Y \times Z$.

Since $\overline{\Gamma} \setminus \Gamma^0$ is nowhere dense and Zariski closed, we can choose an open disc $U \subset \mathbb{C}$ around 0 and a holomorphic curve $\gamma: U \to \overline{\Gamma}$ such that the Zariski-closure of $\gamma(U)$ is an algebraic curve, $\gamma(0) = (y, z)$ holds, and the complement of $U_1 := \gamma^{-1}(\Gamma)$ in U is discrete and closed (see, e.g., [6]). Since p is weakly proper, we find an open neighborhood V of $0 \in \mathbb{C}$, a regular curve $\tilde{\gamma}: V \to X$, and a non-constant regular map $\alpha: V \to U$ such that

$$\alpha(0) = 0, \qquad p \circ \tilde{\gamma} = \mathrm{pr}_Y \circ \gamma \circ \alpha.$$

Since α is non-constant, $\alpha^{-1}(U_1)$ has a closed discrete complement in V. For any $s \in \alpha^{-1}(U_1)$ we have

$$\gamma(\alpha(s)) = (p(\tilde{\gamma}(s)), f(\tilde{\gamma}(s))).$$

Thus, for continuity reasons, we obtain

$$(y, z) = \gamma(\alpha(0)) = (p(\tilde{\gamma}(0)), f(\tilde{\gamma}(0))) \in \Gamma.$$

∎

For open surjections p the above result is well known (see, e.g., [3, II.6.2]). We will apply Proposition 1.1. in the following situation. Let $\mathscr{S}$ be a system of fans in a lattice N', let Δ be a fan in a lattice N, and assume that $P\colon N' \to N$ is surjective and defines a map of systems of fans from $\mathscr{S}$ to Δ. Denote by $p\colon X_{\mathscr{S}} \to X_{\Delta}$ the toric morphism associated to P. Then we obtain the following characterization of weak properness:

1.2. Proposition. *The map p is weakly proper if and only if* $P_{\mathbb{R}}(|\mathscr{S}|) = |\Delta|$.

For the proof we formulate some auxiliary results. Let T be the acting torus of X_{Δ} and denote by x_0 the base point of X_{Δ}. Call a local curve in X_{Δ} *generic* if its image intersects the open orbit $T \cdot x_0$. For any $v \in N$, we denote by $\lambda_v\colon \mathbb{C}^* \to T$ the associated one-parameter subgroup.

1.3. Lemma. *Let γ be a generic local curve in X_{Δ}. Then there exists a local curve β in $t \in T$ and a point $v \in |\Delta| \cap N$ such that near* 0 *we have* $\gamma(s) = \beta(s)\lambda_v(s) \cdot x_0$.

Proof. We may assume that $N = \mathbb{Z}^n$ and hence $T = (\mathbb{C}^*)^n$. Let γ be defined on some open disc $U \subset \mathbb{C}$ around 0. Set $V := \gamma^{-1}(T \cdot x_0)$. Then $U \setminus V$ is a proper analytic subset of U and hence it is discrete and closed. Thus, after shrinking U, we may assume that either $U = V$ or $U = V \cup \{0\}$ holds. On V there is a representation

$$\gamma(s) = (g_1(s), \ldots, g_n(s)) \cdot x_0$$

with holomorphic functions $g_i \in \mathscr{O}^*_{\text{an}}(V)$. Using Laurent series expansion, we obtain $g_i(s) = s^{v_i}\beta_i(s)$ with an integer v_i and a function $\beta_i \in \mathscr{O}^*_{\text{an}}(U)$. Let $v := (v_1, \ldots, v_n)$ and $\beta := (\beta_1, \ldots, \beta_n)$. Then

$$\lim_{s \to 0} \lambda_v(s) \cdot x_0 = \beta(0)^{-1} \cdot \gamma(0).$$

Consequently, the point v lies in $|\Delta|$ and the desired decomposition of γ is given by $\gamma = \beta(s)\lambda_v(s) \cdot x_0$. ∎

Now, let $\Delta_1, \ldots, \Delta_r$ be fans in lattices N_i' and let surjective $P_i\colon N_i' \to N$ be given that are maps of the fans Δ_i and Δ. Let $p_i\colon X_{\Delta_i} \to X_{\Delta}$ be the associated toric morphisms.

1.4. Lemma. *Let γ be a generic local curve in X_{Δ}. If* $|\Delta| = P_{\mathbb{R}}(|\Delta_1|) \cup \cdots \cup P_{\mathbb{R}}(|\Delta_r|)$, *then, for some i, there is a weak p_i-lifting of γ.*

Proof. Choose β and $v \in N \cap |\Delta|$ as in Lemma 1.3. By assumption, for some i, there is a $v' \in |\Delta_i|$ such that $P_i(v') = lv$ with a positive integer l. Moreover, since P_i is surjective, we have a splitting

$$\begin{array}{ccc} T_i' & \xrightarrow{\cong} & T \times \ker(\pi_i) \\ & {\scriptstyle \pi_i} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_T} & \\ & T & \end{array}$$

where T_i' is the acting torus of X_{Δ_i} and $\pi_i\colon T_i' \to T$ is the homomorphism associated to p_i. In particular, there is a lifting $\tilde{\beta}$ with respect to π_i of the local curve $s \mapsto \beta(s^l)$ in $t \in T$. Now, let x_0' be the base point of X_{Δ_i}. Then the desired weak p_i-lifting of γ is given by

$$\tilde{\gamma}(s) := \tilde{\beta}(s)\lambda_{v'}(s) \cdot x_0', \qquad \alpha(s) := s^l.$$

∎

Finally, we need an elementary statement from convex geometry. Let σ denote a strictly convex polyhedral cone in some real vector space, let τ be a face of σ, and let $P\colon V \to V/\mathrm{lin}(\tau)$ be the projection.

1.5. LEMMA. *If $\sigma = \sigma_1 \cup \cdots \cup \sigma_r$ with polyhedral cones σ_i, then $P(\sigma)$ is the union of all $P(\sigma_i)$, where $\tau^\circ \cap \sigma_i \neq \emptyset$.*

Proof. We prove the assertion by induction on r. For $r = 1$ or $\tau = \{0\}$ there is nothing to show, so assume that $r > 1$ and τ is not trivial. Suppose that for some j and some $v \in \sigma_j$ we had $P(v) \notin P(\sigma_i)$ for all σ_i meeting τ°. Then v does not lie in τ. Hence there is a linear form $u \in \sigma_j^{\vee}$ with $u(v) > 0$ and $u(w) < 0$ for some $w \in \tau^\circ$. Fix a large $n \in \mathbb{N}$ such that $u(v + nw) < 0$. Now consider the cone

$$\sigma' := \sigma \cap \{v \in N_{\mathbb{R}};\, u(v) \leq 0\}.$$

Note that σ' contains $v + nw$ and is covered by less than r of the cones $\sigma_i' := \sigma_i \cap \sigma$. Moreover, $\tau' := \sigma' \cap \tau$ is a face of σ' and has the same dimension as τ. The induction hypothesis provides an i and a $w' \in \mathrm{lin}(\tau') = \mathrm{lin}(\tau)$ such that $v + nw + w' \in \sigma_i'$ and $\emptyset \neq \sigma_i' \cap (\tau')^\circ \subset \tau^\circ$.

Proof of Proposition 1.2. First assume that p is weakly proper. Clearly $P_{\mathbb{R}}(|\mathscr{S}|) \subset |\Delta|$. To obtain the reverse inclusion, assume that there are points $v \in |\Delta| \setminus P_{\mathbb{R}}(|\mathscr{S}|)$. Then we even find such a v lying in N. For this v, the curve $\lambda_v(s) \cdot x_0$ admits no weak p-lifting, contradicting our assumption on p.

Now, assume that $P_{\mathbb{R}}(|\mathscr{S}|)$ equals $|\Delta|$. Let $U \subset \mathbb{C}$ be an open disc around zero and let $\gamma\colon U \to X_\Delta$ be a holomorphic curve such that the Zariski-closure of $\gamma(U)$ is an algebraic curve. Then there is a unique T-orbit $T \cdot x_\tau$

of minimal dimension such that $\gamma(U)$ is contained in $V_\tau := \overline{T \cdot x_\tau}$. Note that V_τ is itself a toric variety.

We will use the fact that $p^{-1}(V_\tau)$ is a union of toric varieties to apply Lemma 1.4. Let $x_\sigma \in V_\tau$, i.e., σ is a cone of Δ with $\tau \prec \sigma$. By our assumption, we have $\sigma = P_{\mathbb{R}}(\sigma_1) \cup \cdots \cup P_{\mathbb{R}}(\sigma_s)$ with certain $(\sigma_i, k_i) \in \mathfrak{F}(\mathscr{S})$. By suitable ordering we achieve that $P_{\mathbb{R}}(\sigma_i)$ meets τ° if and only if $i \le r$ with some $r \le s$. Now, the above Lemma 1.5 implies

$$\sigma + \mathrm{lin}(\tau) = \bigcup_{i=1}^{r} P_{\mathbb{R}}(\sigma_i) + \mathrm{lin}(\tau) \subset N_{\mathbb{R}}/\mathrm{lin}(\tau).$$

Set $\tau_i := P_{\mathbb{R}}^{-1}(\tau) \cap \sigma_i$ and consider the orbit closures V_{τ_i} in X_{σ_i}. Note that $p(x_{\tau_i}) = x_\tau$ since $P_{\mathbb{R}}(\tau_i) \cap \tau^\circ \neq \emptyset$. Therefore p induces toric morphisms p_i: $V_{\tau_i} \to V_\tau$. Applying this procedure to all the other $\sigma \in \Delta$ with $\tau \prec \sigma$, we obtain a family of locally closed toric varieties $V_{\tau_j} \subset p^{-1}(V_\tau)$ and toric morphisms $V_{\tau_j} \to V_\tau$. According to [1, Example 2.7] these toric morphisms satisfy the assumptions of Lemma 1.4. ∎

2. TWO CONES

In this section we consider the special case of a toric prevariety X arising from an affine system $\mathscr{S}$ of fans in a lattice N with two maximal cones $\sigma(1)$ and $\sigma(2)$. Let L be a primitive sublattice of N. Throughout this section we assume that the projection P: $N \to N/L$ satisfies

$$P_{\mathbb{R}}(\sigma(1))^\circ \cap P_{\mathbb{R}}(\sigma(2))^\circ \neq \emptyset. \qquad (*)$$

Let H be the subtorus of the big torus T of X corresponding to $L \subset N$ and suppose that f: $X \to Z$ is an H-invariant regular map to a (not necessarily toric) variety Z. A first simple observation is

2.1. LEMMA. *Let $t \in T$. Then we have:*

(i) *There are regular curves C_1, C_2: $\mathbb{C} \to X$ and C: $\mathbb{C}^* \to H$ with $C_1(s) = C(s)C_2(s)$ for all $s \in \mathbb{C}^*$ and $C_i(0) = t \cdot x_{[\sigma(i), i]}$.*

(ii) *$f(t \cdot x_{[\sigma(1), 1]}) = f(t \cdot x_{[\sigma(2), 2]})$. In particular, f is constant on the orbit $T' \cdot x_{[\sigma(1), 1]}$, where $T' := T_{x_{[\sigma(1), 1]}} T_{x_{[\sigma(2), 2]}}$.*

Proof. By assumption $(*)$, there are $w_i \in \sigma(i)^\circ \cap N$ such that $w_1 = v_L + w_2$ holds for some $v_L \in L$. Let λ_{w_i} and λ_{v_L} denote the one-parameter subgroups of T corresponding to these lattice vectors. The curves

$$C_i: \mathbb{C}^* \to X, \qquad s \mapsto t \cdot \lambda_{w_i}(s) \cdot x_0$$

can be extended regularly to $\mathbb{C}$ by setting $C_i(0) := t \cdot x_{[\sigma(i), i]}$. Together with the curve $C: \mathbb{C}^* \to H$, $s \mapsto \lambda_{v_L}(s)$ the C_i satisfy (i).

In order to check (ii) note that according to (i) the points $x_{[\sigma(1), 1]}$ and $x_{[\sigma(2), 2]}$ cannot be separated by H-stable complex open neighborhoods. Since f is continuous with respect to the complex topology and Z is hausdorff, the claim follows. ∎

Note that in the proof of assertion (ii), we only used that f is H-invariant and continuous with respect to the complex topology. Hence the statement holds also for holomorphic f.

2.2. PROPOSITION. *Assume that there are faces $\tau_i \prec \sigma(i)$ and $v_i \in \tau_i^\circ \cap N$ such that the cone generated by $P(v_1)$ and $P(v_2)$ is a line. Then $f(\lambda_{v_i}(\mathbb{C}^*) \cdot x) = f(x)$ for $i = 1, 2$ and for all $x \in X$.*

Proof. Since X is covered by affine T-stable open subspaces, it suffices to show that with some nonempty open subset V of T we have $f(\lambda_{v_i}(s) \cdot t \cdot x_0) = f(t \cdot x_0)$ for all $s \in \mathbb{C}^*$ and all t contained in V.

By appropriate scaling we achieve that $v_1 + v_2 \in L$ holds. Assumption $(*)$ provides $w_i \in \sigma(i)^\circ \cap N$ such that $w_1 - w_2 \in L$. Let X_i denote the affine chart of X corresponding to $\sigma(i)$. We now want to define toric morphisms $\varphi_i: \mathbb{C} \times \mathbb{C} \times T \to X_i$.

We consider the lattice homomorphisms: $F_i: \mathbb{Z}^2 \times N \to N$, defined by $F_i(e_1) = v_i$, $F_i(e_2) = w_i$, and $F_i(v) = v$ for all $v \in N$. The corresponding toric morphisms are the maps

$$\varphi_i: \mathbb{C} \times \mathbb{C} \times T \to X_i,$$

$$(s, r, t) \mapsto \begin{cases} t \cdot \lambda_{v_i}(s) \cdot \lambda_{w_i}(r) \cdot x_0 & \text{if } r \neq 0 \neq s, \\ t \cdot \lambda_{w_i}(r) \cdot x_{[\tau_i, i]} & \text{if } r \neq 0,\ s = 0, \\ t \cdot x_{[\sigma(i), i]} & \text{if } r = 0. \end{cases}$$

We use the regular maps φ_i to define a regular map $\psi: \mathbb{C} \times \mathbb{P}_1 \times T \to Z$. First notice that H-invariance of f yields for all $s \in \mathbb{C}^*$, $r \in \mathbb{C}$, and $t \in T$ the identity

$$\begin{aligned} f(\varphi_1(s, r, t)) &= f\big(t \cdot \lambda_{v_1}(s) \cdot \lambda_{w_1}(r) \cdot x_0\big) \\ &= f\big(t \cdot \lambda_{v_2}(1/s) \cdot \lambda_{w_2}(r) \cdot x_0\big) = f(\varphi_2(1/s, r, t)). \end{aligned}$$

So the rational map $\mathbb{P}_1 \times \mathbb{C} \times T \to Z$ given by

$$([s_0, s_1], r, t) \mapsto f(\varphi_1(s_1/s_0, r, t))$$

extends to a morphism ψ. The fibre $\psi^{-1}(z)$ of $z := \psi(0, 0, e_T)$ contains $\mathbb{P}_1 \times \{(0, e_T)\}$, where e_T denotes the neutral element of T.

Now choose an open affine neighborhood W of z in Z and set $Y := \mathbb{P}_1 \times \mathbb{C} \times T \setminus \psi^{-1}(W)$. Consider the projection pr: $\mathbb{P}_1 \times \mathbb{C} \times T \to \mathbb{C} \times T$. Since $\mathbb{P}_1$ is complete, pr(Y) is closed in $\mathbb{C} \times T$. Moreover, we have pr$(z) \notin$ pr(Y). Thus for $W_0 := \mathbb{C} \times T \setminus \mathrm{pr}(Y)$ we have

$$\psi^{-1}(z) \subset \mathbb{P}_1 \times W_0 = \mathbb{P}_1 \times \mathbb{C} \times T \setminus \mathrm{pr}^{-1}(\mathrm{pr}(Y)) \subset \psi^{-1}(W).$$

Since we chose W to be affine, ψ maps $\mathbb{P}_1 \times \{w\}$ to a point for every $w \in W_0$. In particular, for every point $(r, t) \in W_0 \cap \mathbb{C}^* \times T$ we have

$$f(\lambda_{w_1}(r) \cdot \lambda_{v_1}(s) \cdot t \cdot x_0) = f(\lambda_{w_1}(r) \cdot t \cdot x_0)$$

for all $s \in \mathbb{C}^*$. So f is constant on orbits of the one-parameter subgroup λ_{v_1} on a dense subset of $T \cdot x_0 \subset X_i$ and hence this is true everywhere. Since $v_1 + v_2 \in L$, this also holds for λ_{v_2}. ∎

3. A CRITERION FOR THE EXISTENCE OF AN INVARIANT SEPARATION

Let $\mathscr{S} = (\Delta_{ij})_{i,j \in I}$ denote a system of fans in a lattice N and let L be a sublattice of N. Suppose that the projection P: $N \to N/L =: \tilde{N}$ fulfills the following conditions:

(i) $\tau := \bigcup_{i \in I} P_{\mathbb{R}}(\sigma(i))$ is a strictly convex cone.

(ii) For every face $\varrho \prec \tau$ and any two (σ, i), $(\sigma', i') \in \mathfrak{F}(\mathscr{S})$ with $P_{\mathbb{R}}(\sigma)^\circ \cup P_{\mathbb{R}}(\sigma')^\circ \subset \varrho^\circ$, there is a chain

$$(\sigma, i) = (\sigma_{i_1}, i_1), \ldots, (\sigma_{i_r}, i_r) = (\sigma', i')$$

in $\mathfrak{F}(\mathscr{S})$ such that each $P_{\mathbb{R}}(\sigma_{i_k})^\circ$ is contained in ϱ and $P_{\mathbb{R}}(\sigma_{i_k})^\circ \cap P_{\mathbb{R}}(\sigma_{i_{k+1}})^\circ \neq \emptyset$.

3.1. *Remark.* If $\dim(\tilde{N}) \leq 1$ then (i) implies (ii). If $\dim \tilde{N} = 2$, then (ii) is equivalent to $\tau^\circ = \bigcup_{\dim(P_{\mathbb{R}}(\sigma(i)) = 2} P_{\mathbb{R}}(\sigma(i))^\circ$.

The projection P defines a map of systems of fans from $\mathscr{S}$ to the fan of faces of τ. Moreover, denoting by H the subtorus of T that corresponds to L, we have

3.2. PROPOSITION. *The toric morphism p: $X_{\mathscr{S}} \to X_\tau$ defined by P is an H-invariant separation.*

Proof. By Propositions 1.1 and 1.2 it suffices to show that every H-invariant morphism $f\colon X_{\mathscr{S}} \to Z$ to a variety is constant on the fibres of p. So let $\pi\colon T \to \tilde{T}$ denote the homomorphism of the acting tori associated to p. Then the p-fibre of a point $\tilde{t} \cdot x_{\varrho} \in X_{\tau}$ is

$$p^{-1}\left(\tilde{t} \cdot x_{\varrho}\right) = \bigcup_{P_{\mathbb{R}}(\sigma)^{\circ} \subset \varrho^{\circ}} \pi^{-1}\left(\tilde{t} \cdot \tilde{T}_{x_{\varrho}}\right) \cdot x_{[\sigma, i]}$$

(see [2, Proposition 3.5]). Let T' denote the subtorus of T, generated by all isotropy groups $T_{x_{[\sigma, i]}}$, where $P_{\mathbb{R}}(\sigma^{\circ}) \subset \varrho^{\circ}$, i.e., T' corresponds to the maximal sublattice in the vector subspace spanned by the $\mathrm{lin}(\sigma)$. Then $T' \cdot H = \pi^{-1}(\tilde{T}_{x_{\varrho}})$.

Now, for (σ, i) and $(\sigma', i') \in \mathfrak{F}(\mathscr{S})$ with $P_{\mathbb{R}}(\sigma^{\circ}) \cup P_{\mathbb{R}}(\sigma'^{\circ}) \subset \varrho^{\circ}$, the chain condition (ii) implies by Lemma 2.1 that $f(t \cdot x_{[\sigma, i]}) = f(t \cdot x_{[\sigma', i']})$ for all t, and hence that f is constant on $T' \cdot t \cdot x_{[\sigma, i]}$. That shows that f is constant on the fibre $p^{-1}(\tilde{t} \cdot x_{\varrho})$. ∎

4. CODIMENSION TWO

Let X be a toric prevariety and let H be a subtorus of the acting torus T of X. Denote by $\hat{H}$ the maximal subtorus of T such that every H-invariant regular map from X to a variety Z is invariant by $\hat{H}$. In this section we prove

4.1. THEOREM. *If $\hat{H}$ is of codimension at most two in T, there exists an H-invariant separation for X.*

4.2. COROLLARY. *Every toric presurface admits a separation.*

4.3. COROLLARY. *If X is a toric variety and H is of codimension at most two in T, then the TV-quotient is also an AV-quotient.*

As we shall see in Section 7, a PV-quotient need not exist even in small codimension, and even if it exists, the AV-quotient and the PV-quotient may be different (see Example 4.5).

For the proof of Theorem 4.1, we may assume that $H = \hat{H}$. In particular, H itself is of codimension at most two in T. Moreover, we may assume that $X = X_{\mathscr{S}}$ for some affine system of fans $\mathscr{S}$ in a lattice N.

Let L denote the sublattice of N that corresponds to H and let P: $N \to N/L =: \tilde{N}$ be the projection. Define an equivalence relation on the index set I by

$$i \sim j :\Leftrightarrow \exists\, i = i_1, \dots, i_r = j \qquad \text{with } P_{\mathbb{R}}(\sigma(i_k))^{\circ} \cap P_{\mathbb{R}}(\sigma(i_{k+1}))^{\circ} \neq \emptyset.$$

4.4. LEMMA. *For each equivalence class $E \subset I$ the set*

$$\tau_E := \bigcup_{\sigma \in E} P_{\mathbb{R}}(\sigma(i))$$

is a strictly convex cone.

Proof. Maximality of H and Proposition 2.2 imply that every cone $P_{\mathbb{R}}(\sigma(i))$ is strictly convex. In particular the assertion is verified in the case $\dim(T/H) \le 1$. Now suppose that $\tilde{N}$ is of dimension two and there is an equivalence class E such that τ_E is not strictly convex. Then we find subsets E_1, E_2 of E such that each

$$\tau_k := \bigcup_{\sigma \in E_k} P_{\mathbb{R}}(\sigma)$$

is strictly convex, $\tau_1^\circ \cap \tau_2^\circ \neq \emptyset$ and $\tau_1 \cup \tau_2$ is not strictly convex. Let X_k be the open T-stable subspace of $X_{\mathscr{S}}$ defined by the cones $\sigma(i)$ with $P_{\mathbb{R}}(\sigma(i)) \subset \tau_k$. According to Proposition 3.2, the map P defines H-invariant separations p_k: $X_k \to X_{\tau_k}$, $k = 1, 2$.

Now let f: $X \to Z$ be any H-invariant regular map. Set $f_k := f|_{X_k}$. Then we obtain the following commutative diagram of regular maps,

$$\begin{array}{ccc} X_1 \cup_T X_2 & \xrightarrow{f_1 \cup_T f_2} & Z \\ {\scriptstyle p_1 \cup_T p_2} \searrow & & \nearrow {\scriptstyle \tilde{f}} \\ & X_{\tau_1} \cup_{\tilde{T}} X_{\tau_2} & \end{array}$$

Here $\cup_T$ indicates gluing along T. Let $\tilde{L} \subset \tilde{N}$ be a line contained in $\tau_1 \cup \tau_2$. Then Proposition 2.2 yields that $\tilde{f}$ is invariant with respect to the action of the subtorus $\tilde{H} \subset \tilde{T}$ corresponding to $\tilde{L}$. Now let π: $T \to \tilde{T}$ denote the homomorphism of the acting tori determined by p. Then $f_1 \cup_T f_2$ and hence f is invariant by $\pi^{-1}(\tilde{H})$. This contradicts the maximality of H. ∎

Proof of Theorem 4.1. By construction, the cones τ_E, where E runs through the equivalence classes of $\sim$, form a fan Δ in $\tilde{N}$. Moreover, the projection P determines a map of systems of fans from $\mathscr{S}$ to Δ. It follows directly from Proposition 3.2 that the associated toric morphism p is an H-invariant separation of X. ∎

4.5. EXAMPLE. *A $\mathbb{C}^*$-action with AV- and PV-quotients different from each other*. Let $X := \mathbb{C}^2 \backslash \{0\}$ and consider the action of

$$H := \{(t, t^{-1});\ t \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^2$$

on X. Then the AV-quotient for this action is by Corollary 4.3 just the map

$$X \to \mathbb{C}, \qquad (z, w) \mapsto zw.$$

On the other hand, the PV-quotient is given by the following map from X onto the line $\mathbb{C}_{00}$ with doubled zero

$$(z, w) \mapsto \begin{cases} zw & \text{if } zw \neq 0, \\ 0_1 & \text{if } w = 0, \\ 0_2 & \text{if } z = 0. \end{cases}$$

4.6. EXAMPLE. *An AV-quotient without base-change property.* Let Δ be the fan in $\mathbb{R}^3$ that has the maximal cones

$$\sigma_1 := \operatorname{cone}(-e_1, e_2, e_1 + e_2 + e_3), \qquad \sigma_2 := \operatorname{cone}(e_1, e_2, e_1 + e_2 + e_3).$$

Consider the projection P_0: $\mathbb{Z}^3 \to \mathbb{Z}^2$, $(u, v, w) \mapsto (u, v)$. Let H be the subtorus of the acting torus of X_Δ corresponding to the kernel of P_0. Then the TV-quotient p: $X_\Delta \to X_\Delta \mathbin{/\!\!\!/_{\text{tor}}} H$ for the action of H arises from the map P: $\mathbb{Z}^3 \to \mathbb{Z}$, $(u, v, w) \mapsto v$ from Δ to the fan of faces of $\mathbb{R}_{\geq 0}$.

In particular, $X_\Delta \mathbin{/\!\!\!/_{\text{tor}}} H = \mathbb{C}$ is of dimension one. According to Corollary 4.3, p is also an AV-quotient. But for the acting torus $\mathbb{C}^*$ of $X_\Delta \mathbin{/\!\!\!/_{\text{tor}}} H$, the open set $p^{-1}(\mathbb{C}^*)$ has, again by Corollary 4.3, a two-dimensional AV-quotient.

5. A $\mathbb{C}^*$-ACTION WITHOUT AV-QUOTIENT BUT WITH PV-QUOTIENT

We consider the open toric subvariety $X := \mathbb{C}^2 \times (\mathbb{C}^*)^2 \cup (\mathbb{C}^*)^2 \times \mathbb{C}^2$ of $\mathbb{C}^4$ and the action of the one-dimensional subtorus

$$H := \{(t, t, 1, t^{-1});\, t \in \mathbb{C}^*\} \subset (\mathbb{C}^*)^4.$$

5.1. PROPOSITION. (i) *There is a PV-quotient for the action of H on X.*

(ii) *The action of H on X admits no AV-quotient.*

Proof. Note that X arises from the fan Δ in $\mathbb{Z}^4$ that has $\sigma_1 := \operatorname{cone}(e_1, e_2)$ and $\sigma_2 := \operatorname{cone}(e_3, e_4)$ as its maximal cones. Let F: $\mathbb{Z}^4 \to \mathbb{Z}^3$ denote the lattice homomorphism defined by

$$F(e_1) := e_1, \qquad F(e_2) := e_2, \qquad F(e_3) := e_3, \qquad F(e_4) := e_1 + e_2.$$

To prove (i), set $\tau_i := F_{\mathbb{R}}(\sigma_i)$ and define a system $\mathscr{S}$ of fans in $\mathbb{Z}^3$ by $\Delta_{ii} := \mathfrak{F}(\tau_i)$, where $i = 1, 2$ and $\Delta_{12} := \Delta_{21} := \{\{0\}\}$ (see Fig. 1).

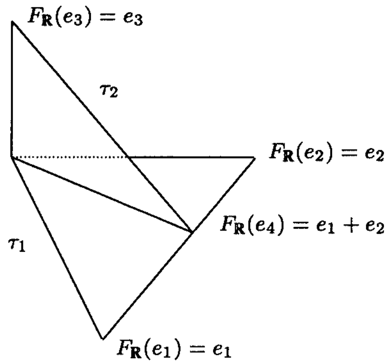

FIGURE 1

According to [2, Theorem 6.7], the toric morphism $X \to X_{\mathscr{S}}$ defined by F is a good prequotient for the action of H on X. In particular, it is a PV-quotient.

We prove (ii). Assume that there exists an AV-quotient p: $X \to Y$ for the action of H on X. We lead this to a contradiction by presenting an H-invariant map that does not factor through p. Consider

$$f: X \to \mathbb{C}^3, \qquad (x_1, x_2, x_3, x_4) \mapsto (x_1 x_4, x_2 x_4, x_3).$$

Note that

$$f(X) = \mathbb{C}^3 \setminus (\{0\} \times \mathbb{C}^* \times \{0\} \cup \mathbb{C}^* \times \{0\} \times \{0\}).$$

In particular, $f(X)$ is not open in $\mathbb{C}^3$. We describe f in terms of fans. Let $\tau := \operatorname{cone}(e_1, e_2, e_3) \subset \mathbb{R}^3$. Then f is just the toric morphism $X \to X_\tau = \mathbb{C}^3$ defined by the lattice homomorphism F. Thus it follows from [1] that f is the TV-quotient for the action of H on X.

By its universal property, p is surjective and there is a regular map $\tilde{f}$: $Y \to X_\tau$ such that $f = \tilde{f} \circ p$. We claim that all fibres of $\tilde{f}$ are of dimension zero. To see this let $\varrho := \operatorname{cone}(e_3) \in \mathbb{R}^3$ and note that surjectivity of p implies

$$Y = \tilde{f}^{-1}(X_{\tau_1}) \cup \tilde{f}^{-1}(X_\varrho) \cup \tilde{f}^{-1}(x_\tau).$$

By [1, Example 3.1], the map $f_1 := f|_{X_{\sigma_1}}$: $X_{\sigma_1} \to X_{\tau_1}$ is an algebraic quotient for the action of H on X_{σ_1}. Hence we obtain a regular map g: $X_{\tau_1} \to Y$ and a commutative diagram

$$\begin{array}{ccccc} X_{\sigma_1} & \subset & X & \xrightarrow{f} & X_\tau \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle p} & \nearrow{\scriptstyle \tilde{f}} & \\ X_{\tau_1} & \xrightarrow{g} & Y & & \end{array}$$

Note that $\tilde{f} \circ g$ is necessarily an isomorphism and hence g is an open embedding. Let $T := (\mathbb{C}^*)^4$ be the acting torus of p and set $\varrho_4 := \mathrm{cone}(e_4) \subset \mathbb{R}^4$. By surjectivity of p and [2, Fibre Formula 3.5], we have

$$\tilde{f}^{-1}(X_{\tau_1}) = p\big(f^{-1}(X_{\tau_1})\big) = p\big(X_{\sigma_1} \cup T \cdot x_{\varrho_4}\big) = p\big(X_{\sigma_1}\big) = g\big(X_{\tau_1}\big).$$

Here the third equality is a consequence of Lemma 2.1(ii). So $\tilde{f}$ is injective on $\tilde{f}^{-1}(X_{\tau_1})$. A similar argument shows that $\tilde{f}$ is injective on $\tilde{f}^{-1}(X_\varrho)$. To verify the claim, we still have to consider the fibre $\tilde{f}^{-1}(x_\tau)$. Again by Lemma 2.1(ii) one has

$$\begin{aligned}\tilde{f}^{-1}(x_\tau) &= p(T \cdot x_{\sigma_2}) \subset p\big(\overline{T \cdot x_{\varrho_4}}\big) \subset \overline{p(T \cdot x_{\varrho_4})} = \overline{p(T \cdot x_{\sigma_1})}\\ &= \overline{\tilde{f}^{-1}(T_1 \cdot x_{\tau_1})}.\end{aligned}$$

Here T_1 denotes the acting torus of X_τ. Since the closure of $\tilde{f}^{-1}(T_1 \cdot x_\tau)$ is contained in $\tilde{f}^{-1}(\overline{T_1 \cdot x_\tau})$, the above inclusion yields

$$\tilde{f}^{-1}\big(\overline{T_1 \cdot x_{\tau_1}}\big) = \tilde{f}^{-1}\big(T_1 \cdot x_{\tau_1}\big) \cup \tilde{f}^{-1}(x_\tau) = \overline{\tilde{f}^{-1}\big(T_1 \cdot x_{\tau_1}\big)},$$

i.e., $\tilde{f}^{-1}(x_\tau)$ is contained in the closure of $\tilde{f}^{-1}(T_1 \cdot x_{\tau_1})$. Moreover, we know that $\tilde{f}^{-1}(T \cdot x_{\tau_1}) = g(T \cdot x_{\tau_1})$ is locally closed of dimension one. Thus $\tilde{f}^{-1}(x_\tau)$ is of dimension zero and our claim is proved.

To conclude the proof, observe that by Zariski's Main Theorem, $\tilde{f}$ is an open embedding. This contradicts the fact that $f(X)$ is not open in X_τ. ∎

In fact the arguments used in our proof are chosen to work also in the category of analytic spaces (for the existence of g use [7]). Thus we obtain:

5.2. PROPOSITION. *The action of H on X does not admit categorical quotients in the categories of analytic and algebraic spaces.*

6. A $\mathbb{C}^*$-ACTION ADMITTING NEITHER AN AV-QUOTIENT NOR A PV-QUOTIENT

Let X denote the smooth four-dimensional toric variety obtained by gluing the two affine charts $X_1 = \mathbb{C}^4$ and $X_2 = \mathbb{C}^3 \times \mathbb{C}^*$ along the common subset $(\mathbb{C} \times \mathbb{C}^*)^2$, using the gluing map

$$(t_1, t_2, t_3, t_4) \mapsto \big(t_1 t_2^2, t_2^{-1}, t_3, t_4\big).$$

Let $T := (\mathbb{C}^*)^4$ denote the acting torus of X. We consider the action of the one-dimensional subtorus $H \subset T$ on X, where

$$H := \{(t^{-2}, 1, t, t);\ t \in \mathbb{C}^*\}.$$

6.1. PROPOSITION. *There is neither an AV-quotient nor a PV-quotient for the action of H on X.*

Proof. We will consider the TV-quotient $f: X \to X'$ for the action of H on X, which will turn out to be non-surjective. The assumption that f factors through a surjective H-invariant regular map onto a complex prevariety will then lead to a contradiction.

As before, we first describe the situation in terms of fans. Let $e_1, \dots, e_4$ denote the canonical basis vectors of $\mathbb{R}^4$ and let Δ be the fan in $\mathbb{R}^4$ with the maximal cones

$$\sigma_1 := \operatorname{cone}(e_1, e_2, e_3, e_4) \qquad \text{and} \qquad \sigma_2 := \operatorname{cone}(e_1, 2e_1 - e_2, e_3).$$

Note that $\sigma_1 \cap \sigma_2$ is the cone spanned by e_1 and e_3. The toric variety X_Δ associated to Δ equals X. In order to describe f, consider the fan Δ' in $\mathbb{R}^3$ with the maximal cones

$$\tau_1 := \operatorname{cone}(e_1 - e_2, e_1 + e_3, e_1 - e_3),$$
$$\tau_2 := \operatorname{cone}(e_1 + e_2, e_1 + e_3, e_1 - e_3).$$

Then $f: X \to X'$ arises from the map $F: \mathbb{Z}^4 \to \mathbb{Z}^3$ of the fans Δ and Δ' that, with respect to the canonical bases, is given by the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Note that $F_{\mathbb{R}}(\sigma_1) = \tau_1$, whereas $F_{\mathbb{R}}(\sigma_2) \neq \tau_2$. More precisely, there is exactly one cone in Δ' whose relative interior does not intersect $F_{\mathbb{R}}(\sigma_1) \cup F(\sigma_2)$, namely the face τ of τ_2 spanned by $(1,1,0)$ and $(1,0,-1)$.

Let T' denote the acting torus of $X_{\Delta'}$. Then we obtain $f(X_\Delta) = X_{\Delta'} \setminus T' \cdot x_\tau$. In particular, f is not surjective and $f(X_\Delta)$ is not open in $X_{\Delta'}$.

Now, assume that there is an AV- or a PV-quotient for the action of H on X_Δ. Then, in both cases, we have a surjective regular H-invariant map $p: X_\Delta \to Y$ onto a complex prevariety Y and a regular map $\tilde{f}: Y \to X_{\Delta'}$ such that the diagram $f = \tilde{f} \circ p$.

Note that $\tilde{f}$ is compatible with the induced (set theoretical) action of T on Y, i.e., if $\varphi: T \to T'$ denotes the homomorphism of the acting tori associated to f, then we have $\tilde{f}(t \cdot y) = \varphi(t) \cdot \tilde{f}(y)$ for all $t \in T$ and $y \in Y$.

We claim that $\tilde{f}$ has finite fibres and is injective over an open set of $X_{\Delta'}$. First consider the open affine toric subvariety X_{σ_1} of X_Δ. By [1, Example 3.1], the toric morphism $f_1: X_{\sigma_1} \to X_{\tau_1}$ defined by F is the algebraic quotient for the action of H on X_{σ_1}. Thus, there is a regular map g:

$X_{\tau_1} \to Y$ such that the diagram

$$\begin{array}{ccccc} X_{\sigma_1} & \subset & X_\Delta & \xrightarrow{f} & X_{\Delta'} \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle p} & \nearrow{\scriptstyle \tilde{f}} & \\ X_{\tau_1} & \xrightarrow{g} & Y & & \end{array}$$

is commutative (see, e.g., [2, Proposition 6.4]). It follows that $\tilde{f} \circ g$ defines an automorphism of X_{τ_1}. Since p is surjective we have

$$\tilde{f}^{-1}(X_{\tau_1}) = p\big(f^{-1}(X_{\tau_1})\big) = p(X_{\sigma_1}) = g(X_{\tau_1}).$$

Consequently, $\tilde{f}$ is injective on the set $\tilde{f}^{-1}(X_{\tau_1})$. Now consider $\sigma' := \mathrm{cone}(e_3, 2e_1 - e_2) \in \Delta$ and set $\tau' := F_{\mathbb{R}}(\sigma')$. By looking at the toric morphism $f_2\colon X_{\sigma'} \to X_{\tau'}$ induced by f, we obtain with similar arguments as above that $\tilde{f}$ is injective over $X_{\tau'}$ (see Fig. 2).

Thus, to obtain our claim it remains to consider the fibre $\tilde{f}^{-1}(x_{\tau_2})$. Note that according to [2, Fibre Formula 3.5] one has

$$f^{-1}(x_{\tau_2}) = T \cdot x_{\sigma_2} \cup T \cdot x_{\sigma},$$

where $\sigma := \mathrm{cone}(e_1, 2e_1 - e_2) \in \Delta$. Let $\varrho := \mathrm{cone}(e_1) \subset \mathbb{R}^4$. We claim that $T \cdot p(x_\varrho)$ is locally closed of dimension one. This follows from the fact that $T_1 \cdot f_1(x_\varrho)$ and $T' \cdot f(x_\varrho)$ are locally closed of dimension one.

Now note that $\varrho \prec \sigma_2$ and $\varrho \prec \sigma$. Consequently $T \cdot x_{\sigma_2}$ and $T \cdot x_\sigma$ are contained in the closure of the orbit $T \cdot x_\varrho$. This implies

$$T \cdot p(x_{\sigma_2}) \cup T \cdot p(x_\sigma) \subset \overline{T \cdot p(x_\varrho)}.$$

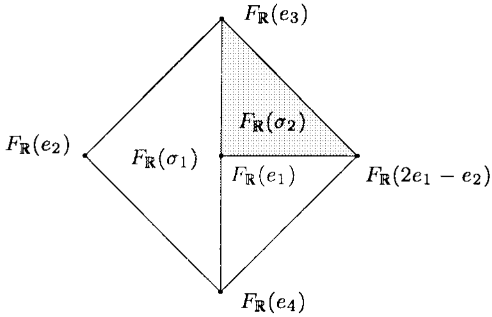

FIG. 2. Intersection of Δ' with the plane defined by $x = 1$ in $\mathbb{R}^3$.

Since $f(x_{\sigma_2}) \neq f(x_\varrho) \neq f(x_\sigma)$, we obtain that the T'-orbits through $f(x_{\sigma_2})$ and $f(x_\sigma)$ do not meet $T' \cdot f(x_\varrho)$. Thus we have even

$$T \cdot p(x_{\sigma_2}) \cup T \cdot p(x_\sigma) \subset \overline{T \cdot p(x_\varrho)} \setminus T \cdot p(x_\varrho).$$

In other words, $\tilde{f}^{-1}(x_{\tau_2}) = p(f^{-1}(x_{\tau_2}))$ consists of finitely many points. Thus we verified that $\tilde{f}$ has finite fibres.

Now, cover Y by open affine charts $U_1, \ldots, U_r$ and set $U := T \cdot p(x_0) = \tilde{f}^{-1}(T' \cdot x_0')$. Then each restriction $\tilde{f}_i := \tilde{f}|_{U_i}$ has finite fibres and is injective along the nonempty open set $U_i \cap U$. Since Y and $X_{\Delta'}$ are normal, we obtain that the $\tilde{f}_i$ are open maps. This yields openness of $f(X) = \bigcup \tilde{f}_i(U_i)$ and we arrive at a contradiction. ∎

7. A $\mathbb{C}^*$-ACTION WITH AV-QUOTIENT BUT WITHOUT PV-QUOTIENT

We consider the smooth three-dimensional toric variety X obtained from gluing two copies of $\mathbb{C}^3$ along the open subset $\mathbb{C} \times (\mathbb{C}^*)^2$ by the map

$$(x_1, x_2, x_3) \mapsto \left(x_1 x_2^2 x_3^2, x_2^{-1}, x_3^{-1}\right).$$

In terms of convex geometry, X is the toric variety arising from the fan Δ in $\mathbb{Z}^3$ that has the maximal cones

$$\sigma_1 := \operatorname{cone}(e_1, e_1 - e_2, e_1 + e_2 + e_3),$$
$$\sigma_2 := \operatorname{cone}(e_1, e_1 + e_2, e_1 - e_2 - e_3).$$

Moreover, let $\tilde{X}$ denote the affine toric variety defined by the fan $\tilde{\Delta}$ of faces of $\sigma := \operatorname{cone}(e_1 + e_2, e_1 - e_2)$ in $\mathbb{Z}^2$. Let $P\colon \mathbb{Z}^3 \to \mathbb{Z}^2$, $(x, y, z) \mapsto (x, y)$ denote the projection (see Fig. 3).

Then $P_{\mathbb{R}}(\sigma_i) = \sigma$, so P is a map of the fans Δ and $\tilde{\Delta}$. Set $L := \ker(P)$. Note that P is universal with respect to L-invariant maps of fans (see [1, Sect. 2]) and also with respect to L-invariant maps of systems of fans (see [2, Sect. 7]).

Now let H be the one-dimensional subtorus of the acting torus $T = (\mathbb{C}^*)^3$ of X that corresponds to L. Moreover, let $X_i \subset X$ be the affine open subset corresponding to σ_i. Then Proposition 3.2 and [1, Example 3.1; 2, Sect. 7] yield the following

7.1. PROPOSITION. *The toric morphism $p\colon X \to \tilde{X}$ associated to P satisfies*

(i) *p is the AV-quotient for the action of H.*

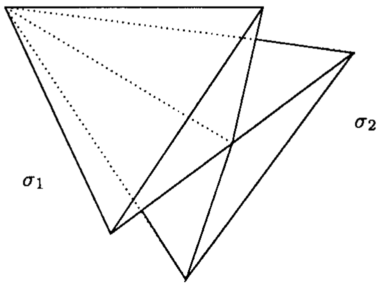

FIGURE 3

(ii) *p is the TP-quotient for the action of H.*

(iii) *The restriction $p_i := p|_{X_i}\colon X_i \to \tilde{X}$ is the algebraic quotient.*

But as we will show below, p does not satisfy the universal property of a PV-quotient in the category of arbitrary prevarieties. In fact, we even obtain

7.2. PROPOSITION. *The action of H on X admits no PV-quotient.*

Proof. Assume that there is a PV-quotient $q\colon X \to Y$. We claim that Y is a toric prevariety and q a toric morphism. Note that there is an induced (set theoretical) T-action on Y such that q is equivariant. By the universal property of q and Proposition 7.1, there are commutative diagrams

$$\begin{array}{ccccc} X_i & \subset & X & & \\ {\scriptstyle p_i}\downarrow & & {\scriptstyle q}\downarrow & \searrow^{p} & \\ \tilde{X} & \xrightarrow[r_i]{} & Y & \xrightarrow[r]{} & \tilde{X} \end{array}$$

of T-equivalent regular maps. Since $r \circ r_i = \mathrm{id}_{\tilde{X}}$ holds, each r_i is injective, so Zariski's Main Theorem implies that the r_i are open embeddings. Since X is covered by the X_i and q is surjective, we obtain that Y is covered by the T-stable affine open subspaces

$$Y_i := r_i(\tilde{X}) = q(X_i).$$

In particular it follows that the induced T-action on Y is regular. So Y is a toric prevariety and q is a toric morphism. This readily implies that q satisfies the universal property of a TP-quotient for the action of H on X. According to Proposition 7.1, we may assume $q = p$.

In order to show that p is not a PV-quotient we construct a map f: $X \to Z$ of prevarieties that does not factor through p. Consider the maps p_i defined above and the distinguished points.

$$x_1 := x_{\varrho_1} \in X_1, \qquad x_2 := x_{\varrho_2} \in X_2,$$

where $\varrho_1 := \mathbb{R}_{\geq 0}(e_1 - e_2)$ and $\varrho_2 := \mathbb{R}_{\geq 0}(e_1 - e_2 - e_3)$. Note that the point $z := p(x_1) = p(x_2)$ does not lie in $p(X_1 \cap X_2)$. Consequently the maps p_i glue together to a regular map

$$f\colon X = X_1 \cup_{X_1 \cap X_2} X_2 \to \tilde{X} \cup_{\tilde{X} \setminus \{z\}} \tilde{X} =: Z$$

of prevarieties. Since f separates the points x_1 and x_2, there is no set-theoretical factorization of f through p. ∎

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