Europ. J. Combinatorics (1982) 3, 219–231

The Unimodular Lattices of Dimension up to 23 and the Minkowski–Siegel Mass Constants

J. H. CONWAY AND N. J. A. SLOANE

In an earlier paper we enumerated the integral lattices of determinant one and dimension not exceeding 20. The present paper extends this enumeration to dimension 23, finding 40 lattices of dimension 21, 68 of dimension 22, and 117 of dimension 23. We also give explicit formulae for the Minkowski–Siegel mass constants for unimodular lattices (apparently not stated correctly elsewhere in the literature) and an exact table of the mass constants up to 32 dimensions, which provided a valuable check on our enumeration.

1. INTRODUCTION

The norm of a vector x is its squared length $x.x$, and is written $N(x)$. An integral lattice Λ in real n-dimensional Euclidean space $\mathbb{R}^n$ is said to be unimodular if it has determinant ± 1, and an integral unimodular lattice is *even*, or of genus II, if the norm of every lattice vector is an even integer; otherwise it is *odd*, or of genus I. Even unimodular lattices exist only in dimensions which are multiples of eight, whereas odd lattices exist in all dimensions greater than zero. The history of the enumeration of integral unimodular lattices is described briefly in our earlier paper [9]. It is worth mentioning, however, that some of the recent interest in these lattices is due to their connections, as yet not fully understood, with the Leech lattice and with hyperbolic geometry (see [6, 10–12, 15, 28, 29] and Section 2.1 below). The even unimodular lattices in dimension 24 were found by Niemeier in 1968 [16], and in [9] we enumerated the odd lattices in up to 20 dimensions. In the present paper we complete the enumeration of the odd lattices in dimensions below 24 by deriving them from Niemeier's lattices. We shall prove the following theorem.

THEOREM 1: THE CLASSIFICATION. *The unimodular lattices of dimension not exceeding* 23 *and containing no vectors of norm* 1 *are those shown in Table* 1.

TABLE 1
The unimodular lattices of dimension ≤23 and containing no vectors of norm 1. All are odd (or of genus I) except for four lattices: the empty lattice, E_8, E_8^2, and D_{16}, which are even (or of genus II). The notation is explained in Section 2.8

dim	N	w	V	$c(w)$	$g_1 g_2$	t_2
0	D_{24}	–	∅	1	1	0
8	$D_{16}E_8$	–0	E_8	1	1	240
12	D_{12}^2	–0	D_{12}	2	1	264
14	$D_{10}E_7^2$	–00	E_7^2	1	2	252
15	$A_{15}D_9$	0–	A_{15}	1	2	240

0195–6698/82/030219+13 $02.00/0

TABLE 1 (*cont.*)

dim	N	w	V	$c(w)$	g_1g_2	t_2
16	E_8^3	−00	E_8^2	3	2	480
16	$D_{16}E_8$	0−	D_{16}	1	1	480
16	D_8^3	−00	D_8^2	3	2	224
17	$A_{11}D_7E_6$	0−0	$A_{11}E_6$	1	2	204
18	$A_{17}E_7$	0−	$A_{17}A_1$	1	2	308
18	$D_{10}E_7^2$	0−0	$D_{10}E_7A_1$	2	1	308
18	D_6^4	−000	D_6^3	4	6	180
18	$A_9^2D_6$	00−	A_9^2	1	4	180
19	E_6^4	−000	$E_6^3O_1$	4	12	216
19	$A_{11}D_7E_6$	00−	$A_{11}D_7O_1$	1	2	216
19	$A_7^2D_5^2$	00−0	$A_7^2D_5$	2	4	152
20	D_{24}	+	D_{20}	1	1	760
20	$D_{16}E_8$	+0	$D_{12}E_8$	1	1	504
20	D_{12}^2	+0	$D_{12}D_8$	2	1	376
20	$D_{10}E_7^2$	+00	$E_7^2D_6$	1	2	312
20	$A_{15}D_9$	0+	$A_{15}D_5$	1	2	280
20	D_8^3	+00	$D_8^2D_4$	3	2	248
20	$A_{11}D_7E_6$	0+0	$A_{11}E_6A_3$	1	2	216
20	D_6^4	+000	$D_6^3A_1^2$	4	6	184
20	$A_9^2D_6$	00+	$A_9^2A_1^2$	1	4	184
20	$A_7^2D_5^2$	00+0	$A_7^2D_5O_1$	2	4	152
20	D_4^6	$+0^5$	D_4^5	6	120	120
20	$A_5^4D_4$	0^4+	A_5^4	1	16	120
21	A_{24}	+	$A_{20}O_1$	1	2	420
21	$A_{17}E_7$	+0	$A_{13}E_7O_1$	1	2	308
21	$A_{15}D_9$	+0	$A_{11}D_9O_1$	1	2	276
21	A_{12}^2	+0	$A_{12}A_8O_1$	2	2	228
21	$A_{11}D_7E_6$	+00	$D_7A_7E_6O_1$	1	2	212
21	$A_9^2D_6$	+00	$A_9D_6A_5O_1$	2	2	180
21	A_8^3	+00	$A_8^3A_4O_1$	3	4	164
21	$A_7^2D_5^2$	+000	$A_7D_5^2A_3O_1$	2	4	148
21	A_6^4	+000	$A_6^3A_2O_1$	4	6	132
21	$A_5^4D_4$	+0000	$A_5^3D_4A_1O_1$	4	12	116
21	A_4^6	$+0^5$	$A_4^5O_1$	6	40	100
21	A_3^8	$+0^7$	A_3^7	8	336	84
22	$D_{16}E_8$	**	$D_{14}E_7A_1$	1	1	492
22	E_8^3	**0	$E_8E_7^2$	3	2	492
22	D_{12}^2	**	$D_{10}^2A_1^2$	1	2	364
22	$A_{17}E_7$	**	$A_{15}D_6O_1$	1	2	300
22	$D_{10}E_7^2$	0**	$D_{10}D_6^2$	1	2	300
22	$D_{10}E_7^2$	**0	$D_8E_7D_6A_1$	2	1	300
22	$A_{15}D_9$	**	$A_{13}D_7A_1O_1$	1	2	268
22	D_8^3	**0	$D_8D_6^2A_1^2$	3	2	236
22	A_{12}^2	**	$A_{10}^2O_2$	1	4	220
22	E_6^4	**00	$E_6^2A_5^2$	6	8	204
22	$A_{11}D_7E_6$	0**	$A_{11}D_5A_5A_1$	1	2	204
22	$A_{11}D_7E_6$	*0*	$A_9D_7A_5O_1$	1	2	204
22	$A_{11}D_7E_6$	**0	$A_9E_6D_5A_1O_1$	1	2	204
22	D_6^4	**00	$D_6^2D_4^2A_1^2$	6	4	172
22	$A_9^2D_6$	**0	$A_7^2D_6O_2$	1	4	172

TABLE 1 (*cont.*)

dim	N	w	V	$c(w)$	g_1g_2	t_2
22	$A_9^2D_6$	*0*	$A_9A_7D_4A_1O_2$	2	2	172
22	A_8^3	**0	$A_8A_6^2O_2$	3	4	156
22	$A_7^2D_5^2$	00**	$A_7^2A_3^2A_1^2$	1	8	140
22	$A_7^2D_5^2$	**00	$D_5^2A_5^2O_2$	1	8	140
22	$A_7^2D_5^2$	*0*0	$A_7D_5A_5A_3A_1O_1$	4	2	140
22	A_6^4	**00	$A_6^2A_4^2O_2$	6	4	124
22	D_4^6	$**0^4$	$D_4^4A_1^6$	15	144	108
22	$A_5^4D_4$	*000*	$A_5^3A_3A_1^3O_1$	4	12	108
22	$A_5^4D_4$	**000	$A_5^2D_4A_3^2O_2$	6	8	108
22	A_4^6	$**0^4$	$A_4^4A_2^2O_2$	15	16	92
22	A_3^8	$**0^6$	$A_3^6A_1^2O_2$	28	96	76
22	A_2^{12}	$**0^{10}$	$A_2^{10}O_2$	66	2880	60
22	A_1^{24}	$**0^{22}$	A_1^{22}	276	887040	44
23	$D_{16}E_8$	10	$A_{15}E_8$	1	2	480
23	A_{24}	5	$A_{19}A_4$	2	2	400
23	D_{12}^2	12	$D_{11}A_{11}O_1$	2	2	352
23	$A_{17}E_7$	60	$A_{11}E_7A_5$	2	2	288
23	$D_{10}E_7^2$	110	$A_9E_7E_6O_1$	2	2	288
23	$D_{10}E_7^2$	211	$D_9E_6^2O_2$	1	4	288
23	$A_{17}E_7$	31	$A_{14}E_6A_2O_1$	2	2	288
23	$A_{15}D_9$	80	$D_9A_7^2$	1	4	256
23	$A_{15}D_9$	21	$A_{13}A_8A_1O_1$	2	2	256
23	$A_{15}D_9$	42	$A_{11}D_8A_3O_1$	2	2	256
23	D_8^3	033	$D_8A_7^2O_1$	3	4	224
23	D_8^3	122	$D_7^2A_7O_2$	3	4	224
23	A_{12}^2	15	$A_{11}A_7A_4O_1$	4	2	208
23	A_{12}^2	32	$A_{10}A_9A_2A_1O_1$	4	2	208
23	E_6^4	0111	$E_6D_5^3O_2$	8	12	192
23	$A_{11}D_7E_6$	620	$E_6D_6A_5^2O_1$	1	4	192
23	$A_{11}D_7E_6$	401	$D_7A_7D_5A_3O_1$	2	2	192
23	$A_{11}D_7E_6$	330	$A_8E_6A_6A_2O_1$	2	2	192
23	$A_{11}D_7E_6$	111	$A_{10}A_6D_5O_2$	2	2	192
23	$A_{11}D_7E_6$	222	$A_9D_6D_5A_1O_2$	2	2	192
23	D_6^4	2222	$D_5^4O_3$	1	48	160
23	$A_9^2D_6$	501	$A_9A_5A_4^2O_1$	2	4	160
23	D_6^4	0123	$D_6D_5A_5^2O_2$	12	4	160
23	$A_9^2D_6$	240	$A_7D_6A_5A_3A_1O_1$	4	2	160
23	$A_9^2D_6$	312	$A_8A_6D_5A_2O_2$	4	2	160
23	$A_9^2D_6$	121	$A_8A_7A_5A_1O_2$	4	2	160
23	A_8^3	036	$A_8A_5^2A_2^2O_1$	6	4	144
23	A_8^3	411	$A_7^2A_4A_3O_2$	6	4	144
23	A_8^3	177	$A_7A_6^2A_1^2O_2$	6	4	144
23	$A_7^2D_5^2$	4400	$D_5^2A_3^4O_1$	1	16	128
23	$A_7^2D_5^2$	4022	$A_7D_4^2A_3^2O_2$	2	8	128
23	$A_7^2D_5^2$	2031	$A_7A_5A_4^2A_1O_2$	4	4	128
23	$A_7^2D_5^2$	2220	$D_5A_5^2D_4A_1^2O_2$	4	4	128
23	$A_7^2D_5^2$	1112	$A_6^2D_4A_4O_3$	4	4	128
23	$A_7^2D_5^2$	1303	$A_6D_5A_4^2A_2O_2$	8	2	128
23	A_6^4	5111	$A_5^3A_4A_1O_3$	8	6	112
23	A_6^4	0124	$A_6A_5A_4A_3A_2A_1O_2$	24	2	112
23	D_4^6	002332	$D_4^2A_3^4O_3$	45	96	96
23	$A_5^4D_4$	00331	$A_5^2A_3A_2^4O_2$	6	16	96
23	$A_5^4D_4$	02220	$A_5D_4A_3^3A_1^3O_2$	8	12	96
23	$A_5^4D_4$	31110	$D_4A_4^3A_2^2O_3$	8	12	96
23	$A_5^4D_4$	04111	$A_5A_4^2A_3^2A_1O_3$	24	4	96

TABLE 1 (*cont.*)

dim	N	w	V	$c(w)$	g_1g_2	t_2
23	A_4^6	011111	$A_4A_3^5O_4$	12	40	80
23	A_4^6	001234	$A_4^2A_3^2A_2^2A_1^2O_3$	60	8	80
23	A_3^8	0^42^4	$A_3^4A_1^8O_3$	14	384	64
23	A_3^8	0^321^33	$A_3^3A_2^4A_1^2O_4$	112	48	64
23	A_2^{12}	0^61^6	$A_2^6A_1^6O_5$	264	1440	48
23	A_1^{24}	$0^{16}1^8$	$A_1^{16}O_7$	759	645120	32
23	Λ_{24}	min	O_{23}	196560	84610842624000	0

The lattices of minimum norm 1 are easily determined, as we shall see in Section 2, and the numbers of lattices in each dimension may be found in Table 6 below. The entries in Table 1 are explained in Section 2.8. The proof of Theorem 1 is based on the remark (justified in Section 2.3) that any such lattice is associated with a certain Niemeier lattice, and consists in finding all lattices that can be obtained from the Niemeier lattices. As a valuable check we determined the automorphism groups of the lattices in Table 1, and verified that the sum of the reciprocals of the group orders of the lattices of each dimension is equal to the Minkowski–Siegel mass constant. Since we have not been able to find the mass constants stated correctly anywhere in the literature we give them here. (There are unfortunately still three errors even in the formulae as given in [26].) For any undefined terms the reader is referred to [3, 14, 17, 22].

THEOREM 2: THE MINKOWSKI–SIEGEL MASS CONSTANTS. *Let $\Phi_X(n)$ denote the family of all inequivalent lattices of dimension n and genus* X. Then

$$\sum_{\Lambda\in\Phi_I(n)}\frac{1}{|\mathrm{Aut}(\Lambda)|}$$

is given by

$$\frac{(1-2^{-k})(1+2^{1-k})}{2.k!}|B_k.B_2B_4\cdots B_{2k-2}|,\ \text{if } n=2k\equiv 0 \pmod 8,$$

$$\frac{2^k+1}{k!2^{2k+1}}|B_2B_4\cdots B_{2k}|,\ \text{if } n=2k+1\equiv \pm 1 \pmod 8,$$

$$\frac{1}{(k-1)!2^{2k+1}}|E_{k-1}.B_2B_4\cdots B_{2k-2}|,\ \text{if } n=2k\equiv \pm 2 \pmod 8,$$

$$\frac{2^k-1}{k!2^{2k+1}}|B_2B_4\cdots B_{2k}|,\ \text{if } n=2k+1\equiv \pm 3 \pmod 8,$$

$$\frac{(1-2^{-k})(1-2^{1-k})}{2.k!}|B_k.B_2B_4\cdots B_{2k-2}|,\ \text{if } n=2k\equiv 4 \pmod 8.$$

Here B_k and E_k are the k-th Bernoulli and Euler numbers, respectively: $B_0=1$, $B_1=-1/2$, $B_2=1/6,\ldots$, $E_0=1$, $E_2=-1$, $E_4=5,\ldots$ [1, *p.* 810]. *Also*

$$\sum_{\Lambda\in\Phi_{II}(n)}\frac{1}{|\mathrm{Aut}(\Lambda)|}=\frac{|B_k|}{2k}\prod_{j=1}^{k-1}\frac{|B_{2j}|}{4j},$$

if $n=2k\equiv 0 \pmod 8$.

(The formulae for genus I may be derived from [18–21, 23–25], while the formula for genus II is given in [22].) The numerical values of these expressions for $n \leqslant 32$ are shown in Tables 2–4.

2. The Proof of Theorem 1

2.1. Niemeier's Result. Niemeier [16] showed that there are exactly 24 even unimodular lattices in $\mathbb{R}^{24}$, 23 of which have minimum norm 2, and one, the Leech lattice Λ_{24}, having minimum norm 4. We shall refer to these 24 as the *Niemeier lattices.*

The Niemeier lattices of minimum norm 2 (and more generally any integral lattice of minimum norm 1 or 2) are composed of various *Witt components* held together by *glue.* This glueing theory has been adequately described elsewhere ([8, 9]), and so we shall just recall that the Witt component lattices are taken from the list $O_n (n \geqslant 1)$, an empty component; $\mathbb{Z}$, the one-dimensional lattice of integers; and the root lattices $A_n (n \geqslant 1)$, $D_n (n \geqslant 1)$, E_6, E_7 and E_8. The subscript on a lattice indicates its dimension. We sometimes use I_m to denote the m-dimensional integer lattice $\mathbb{Z}^m$. The enumeration of these lattices is usually carried out up to equivalence, two lattices being called *equivalent* if one can be obtained from the other by a rotation and change of scale.

Alternative proofs of Niemeier's result have been given by Venkov [27] and in our earlier paper [9]. It is perhaps worth outlining a fourth proof, that uses hyperbolic geometry and throws some light on the connections between [7], [10] and [16].

Let G denote the group of all autochronous automorphisms of the even unimodular 26-dimensional Lorentzian lattice $\mathrm{II}_{25,1}$, and let H be the reflection subgroup of G. The groups G and H were found in [6], where in particular it was shown that H is a Coxeter group whose graph is isomorphic to the Leech lattice. From Vinberg's work [28] it follows that the even unimodular 24-dimensional Euclidean lattices of minimum norm 2 are described by those maximal subdiagrams of the Leech lattice that are unions of the extended Coxeter–Dynkin diagrams $A_n (n \geqslant 1)$, $D_n (n \geqslant 4)$, E_6, E_7 and E_8. But in [7] it was shown that there are precisely 23 such subdiagrams, which are exactly those corresponding to the Niemeier lattices of minimum norm 2. This result, plus the fact [4] that the Leech lattice is the unique even unimodular lattice of minimum norm $\geqslant 4$, provides another proof of Niemeier's result. It also clarifies the one-to-one correspondence between the deep holes in the Leech lattice [7] and the Niemeier lattices. We must emphasize, however, that in no sense is this a short-cut to Niemeier's result, for the proofs in [7] required extensive computations.

Remark. The notation used in this paper to specify the components of a lattice differs slightly from that used in [9]. For example the lattice $E_6^3 O_1$ was called $E_6^3[3]$ in reference [9]. The latter notation is more informative but less general (it fails when the empty component has dimension greater than one), and somewhat confusing, since [3] is also the name of a glue digit. We therefore recommend the ... O_n notation for general use.

2.2. Chains of Lattices. Every unimodular lattice Λ appears in a uniquely determined chain of the form

$$\Lambda_n, \Lambda_n \oplus I_1, \Lambda_n \oplus I_2, \dots, \Lambda = \Lambda_n \oplus I_m, \dots$$

where the initial lattice Λ_n does not represent 1. We call Λ_n the *reduced* version of Λ. The summand I_m is the sublattice of Λ generated by vectors of norm 1, and Λ_n is its orthogonal complement. In these circumstances we have

$$|\mathrm{Aut}(\Lambda_n \oplus I_m)| = |\mathrm{Aut}(\Lambda_n)| . 2^m m!.$$

TABLE 2
Exact values of the Minkowski–Siegel mass constants for lattices of genus I

n	mass
1	1
2	$\frac{1}{8}$
3	$\frac{1}{48}$
4	$\frac{1}{384}$
5	$\frac{1}{3840}$
6	$\frac{1}{46080}$
7	$\frac{1}{645120}$
8	$\frac{1}{10321920}$
9	$\frac{17}{2786918400}$
10	$\frac{1}{2229534720}$
11	$\frac{31}{735746457600}$
12	$\frac{31}{5885971660800}$
13	$\frac{691}{765176315904000}$
14	$\frac{42151}{192824431607808000}$
15	$\frac{29713}{385648863215616000}$
16	$\frac{505121}{12340763622899712000}$
17	$\frac{642332179}{18881368343036559360000}$
18	$\frac{692319119}{15105094674429247488000}$
19	$\frac{8003636403977}{77489135679822039613440000}$
20	$\frac{248112728523287}{619913085438576316907520000}$
21	$\frac{593468652605200909}{216969579903501710917632000000}$
22	$\frac{50904295073459007001}{1507367607750643465322496000000}$
23	$\frac{10157405324982344470066371}{1317439289174062388691861504000000}$

TABLE 2 (*cont.*)

n	mass
24	$\dfrac{7018767079562800188158623 61}{21079028626784998219069784064000000}$
25	$\dfrac{847150594803046516236122728421 47}{3046539608000631801426732908544 0000000}$
26	$\dfrac{1461633563589438887618847268485 1927}{318714912836989173072335135047 68000000}$
27	$\dfrac{18943527517721468674304869954629232 65007}{124298816006425777498210702668595 20000000}$
28	$\dfrac{10345060377427694043037889482223023950203227}{99439052805140621998568562134876160000000}$
29	$\dfrac{428500982395959068211562873935616958668722075 2159}{288373253134907803795848830191140864 00000000}$
30	$\dfrac{15642991431957907027010271029295720146572585045 1195039}{3460479037618893645550185962293690368 00000000}$
31	$\dfrac{44754357270087840423277214992727557304272563905958558 7715489}{1501847902326599842168780707635461619712 00000000}$
32	$\dfrac{41602263133198283734425377463574762715775342957497991518709 9394241}{96118265748902389898801965288669543661568 00000000}$

TABLE 3
Decimal expansions of the mass constants for lattices of genus I

n	mass	n	mass	n	mass	n	mass
1	1·0	9	$6{\cdot}100\times10^{-9}$	17	$3{\cdot}402\times10^{-14}$	25	$2{\cdot}781\times10^{-6}$
2	1·25	10	$4{\cdot}485\times10^{-10}$	18	$4{\cdot}583\times10^{-14}$	26	$4{\cdot}586\times10^{-4}$
3	$2{\cdot}083\times10^{-2}$	11	$4{\cdot}213\times10^{-11}$	19	$1{\cdot}033\times10^{-13}$	27	$1{\cdot}524\times10^{-1}$
4	$2{\cdot}604\times10^{-3}$	12	$5{\cdot}267\times10^{-12}$	20	$4{\cdot}002\times10^{-13}$	28	$1{\cdot}040\times10^{2}$
5	$2{\cdot}604\times10^{-4}$	13	$9{\cdot}031\times10^{-13}$	21	$2{\cdot}735\times10^{-12}$	29	$1{\cdot}486\times10^{5}$
6	$2{\cdot}170\times10^{-5}$	14	$2{\cdot}186\times10^{-13}$	22	$3{\cdot}377\times10^{-11}$	30	$4{\cdot}520\times10^{8}$
7	$1{\cdot}551\times10^{-6}$	15	$7{\cdot}705\times10^{-14}$	23	$7{\cdot}710\times10^{-10}$	31	$2{\cdot}980\times10^{12}$
8	$9{\cdot}688\times10^{-8}$	16	$4{\cdot}093\times10^{-14}$	24	$3{\cdot}330\times10^{-8}$	32	$4{\cdot}328\times10^{16}$

Our enumeration process considers all lattices of a chain simultaneously. For some purposes the most appropriate lattice to consider is the initial lattice Λ_n, but for other purposes it is the 23-dimensional lattice $\Lambda_n \oplus I_{23-n}$.

2.3. THE ASSOCIATED NIEMEIER LATTICE. We now describe how an odd unimodular lattice Λ_n of dimension $n \leq 23$ is associated with a uniquely determined Niemeier lattice N. For brevity we omit the easy justifications of some statements, since in any case the mass formula provides an independent verification. Let $m = 24 - n$. Both Λ_n and I_m have sublattices of index 2 containing only vectors of even norm, denoted by Λ_n^0 and I_m^0, respectively. The dual of Λ_n^0 consists of Λ_n^0 and three cosets, say Λ_n^1, Λ_n^2, Λ_n^3, and similarly the dual of I_m^0 consists of the cosets I_m^0, I_m^1, I_m^2, I_m^3. The notation can be chosen

TABLE 4
The mass constants for lattices of genus II

n	mass
0	1
8	$\dfrac{1}{696729600}$
16	$\dfrac{691}{277667181515243520000}$
24	$\dfrac{1027637932586061520960267}{129477933340026851560636148613120000000}$
32	$\dfrac{4890529010450384254108570593011950899382291953107314413193123}{121325280941552041649762780685623131486814208000000000}$

so that

$$\Lambda_n = \Lambda_n^0 \cup \Lambda_n^2,$$
$$I_m = I_m^0 \cup I_m^2,$$

and the set of all vectors

$$\{x + y : x \in \Lambda_n^k,\ y \in I_m^k,\ k = 0, 1, 2, 3\}$$

is an even unimodular 24-dimensional lattice. The latter is the Niemeier lattice N associated with Λ_n.

2.4. CONSTRUCTING Λ_n FROM N. The lattice I_m^0 is better known as D_m, and seen from the point of view of N the process we have just described appears as follows. We look for a copy of the lattice D_m in N and observe that Λ_n is the set of vectors x in the orthogonal complement $D_m^{\perp}$ for which either $x+[0]$ or $x+[2]$ is in N, where [0], [1], [2], [3] are the glue vectors for D_m in $D_m^{\perp}$, [2] being a glue vector of norm 1. (For further information about these glue vectors see reference [9].) Λ_n will be the initial member of its chain if D_m is not contained in a $D_{m'}$ in N with $m' > m$ using the same glue vector [2].

Alternatively we can refer to the 23-dimensional member of the chain, for which the appropriate D_m is a D_1 generated by a single vector $v = 2e$ of norm 4 and $[2] = e$. This 23-dimensional lattice is the set of vectors $x \in e^{\perp}$ for which $x + ne \in N$ for some integer n. The other members of the chain can be obtained by removing summands I_k.

Thus our process can be viewed from either end of the chain containing Λ_n. On the one hand we search for a maximal sublattice D_m (and glue vector [2]) in the Niemeier lattice N, and locate the initial member of the chain in $D_m^{\perp}$. Alternatively we find all possible vectors $v = 2e$ of norm 4 in N and locate the 23-dimensional member of the chain in the subspace $v^{\perp}$.

2.5. THE NIEMEIER LATTICES OF MINIMUM NORM 2. Let N be one of the 23 Niemeier lattices with minimum norm 2, and let $W_1, W_2, \ldots$ be its Witt components. The norm 4 vector v used in the construction may then be written as

$$v = v_1 + v_2 + \cdots$$

where $v_1, v_2, v_3, \ldots$ are

(1) minimal representatives of glue digits corresponding to a glue word of norm 4;

(2) minimal vectors r, s of norm 2 in two distinct Witt components W_i, W_j, and 0 in the other components; or

(≥ 3) 0 except for one v_i which is a vector of norm 4 in its Witt component W_i.

(Again we omit the easy proof of this statement.) The cases have been numbered so as to correspond with the value of m mentioned in Section 2.4.

In case (1) the D_1 generated by v is maximal and we obtain a 23-dimensional lattice not representing 1. Our symbol for this case is the glue word mentioned.

In case (2) the D_2 generated by r and s is maximal with $[2]=\frac{1}{2}(r+s)$, and so the reduced lattice is 22-dimensional. The symbol for this case is a word with $*$'s in positions i and j and 0's elsewhere.

The cases (≥ 3) can be further subdivided as follows. Case (3): $v=v_i$ is the vector $(1, 1, -1, -1, 0^{k-3})$ in a Witt component $W_i=A_k$ for some $k\geq 3$. The maximal D_m is the sublattice A_3 of A_k supported in the four non-zero coordinates of v_i. The reduced lattice is 21-dimensional, and the symbol for this case has a + in position i and 0's elsewhere.

Case (4): $v=v_i$ is a vector of the form $(\pm 1^4, 0^{k-4})$ in a component D_k. In this case the maximal D_m is the D_4 supported in the four non-zero coordinates of v, and the reduced lattice is 20-dimensional. Again the symbol contains 0's except for a + in position i. We remark that the case $k=4$, when there is a Witt component D_4, is rather special because all three non-trivial cosets of D_4 in its dual contain vectors of norm 1 which can serve as the glue vector [2]. Fortunately, in both the cases $N=D_4^6$ and $N=A_5^4D_4$, Aut(N) contains elements permuting all three cosets of any D_4 component, so we may suppose $v=(\pm 1^4, 0^{k-4})$.

Cases (≥ 5): $v=v_i$ is the vector $(2, 0^{k-1})$ in a D_k ($k\geq 5$) or any norm 4 vector in E_6, E_7 or E_8. The maximal D_m in these four cases is D_k, D_5, D_6 or D_8 respectively, and the symbol has a $-$ in position i and 0's elsewhere.

2.6. Inequivalent Lattices. The above remarks make it easy to determine when two lattices constructed in this way are equivalent. Each lattice Λ_n is specified by a Niemeier lattice

$$N=W_1W_2\cdots W_a$$

and a symbol

$$w=d_1d_2\cdots d_a$$

which is either a glue word or a permutation of $**0^{a-2}$, $+0^{a-1}$ or -0^{a-1}. Two lattices are equivalent if and only if they are specified by the same N and their symbols are equivalent under Aut(N). The Witt components V_1, V_2, ... of Λ_n are determined by the W_i and d_i as shown in Table 5.

2.7. The Automorphism Group of Λ_n. To compute the order of the automorphism group of Λ_n we argue as follows. The lattices Λ_n, Λ_{n+1}, ... of a chain whose reduced lattice is Λ_n have

$$|\text{Aut}(\Lambda_{n+m})|=|\text{Aut}(\Lambda_n)|.2^m m!,$$

so that is suffices to compute $|\text{Aut}(\Lambda_{23})|$. Now a 23-dimensional lattice Λ_{23} is completely determined by the associated Niemeier lattice N and either of the norm 4 vectors $\pm v$ of $\Lambda_{23}^{\perp}\cap N$. Hence

$$|\text{Aut}(\Lambda_{23})|=\frac{2|\text{Aut}(N)|}{c(v)}, \tag{1}$$

where Aut(N) is given in [9] and $c(v)$ is the number of images of v under Aut(N). This number is easily computed as the product

$$c(v) = c(d_1)c(d_2)\cdots c(d_a)c(w), \tag{2}$$

where $w = d_1 d_2 \cdots d_a$ is the symbol for Λ_N, $c(w)$ is the number of images of w under Aut(N), and $c(d_i)$ is the number of choices for the component v_i of the vector v which would lead to the digit d_i. For example $c(*)$ is the number of minimal vectors in W_i. The numbers $c(d_i)$ can be found by elementary counting arguments and are given in Table 5. The values of $c(w)$, which are usually small, were found by careful consideration of the group action, and are given in Table 1.

TABLE 5

Each lattice Λ_n is specified by the Witt components $W_1, W_2, \ldots$ of a Niemeier lattice N and a certain symbol $w = d_1, d_2, \ldots$. This table gives the Witt components $V_1, V_2, \ldots$ of Λ_n corresponding to each Witt component W of N and each digit d of w, as well as the numbers $c(d)$ defined in Section 2.7. The four parts of the table correspond to (a) $W = A_n$, (b) $W = D_4$, (c) $w = D_n (n \geqslant 5)$, and (d) $W = E_6$, E_7 or E_8

A_n		
d	$V_1 V_2 \cdots$	$c(d)$
0	A_n	1
i	$A_{i-1}A_{n-i}$	$\binom{n+1}{i}$
*	A_{n-2}	$2\binom{n+1}{2}$
+	A_{n-4}	$6\binom{n+1}{4}$

D_4		
d	$V_1 V_2 \cdots$	$c(d)$
0	D_4	1
1	A_3	8
2	A_3	8
3	A_3	8
*	A_1^3	24
+	–	24

$D_n (n \geqslant 5)$		
d	$V_1 V_2 \cdots$	$c(d)$
0	D_n	1
1	A_{n-1}	2^{n-1}
2	D_{n-1}	$2n$
3	A_{n-1}	2^{n-1}
*	$D_{n-2}A_1$	$4\binom{n}{2}$
+	D_{n-4}	$16\binom{n}{4}$
–	–	$2n$

	E_6		E_7		E_8	
d	$V_1 V_2 \cdots$	$c(d)$	$V_1 V_2 \cdots$	$c(d)$	$V_1 V_2 \cdots$	$c(d)$
0	E_6	1	E_7	1	E_8	1
1	D_5	27	E_6	56		
2	D_5	27				
*	A_5	72	D_6	126	E_7	240
–	–	270	A_1	756	–	2160

The orders of the automorphism groups of the lattices Λ_n in Table 1 were calculated from Equations (1) and (2). The lattices with minimum norm 1 were then found by forming direct sums $\Lambda_n \oplus I_m$, and the mass constants were checked against the formulae given in Theorem 2. (This completes the formal proof of Theorem 1.) The numbers of lattices found in each dimension are shown in Table 6.

TABLE 6
The number of unimodular lattices in $\mathbf{R}^n$ of genus I that do not represent 1 (α_n), the number of genus II (β_n), and the total number of genus I and II (γ_n)

n	0	1	2	3	4	5	6	7	8	9	10	11	12
α_n	0	0	0	0	0	0	0	0	0	0	0	0	1
β_n	1	0	0	0	0	0	0	0	1	0	0	0	0
γ_n	1	1	1	1	1	1	1	1	2	2	2	2	3

n	13	14	15	16	17	18	19	20	21	22	23	24
α_n	0	1	1	1	1	4	3	12	12	28	49	?
β_n	0	0	0	2	0	0	0	0	0	0	0	24
γ_n	3	4	5	8	9	13	16	28	40	68	117	?

The purpose of the column headed g_1g_2 in Table 1 is to provide the reader with an alternative and easier method of computing $|\mathrm{Aut}(\Lambda_n)|$, via the formula

$$|\mathrm{Aut}(\Lambda_n)| = \prod_{i=1}^{b} g_0(V_i).g_1(\Lambda_n)g_2(\Lambda_n). \tag{3}$$

Here $V_1, \ldots, V_b$ are the Witt components of Λ_n (given in the column headed V in Table 1), and the values of $g_0(V_i)$ can be found in Table 7. The product $g_1(\Lambda_n)g_2(\Lambda_n)$ is given in the column headed g_1g_2 in Table 1. For example the penultimate line of Table 1 refers to the lattice $A_1^{16}O_7$, for which

$$|\mathrm{Aut}(\Lambda_n)| = (2!)^{16}.1^7.645120.$$

(The explanation for this decomposition of $|\mathrm{Aut}(\Lambda_n)|$ may be found in [8] or [9], but need not concern us here.) The values of g_1g_2 were actually "back-computed" from Equations (1), (2) and (3) after the mass formulae check had been applied.

TABLE 7
Values of $g_0(W_i)$ for Witt components W_i

W_i	O_n	A_n	D_n	E_6	E_7	E_8
$g_0(W_i)$	1	$(n+1)!$	$2^{n-1}.n!$	$72.6!$	$8.9!$	$192.10!$

2.8. DESCRIPTION OF TABLE 1. Table 1 shows the unimodular lattices Λ_n that contain no vectors of norm 1. The columns give the dimension n, the Witt components $W_1, W_2, \ldots$ of the corresponding Niemeier lattice N, the symbol w, the Witt components $V = V_1, V_2, \ldots$ of Λ_n itself, and the numbers $c(w)$ and $g_1(\Lambda_n)g_2(\Lambda_n)$ (see Section 2.7). The last column gives the number (t_2) of norm 2 vectors in Λ_n. We note that V identifies Λ_n uniquely.

2.9. THE LEECH LATTICE. It remains to consider the unique Niemeier lattice with minimum norm greater than 2, the Leech lattice. In this lattice there is just one orbit of norm 4 vectors ([5]). Our construction produces a certain 23-dimensional lattice with minimum norm 3, which may be called the *shorter Leech lattice*. It appears in the last line of Table 1, and is the only lattice in the table with minimum norm 3.

Acknowledgements

The extensive arithmetical calculations needed to compute the two sides of the mass formulae were performed using ALTRAN [2] and MACSYMA [13]. We should like to thank the M.I.T. Laboratory for Computer Science for allowing us to use MACSYMA.

References

1. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions, Nat. Bur. Standards Appl. Math. Ser.* **55**, U.S. Dept. Commerce, Washington, D.C., 1972.
2. W. S. Brown, *ALTRAN User's Manual*, 4th ed. Bell Laboratories, Murray Hill, 1977.
3. J. W. S. Cassels, *An Introduction to the Geometry of Numbers*, Springer-Verlag, New York, 1971.
4. J. H. Conway, A characterization of Leech's lattice, *Invent. Math.* **7** (1969), 137–142.
5. J. H. Conway, Three lectures on exceptional groups, in *Finite Simple Groups* (M. B. Powell and G. Higman, eds), Academic Press, New York, 1971, pp. 215–247.
6. J. H. Conway, The automorphism group of the 26-dimensional even unimodular Lorentzian lattice, *J. Algebra* (to appear).
7. J. H. Conway, R. A. Parker and N. J. A. Sloane, The covering radius of the Leech lattice, *Proc. Roy. Soc. London Ser. A* **380** (1982), 261–290.
8. J. H. Conway and V. Pless, On the enumeration of self-dual codes, *J. Combin. Theory Ser. A* **28** (1980), 26–53.
9. J. H. Conway and N. J. A. Sloane, On the enumeration of lattices of determinant one, *J. Number Theory* **15** (1982), 83–94.
10. J. H. Conway and N. J. A. Sloane, Lorentzian forms for the Leech lattice, *Bull. Amer. Math. Soc.* **6** (1982), 215–217.
11. J. H. Conway and N. J. A. Sloane, Twenty-three constructions for the Leech lattice, *Proc. Roy. Soc. London Ser. A* **381** (1982), 275–283.
12. J. H. Conway and N. J. A. Sloane, Leech roots and Vinberg groups, *Proc. Roy. Soc. London Ser. A* (to appear).
13. Mathlab Group, *MACSYMA Reference Manual*, Version 9, Laboratory for Computer Science, M.I.T., Cambridge, 1977.
14. J. Milnor and D. Husemoller, *Symmetric Bilinear Forms*, Springer-Verlag, New York, 1973.
15. A Neumaier and J. J. Seidel, Discrete hyperbolic geometry, *Combinatorica* (to appear).
16. H.-V. Niemeier, Definite quadratische Formen der Dimension 24 and Diskriminante 1, *J. Number Theory* **5** (1973), 142–178.
17. O. T. O'Meara, *Introduction to Quadratic Forms*, Springer-Verlag, New York, 1971.
18. G. Pall, The weight of a genus of positive n-ary quadratic forms, pp. 95–105 in *The Theory of Numbers* (A. L. Whiteman, ed.), *Proc. Symp. Pure Math. Vol. 8*, American Mathematical Society, Providence, R.I., 1965.
19. H. Pfeuffer, Bemerkung zur Berechnung dyadischer Darstellungsdichten einer quadratischen Form über algebraischen Zahlkörpern, *J. Reine Angew. Math.* **236** (1969), 219–220.
20. H. Pfeuffer, Quadratsummen in totalreellen algebraischen Zahlkörpern, *J. Reine Angew. Math.* **249** (1971), 208–216.
21. H. Pfeuffer, Einklassige Geschlechter totalpositiver quadratischer Formen in totalreellen algebraischen Zahlkörpern, *J. Number Theory* **3** (1971), 371–411.
22. J.-P. Serre, *Cours d'Arithmétique*, Presses Universitaires de France, Paris, 1970. English translation, Springer-Verlag, New York, 1973.
23. C. L. Siegel, Uber die analytische Theorie der quadratischen Formen, *Ann. of Math.* **36** (1935), 527–606 (Gesam. Abh. **I**, pp. 326–405, Springer-Verlag, New York. N.Y. 1966).
24. C. L. Siegel, Uber die Fourierschen Koeffizienten der Eisensteinschen Reihen, *Danske Vid. Selskab. Mat.-fys. Meddelelser* **34** (1964), Nr. 6 (Gesam. Abh. **III**, pp. 443–458, Springer-Verlag, New York, 1966).
25. C. L. Siegel, Berechnung von Zetafunktionen an ganzzahligen Stellen, *Göttingen Nach.* **10** (1969), 87–102.
26. N. J. A. Sloane, Self-dual codes and lattices, in *Relations Between Combinatorics and Other Parts of Mathematics, Proc. Sympos. Pure Math. Vol. 34*, American Mathematical Society, Providence, R.I., 1979, pp. 273–308.
27. B. B. Venkov, The classification of integral quadratic forms, English translation in *Proc. Steklov Inst. Math.* **4** (1980), 63–74.
28. È. B. Vinberg, On unimodular integral quadratic forms (in Russian), *Funkt. Analiz Prilozen* **6** (No. 2, 1972), 24–31. English translation in *Funct. Anal. Appl.* **6** (1972), 105–111.

29. È. B. Vinberg, Some arithmetical discrete groups in Lobacevskii spaces, in *Discrete Subgroups of Lie Groups and Applications to Moduli*, Oxford University Press, Oxford, 1975, pp. 323–348.

Received 2 April 1982 and in revised form 14 June 1982

J. H. CONWAY
Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, U.K.

N. J. A. SLOANE
Mathematics and Statistics Research Center, Bell Laboratories, Murray Hill, New Jersey 07974, U.S.A.