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The Farahat–Higman ring of wreath products and Hilbert schemes

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Abstract

We study the structure constants of the class algebra $R_{\mathbb{Z}}(\Gamma_n)$ of the wreath products Γ_n associated to an arbitrary finite group Γ with respect to the basis of conjugacy classes. We show that a suitable filtration on $R_{\mathbb{Z}}(\Gamma_n)$ gives rise to the graded ring $\mathcal{G}_\Gamma(n)$ with non-negative integer structure constants independent of n (some of which are computed), which are then encoded in a Farahat–Higman ring $\mathcal{G}_\Gamma$. The real conjugacy classes of Γ come to play a distinguished role and are treated in detail in the case when Γ is a subgroup of $SL_2(\mathbb{C})$. The above results provide new insight to the cohomology rings of Hilbert schemes of points on a quasi-projective surface X.

MSC: primary 20E; secondary 14C05; 14F25

Keywords: Wreath products; Class algebras; Hilbert schemes

1. Introduction

1.1. The wreath products $\Gamma_n = \Gamma^n \rtimes S_n$ associated to a finite group Γ are natural generalizations of the symmetric groups S_n (cf. [Mac,Zel]). Connections of the wreath products with Hilbert schemes of points on surfaces and with Nakajima quiver varieties [Na1,Na3] were first pointed out by the author (cf. [Wa3] for an overview). For example, it has since been expected that all geometric invariants on the Hilbert scheme $(\mathbb{C}^2//\Gamma)^{[n]}$ of n points on the minimal resolution $\mathbb{C}^2//\Gamma$ associated to $\Gamma \leqslant SL_2(\mathbb{C})$ can be entirely described using the wreath product Γ_n. The interrelations

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among wreath products, Hilbert schemes, and Nakajima quiver varieties have been one of the main motivations for us to study various aspects of wreath products, cf. [FJW,Wa1,Wa2,WaZ]. Also see [EG,Wa4] for more recent development.

The first aim of this paper is to develop a new approach of studying the class algebras of the wreath products associated to an arbitrary finite group Γ. We establish various properties of the structure constants of the class algebras $R_{\mathbb{Z}}(\Gamma_n)$ with respect to the basis of conjugacy classes. This allows us to introduce a ring $\mathscr{G}_\Gamma$ which encodes all the structures of the graded algebras associated to a natural filtration on $R_{\mathbb{Z}}(\Gamma_n)$ for all n. We call $\mathscr{G}_\Gamma$ the Farahat–Higman ring, or the FH-ring, since our results specialize to the ones of Farahat–Higman [FH] for the symmetric groups (see [Mac] for an elegant presentation and improvement). While we are much inspired by the approach of [FH], we need to generalize various concepts from the symmetric groups and develop new delicate combinatorial analysis in order to treat the extra complications coming from the presence of the group Γ. The appearance of new phenomena which are not observable in the symmetric group case will be crucial for the subsequent geometric applications.

The second aim of this paper is to point out the geometric implications of these results on wreath products. The FH-ring, when Γ is a finite subgroup of $SL_2(\mathbb{C})$, is isomorphic to the cohomology rings of Hilbert schemes of points on the minimal resolutions $\mathbb{C}^2//\Gamma$ of a simple singularity $\mathbb{C}^2/\Gamma$. In this way, our concrete results on the class algebras of Γ_n provide new insights into the cohomology rings of the corresponding Hilbert schemes. Perhaps no less significantly, the results on the Farahat–Higman ring etc enable us to predict some remarkable general structures of the cohomology rings of Hilbert schemes $X^{[n]}$, most notably for an arbitrary quasi-projective surface X. Let us explain in more detail.

1.2. The conjugacy classes of the wreath product Γ_n are determined by their types, which are in one-to-one correspondence with the set $\mathscr{P}_n(\Gamma_*)$ of partition-valued functions on Γ_* such that the total number of boxes in the corresponding Young diagrams is n. Here and further Γ_* denotes the set of conjugacy classes of Γ. The union $\Gamma_\infty = \bigcup_{n\geqslant 1}\Gamma_n$ carries a group structure by the natural embedding of Γ_n in Γ_{n+1}. We introduce a notion of *modified types*, (cf. [Mac, p. 131] for the symmetric group case), so that the set $\mathscr{P}(\Gamma_*)$ of modified types (which are all partition-valued functions on Γ_*) parametrizes naturally the conjugacy classes of Γ_∞. By assigning a certain non-negative integer to an element or its modified type, we define a function $||\cdot||$ on Γ_∞ or alternatively on $\mathscr{P}(\Gamma_*)$. We introduce a notion of *reduced expression* for an element in Γ_∞, and further identify $||x||$ with the length of a reduced expression for $x\in\Gamma_\infty$. The function $||\cdot||$ is sub-multiplicative and thus defines a filtration on the class algebra $R_{\mathbb{Z}}(\Gamma_n)$ for each n.

Let us denote by $K_\lambda(n)$ the sum of elements in the conjugacy class in Γ_n of modified type λ, and write the multiplication $K_\lambda(n)K_\mu(n) = \sum_\nu a^\nu_{\lambda\mu}(n)K_\nu(n)$ where $a^\nu_{\lambda\mu}(n)\in\mathbb{Z}_+$, and $a^\nu_{\lambda\mu}(n) = 0$ unless $||\nu||\leqslant||\lambda||+||\mu||$. We prove that the structure constants $a^\nu_{\lambda\mu}(n)$ are polynomials in n (whose degree is also described explicitly); and when $||\nu|| = ||\lambda||+||\mu||$, $a^\nu_{\lambda\mu}(n)$ is a constant $a^\nu_{\lambda\mu}\in\mathbb{Z}_+$ independent of n. This allows us to introduce a commutative ring $\mathscr{F}_\Gamma$ with a basis $(K_\lambda)_{\lambda \in \mathscr{P}(\Gamma_*)}$ and a $\mathbb{Z}_+$ filtration, whose associated graded ring $\mathscr{G}_\Gamma$ (called the *Farahat–Higman ring*) has a multiplication $K_\lambda K_\mu = \sum_{||\nu||=||\lambda||+||\mu||} a^\nu_{\lambda\mu} K_\nu$. Similarly, the filtration on $R_\mathbb{Z}(\Gamma_n)$ also gives rise to a graded ring, denoted by $\mathscr{G}_\Gamma(n)$, whose structure constants are non-negative integral and independent of n. The ring $\mathscr{G}_\Gamma$ determines a family of rings $\{\mathscr{G}_\Gamma(n)\}_{n\geqslant 1}$ and vice versa. When Γ is trivial, we write $\mathscr{G}_\Gamma(n)$ as $\mathscr{G}(n)$.

We further calculate explicitly these structure constants $a^\nu_{\lambda\mu}$ when μ is of single-cycle modified type. The calculation requires a rather careful combinatorial analysis of the convolution products for Γ_n, which is made possible by the rigid constraint $||\nu|| = ||\lambda|| + ||\mu||$. Let us mention below five interesting consequences of this computation. First, it provides us an algorithm for computing the general structure constants. Secondly, the ring $\mathbb{Q} \otimes_\mathbb{Z} \mathscr{G}_\Gamma$ is shown to be a polynomial ring generated by K_μ, where μ runs over all possible single-cycle modified types. Thirdly, we obtain a simple set of ring generators for the ring $\mathscr{G}_\Gamma(n)$. Fourthly, we show that the ring $\mathscr{G}_\Gamma(n)$ for any Γ contains the ring $\mathscr{G}(n)$ as a natural quotient. Fifthly, we observe a remarkable fact that the new structure constants of the ring $\mathbb{C} \otimes_\mathbb{Z} \mathscr{G}_\Gamma$ (over $\mathbb{C}$!) with respect to a suitably rescaled basis depend no longer on the group Γ, but only on two integers: the cardinalities of Γ_* and its subset Γ_*^{re} of non-trivial real conjugacy classes.

1.3. It is well known that the set of finite subgroups Γ of $SL_2(\mathbb{C})$ corresponds bijectively to the Dynkin diagrams of ADE types. The dual McKay correspondence of Ito–Reid [IR] (also cf. [Bry]) provides further a canonical bijection between the non-trivial conjugacy classes of Γ and vertices of the associated Dynkin diagram. For every finite subgroup Γ of $SL_2(\mathbb{C})$, we determine the set Γ_*^{re} and further observe that Γ_*^{re} affords an elegant interpretation as the fixed-point set of a distinguished Dynkin diagram automorphism, via the dual McKay correspondence. This automorphism in turn has simple interpretations in surface geometry as well as in Lie theory. To our best knowledge, the significance of the real conjugacy classes was not noted before in the literature on the McKay correspondence. We remark that there has been direct connection between wreath products and McKay correspondence [FJW,Wa2]. However, a direct interplay between McKay and dual McKay correspondences has yet to be discovered.

Note that Γ_n acts on $\mathbb{C}^{2n}$ naturally, and the orbifold $\mathbb{C}^{2n}/\Gamma_n$ can be regarded as a generalization of the simple singularity $\mathbb{C}^2/\Gamma$. The author constructed a crepant resolution of singularities $(\mathbb{C}^2//\Gamma)^{[n]} \to \mathbb{C}^{2n}/\Gamma_n$ and another crepant resolution of singularities of $\mathbb{C}^{2n}/\Gamma_n$ which can be identified with a Nakajima variety (cf. [Wa1,Wa2,Wa3]). A combination of a theorem of Etingof–Ginzburg [EG] and results of Nakajima [Na1] etc[1] says that there is a ring isomorphism between

[1] A quiver variety $M_{\Gamma,n}$ is used in [EG] in place of the Hilbert scheme $(\mathbb{C}^2//\Gamma)^{[n]}$. However, according to Nakajima (unpublished) and Kuznetsov [Kuz], $(\mathbb{C}^2//\Gamma)^{[n]}$ affords a same quiver variety description with different stability conditions. By a theorem in [Na1], $(\mathbb{C}^2//\Gamma)^{[n]}$ and $\mathbb{M}_{\Gamma,n}$ are diffeomorphic, and thus have isomorphic cohomology rings.

the cohomology ring $H^*((\mathbb{C}^2//\Gamma)^{[n]})$ with $\mathbb{C}$-coefficient and the graded ring associated to the orbifold filtration on $\mathbb{C}\otimes_{\mathbb{Z}}R_{\mathbb{Z}}(\Gamma_n)$ defined by the shift numbers [Zas] (also cf. [IR]). These shift numbers for Γ_n were computed earlier in [WaZ], and they coincides with the degree $||\cdot||$ introduced in this paper for $\Gamma \leqslant SL_2(\mathbb{C})$. Thus, there is a ring isomorphism between $H^*((\mathbb{C}^2//\Gamma)^{[n]})$ and $\mathbb{C}\otimes_{\mathbb{Z}}\mathscr{G}_\Gamma(n)$. Note that the ring $\mathbb{C}\otimes_{\mathbb{Z}}\mathscr{G}_\Gamma(n)$ for $\Gamma \leqslant SL_2(\mathbb{C})$ can alternatively be interpreted as the Chen–Ruan cohomology ring of $\mathbb{C}^{2n}/\Gamma_n$, and the above ring isomorphism supports conjectures of Ruan [Ru]. In the important special case $X=\mathbb{C}^2$, i.e. Γ is trivial, the ring isomorphism was due to Lehn–Sorger [LS1] and independently Vasserot [Vas] (but the connections to [FH] was not noticed). In this way, our results on $\mathscr{G}_\Gamma(n)$ and $\mathscr{G}_\Gamma$ for $\Gamma \leqslant SL_2(\mathbb{C})$ can be regarded as new results on the ring $H^*((\mathbb{C}^2//\Gamma)^{[n]})$.

1.4. In recent years, there has been much activity in understanding the cohomology rings of Hilbert schemes $X^{[n]}$ of n points on a projective surface X. However, when X is quasi-projective (the usage of the terminology *quasi-projective* in this paper excludes *projective*), the connections with vertex operators become obscure, and little is known about the cohomology rings of $X^{[n]}$ with the exception of the affine plane. Our guiding principle here (cf. [Wa3]) is that the study of the class algebras of Γ_n associated to an arbitrary finite group Γ (resp. the graded ring associated to some suitable filtration) should mirror the study of the cohomology rings of $X^{[n]}$ of points on an arbitrary surface X which is projective (resp. quasi-projective). From this view, the results on wreath products in this paper should reflect new finer structures of the cohomology rings of $X^{[n]}$ for X quasi-projective. This will be indicated at the end of this paper, and our main conjecture says that the cohomology rings of $X^{[n]}$ for a quasi-projective surface X and all n are governed by a Farahat–Higman-type ring associated to X. We hope that understanding the cohomology ring of $X^{[n]}$ for a quasi-projective surface X may shed light on the same problem for Nakajima quiver varieties, which is typically noncompact. In another direction, we will address elsewhere certain geometric and algebraic deformations of the current work and [Wa4].

The paper is organized as follows. In Section 2 we set up the notations for wreath products, and then establish various properties of the class algebras of the wreath products which lead to the notion of a Farahat–Higman ring $\mathscr{G}_\Gamma$. In Section 3 we compute certain structure constants for the FH-ring $\mathscr{G}_\Gamma$ and describe various implications of this computation. We further describe the connections with the dual McKay correspondence. In Section 4 we present the connections with Hilbert schemes, and describe some predictions on the cohomology rings of $X^{[n]}$ when X is quasi-projective.

2. The Farahat–Higman ring of wreath products

2.1. Basics of the wreath products. Let Γ be a finite group, and Γ_* be the set of conjugacy classes of Γ. The order of the centralizer of an element in a conjugacy class

$c \in \Gamma_*$ is denoted by ζ_c. We will denote the identity conjugacy class in Γ by c^0 and the identity of Γ by 1. Denote by $\bar{c}$ the conjugacy class $\{x | x^{-1} \in c\}$. We will say that two elements of Γ lie in *opposite* conjugacy classes, if one belongs to some conjugacy class $c \in \Gamma_* \backslash c^0$ while the other belongs to $\bar{c}$ (we allow that $\bar{c} = c$). We shall denote by $\langle x \rangle$ the conjugacy class of an element x in Γ.

The symmetric group S_n acts on the product group $\Gamma^n = \Gamma \times \cdots \times \Gamma$ by permutations: $\sigma(g_1, \ldots, g_n) = (g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)})$. The wreath product of Γ with S_n is defined to be the semidirect product

$$\Gamma_n = \{(g, \sigma) | g = (g_1, \ldots, g_n) \in \Gamma^n, \sigma \in S_n\}$$

with the multiplication

$$(g, \sigma) \cdot (h, \tau) = (g\sigma(h), \sigma\tau).$$

The ith factor subgroup of the product group Γ^n will be denoted by $\Gamma^{(i)}$. We have $\Gamma^{(i)} \leqslant \Gamma^n \leqslant \Gamma_n$ and a short exact sequence of groups

$$1 \to \Gamma^n \to \Gamma_n \to S_n \to 1.$$

The space $R_{\mathbb{Z}}(\Gamma)$ of class functions of a finite group Γ (which is often called the *class algebra* of Γ) is closed under the group multiplication (which is often referred to as the *convolution product*). In this way, $R_{\mathbb{Z}}(\Gamma)$ is also identified with the center of the group algebra $\mathbb{Z}[\Gamma]$. We will denote by $R(\Gamma_n)$ (resp. $R_{\mathbb{Z}}(\Gamma_n)$) the *class algebra* of complex (resp. integer) class functions on Γ_n endowed with the convolution product.

We denote by $|\lambda| = \lambda_1 + \cdots + \lambda_\ell$ and the length $\ell(\lambda) = \ell$, for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, where $\lambda_1 \geqslant \cdots \geqslant \lambda_\ell \geqslant 1$. We will also make use of another notation for partitions: $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \cdots)$, where $m_i(\lambda)$ is the number of parts in λ equal to i. Given two partitions λ, μ, we denote by $\lambda \supset \mu$ if μ is consisted of some parts of λ. In particular, we have the empty partition $\emptyset \subset \lambda$ for any λ. For $\lambda \supset \mu$, we denote by $\lambda - \mu$ the partition obtained by removing the parts of μ from λ. We denote by $\lambda \cup \mu$ the partition whose parts are those of λ and μ, arranged in descending order. We denote by $\lambda \geqslant \mu$ if λ dominates μ.

For a finite set Y and $\rho = (\rho(x))_{x \in Y}$ a family of partitions indexed by Y, the *degree* of ρ is $||\rho|| = \sum_{x \in Y} |\rho(x)|$ and the *length* of ρ is $\ell(\rho) = \sum_{c \in Y} \ell(\rho(c))$. It is convenient to regard $\rho = (\rho(x))_{x \in Y}$ as a partition-valued function on Y. We denote by $\mathscr{P}(Y)$ the set of all partition-valued functions on Y and put $\mathscr{P}_n(Y) = \{\rho \in \mathscr{P}(Y) | \, ||\rho|| = n\}$. Given $\rho, \sigma \in \mathscr{P}_n(Y)$, we extend naturally the definitions from partitions to $\mathscr{P}_n(Y)$ by doing so pointwise on Y to define $\rho \supset \sigma$, $\rho - \sigma$ and $\rho \cup \sigma$, etc.

The conjugacy classes of Γ_n can be described in the following way (cf. [Mac]). Let $x = (g, \sigma) \in \Gamma_n$, where $g = (g_1, \ldots, g_n) \in \Gamma^n$, $\sigma \in S_n$. The permutation σ is written as a product of disjoint cycles. For each such cycle $y = (i_1 i_2 \cdots i_k)$, the element $p_y = g_{i_k} g_{i_{k-1}} \cdots g_{i_1} \in \Gamma$ is determined up to conjugacy in Γ by g and y, and will be called the *cycle-product* of x corresponding to the cycle y. For any conjugacy class $c \in \Gamma_*$ and each integer $i \geqslant 1$, the number of i-cycles in σ whose cycle-product lies in c will be denoted by $m_i(\rho(c))$, and $\rho(c)$ denotes the partition $(1^{m_1(c)}2^{m_2(c)}\dots)$. Then each element $x=(g,\sigma)\in\Gamma_n$ gives rise to a partition-valued function $(\rho(c))_{c\in\Gamma_*}\in\mathscr{P}(\Gamma_*)$ such that $\sum_{r,c} rm_r(\rho(c))=n$. The partition-valued function $\rho=(\rho(c))_{c\in G_*}$ is called the *type* of x. It is known that any two elements of Γ_n are conjugate in Γ_n if and only if they have the same type.

For $\lambda=(1^{m_1}2^{m_2}\dots)$, we define $z_\lambda=\prod_{i\geqslant 1} i^{m_i}m_i!$, which is the order of the centralizer of an element of cycle-type λ in the symmetric group $S_{|\lambda|}$. The order of the centralizer of an element $x=(g,\sigma)\in\Gamma_n$ of the type $\rho=(\rho(c))_{c\in\Gamma_*}$ is

$$Z_\rho=\prod_{c\in\Gamma_*} z_{\rho(c)}\zeta_c^{l(\rho(c))}. \tag{2.1}$$

Example 2.1. Let $\Gamma=\mathbb{Z}/2\mathbb{Z}=\{\pm\}$ with identity $+$, and then Γ_n becomes the Weyl group of type B_n or C_n. Let $n=8$ and consider

$$x=((+,+,-,+,+,+,-,-),(1,3)(2,4,7)),$$

$$y=((-,+,-,+,+,+,-,-),(1,7)(3,8)(4,5)).$$

Then the cycle-product in x associated to $(1,3)$ (resp. $(2,4,7)$) is $-\cdot+=-$ (resp. $-\cdot+\cdot+=-$). Thus the type of x is given by the partition $(1,1)$ for the conjugacy class $+$ and the partition $(3,2,1)$ for the conjugacy class $-$. Similarly, the type of y is given by: the partition $(2,2,2,1,1)$ for the conjugacy class $+$ and the empty partition $\emptyset$ for the conjugacy class $-$.

2.2. The wreath products Γ_n and Γ_∞. Let $\mathbb{N}$ be the set of positive integers. The symmetric group S_n acts on the set $\{1,2,\dots,n\}$ by permutations and Γ_n acts on $\{1,2,\dots,n\}$ by its projection onto S_n. The wreath product Γ_n embeds in Γ_{n+1} as the subgroup $\Gamma_n\times 1$. The union $\Gamma_\infty=\bigcup_{n\geqslant 1}\Gamma_n$ carries a natural group structure. When Γ is trivial, Γ_∞ reduces to $S_\infty=\bigcup_{n\geqslant 1}S_n$.

For any set Y of elements in Γ_∞, we define a subset of $\mathbb{N}$

$$\mathbb{N}(Y)=\{j\in\mathbb{N}|\sigma(j)\neq j \text{ or } g_j\neq 1 \text{ for some } x=((g_1,g_2,\dots),\sigma)\in Y\}.$$

Clearly, we have $\mathbb{N}(Y)=\bigcup_{x\in Y}\mathbb{N}(x)$. We denote by $|\mathbb{N}(Y)|$ the cardinality of $\mathbb{N}(Y)$.

Example 2.2. For x,y in Example 2.1, we have $\mathbb{N}(x)=\{1,2,3,4,7,8\}$, $\mathbb{N}(y)=\{1,3,4,5,7,8\}$ and $\mathbb{N}(x,y)=\{1,2,3,4,5,7,8\}$.

If $(x_1,\dots,x_r)$ and $(y_1,\dots,y_r)$ are two r-tuples of elements in Γ_∞, we say that they are *conjugate* in Γ_∞ if $y_i=zx_iz^{-1}$ $(i=1,\dots,r)$ for some z in Γ_∞. In this way, we divide r-tuples into conjugate classes. For a given conjugate class $\mathscr{C}$ of such r-tuples, we denote by $|\mathbb{N}(\mathscr{C})|$ the cardinality of $\mathbb{N}(x_1,\dots,x_r)$ for any element $(x_1,\dots,x_r)$ in $\mathscr{C}$, as apparently this number does not depends on the choice of such an element. The next lemma is a generalization of Lemma 2.1 in [FH] for the symmetric group case.

Lemma 2.3. *The intersection of a conjugate class $\mathscr{C}$ of r-tuples of elements in Γ_∞ with $\overbrace{\Gamma_n \times \cdots \times \Gamma_n}^{r}$ is empty if $n < |\mathbb{N}(\mathscr{C})|$, and is a conjugate class of Γ_n if $n \geqslant |\mathbb{N}(\mathscr{C})|$. The number of r-tuples in this intersection is $n(n-1)\dots(n-|\mathbb{N}(\mathscr{C})|+1)/k(\mathscr{C})$ where $k(\mathscr{C}) \in \mathbb{Q}$ is a constant.*

Proof. The first half of the lemma is obvious. Given an r-tuple $(x_1, \dots, x_r)$ in this intersection, its stabilizer in Γ_n is the direct product of its centralizer H in the wreath product associated to the set $\mathbb{N}(x_1, \dots, x_r)$ and the wreath product associated to the complement of $\mathbb{N}(x_1, \dots, x_r)$ in $\{1, 2, \dots, n\}$, so its order is $A = k_1(\mathscr{C}) \cdot |\Gamma|^{n-|\mathbb{N}(\mathscr{C})|} \cdot (n - |\mathbb{N}(\mathscr{C})|)!$, where the constant $k_1(\mathscr{C})$ is the order of H. Therefore, the number of elements in the intersection (which is a single conjugacy class) is $|\Gamma|^n n!/A$, which can be recast in the form as stated in the lemma, with $k(\mathscr{C}) = k_1(\mathscr{C}) \cdot |\Gamma|^{-|\mathbb{N}(\mathscr{C})|}$. $\square$

2.3. The modified types. Let x be an element of Γ_n of type $\rho = (\rho(c))_{c \in \Gamma_*} \in \mathscr{P}_n(\Gamma_*)$. If we regard it as an element in Γ_{n+k} by the natural inclusion $\Gamma_n \leqslant \Gamma_{n+k}$, then x has the type $\rho \cup (1^k) \in \mathscr{P}_{n+k}(\Gamma_*)$, where $(\rho \cup (1^k))(c) = \rho(c)$ for $c \neq c^0$ and $(\rho \cup (1^k))(c^0) = (\rho(c^0), 1, \dots, 1) = \rho(c^0) \cup (1^k)$. It is convenient to introduce the *modified type* of x to be $\tilde{\rho} \in \mathscr{P}_{n-r}(\Gamma_*)$, where $r = \ell(\rho(c^0))$, as follows: $\tilde{\rho}(c) = \rho(c)$ for $c \neq c^0$ and $\tilde{\rho}(c^0) = (\rho_1 - 1, \dots, \rho_r - 1)$ if we write the partition $\rho(c^0) = (\rho_1, \dots, \rho_r)$. Two elements in Γ_∞ are conjugate if and only if their modified types coincide.

Given $\mu \in \mathscr{P}(\Gamma_*)$, we denote by $\mathscr{K}_\mu$ the conjugacy class in Γ_∞ of elements whose modified type is μ. If $x \in \mathscr{K}_\mu$, the *degree* $||x||$ of x is defined to be the degree $||\mu||$ of its *modified* type. We shall often denote by $(r)_c$, where $r \in \mathbb{Z}_+$ and $c \in \Gamma_*$, an element in $\mathscr{P}(\Gamma_*)$ consisting of a single cycle of degree r whose cycle-product lies in c. We will say an element is of *single-cycle modified type* $(r)_c$ if the modified type of the element consists of a single cycle $(r)_c$.

Example 2.4. The modified type of x in Example 2.1 is given by the partition $\emptyset$ (resp. $(3, 2, 1)$) for the conjugacy class $+$ (resp. $-$). The modified type of y is given by: the partition $(1, 1, 1)$ (resp. $\emptyset$) for $+$ (resp. $-$). Furthermore, $||x|| = 6$ and $||y|| = 3$.

Remark 2.5. For each $n \geqslant 0$ and each $\mu \in \mathscr{P}(\Gamma_*)$, let $K_\mu(n)$ be the characteristic function of the conjugacy class in Γ_n whose modified type is μ, i.e. the sum of all $\sigma \in \Gamma_n \cap \mathscr{K}_\mu$. Noting that $|\mathbb{N}(\mathscr{K}_\mu)| = ||\mu|| + \ell(\mu(c^0))$, we have by Lemma 2.3 that $K_\mu(n) \neq 0$ if and only if $||\mu|| + \ell(\mu(c^0)) \leqslant n$. The nonzero $K_\mu(n)$'s form a $\mathbb{Z}$-basis for $R_\mathbb{Z}(\Gamma_n)$.

Given $g \in \Gamma$ and $1 \leqslant i \neq j$, we denote by $(i \xrightarrow{g} j)$ the element $((g_1, g_2, \dots), (i,j))$ in Γ_∞, where $(i,j) \in S_\infty$ is a transposition, $g_j = g, g_i = g^{-1}$, and $g_k = 1$ for $k \neq i,j$. Note that $(i \xrightarrow{g} j) = (j \xrightarrow{g^{-1}} i)$ and it is of order 2. The elements in Γ_∞ of the form $(i \xrightarrow{g} j)$, where g

runs over Γ and (i,j) (where $i<j$) runs over all transpositions of S_∞, form the single conjugacy class whose cycle-products are all c^0 and the partition corresponding to c^0 is $(2,1,1,\dots)$. Clearly, any element x in Γ_∞ can be written as a product of elements in $\mathscr{K}_{(1)_{c^0}}$ and elements of the form $h^{(i)} \in \Gamma^{(i)},\ 1 \neq h \in \Gamma,\ i \geqslant 1$. We call such a product a *reduced expression* for x if x cannot be written as a product of fewer such elements. For a general element in Γ_∞, a reduced expression can be constructed cycle-by-cycle.

Example 2.6. Let $x = ((g_{i_1}, \dots, g_{i_k}), (i_1 \dots i_k))$ be of single-cycle modified type. The cycle-product $p_x = g_k \cdots g_2 g_1$ lies in some conjugacy class c. The modified type of x is $(k)_c$ if $p_x \neq 1$ and it is $(k-1)_c$ if $p_x = 1$. A reduced expression for x is given by

$$x = \begin{cases} (i_1 \xrightarrow{g_{i_2}} i_2)(i_2 \xrightarrow{g_{i_3}} i_3) \cdots (i_{k-1} \xrightarrow{g_{i_k}} i_k) p_x^{(i_k)} & \text{if } p_x \neq 1, \\ (i_1 \xrightarrow{g_{i_2}} i_2)(i_2 \xrightarrow{g_{i_3}} i_3) \cdots (i_{k-1} \xrightarrow{g_{i_k}} i_k) & \text{if } p_x = 1. \end{cases}$$

where $p_x^{(i_k)}$ denotes the p_x in $\Gamma^{(i_k)}$. For later purpose, it is convenient to introduce the following notation:

$$\delta(p_x) = \begin{cases} 1 & \text{if } p_x \neq 1, \\ 0 & \text{if } p_x = 1. \end{cases}$$

Then, we have $||x|| = (k-1) + \delta(p_x)$.

Lemma 2.7. *Given an element $x \in \Gamma_\infty$ of modified type λ, the number of elements appearing in a reduced expression of x is $||\lambda||$.*

Proof. Follows from the definitions of modified types and reduced expressions (see the above example). $\square$

Lemma 2.8. *Let x, y be elements in Γ_∞ of modified type λ and μ, and let xy be of modified type ν. Then $||xy|| \leqslant ||x|| + ||y||$, or equivalently, $||\nu|| \leqslant ||\lambda|| + ||\mu||$.*

Proof. Follows from Lemma 2.7. $\square$

2.4. Properties of the structure constants. Given $\lambda, \mu \in \mathscr{P}(\Gamma_*)$, we write the convolution product $K_\lambda(n) K_\mu(n)$ in $R_{\mathbb{Z}}(\Gamma_n)$ as a linear combination of $K_\nu(n)$

$$K_\lambda(n) K_\mu(n) = \sum_\nu a_{\lambda\mu}^\nu(n) K_\nu(n),$$

where the structure constants $a_{\lambda\mu}^\nu(n) \in \mathbb{Z}_+$, and they are zero unless $||\nu|| \leqslant ||\lambda|| + ||\mu||$ by Lemma 2.8. For $n \geqslant ||\nu|| + \ell(\nu(c^0))$, the constant $a_{\lambda\mu}^\nu(n)$ is uniquely determined (cf. Remark 2.5). For convenience, we sometimes denote by $[A]B$ the coefficient of A in B, and thus by $[K_\nu(n)] K_\lambda(n) K_\mu(n)$ the structure constant $a_{\lambda\mu}^\nu(n)$.

The following proposition generalizes Theorem 2.2 in [FH] in the symmetric group case.

Proposition 2.9. *Let $\lambda, \mu, \nu \in \mathscr{P}(\Gamma_*)$. There is a unique polynomial $f^{\nu}_{\lambda\mu}(x)$ such that $f^{\nu}_{\lambda\mu}(n) = a^{\nu}_{\lambda\mu}(n)$ for all $n \geqslant \|\nu\| + \ell(\nu(c^0))$. The degree of the polynomial $f^{\nu}_{\lambda\mu}(x)$ is the greatest value of $|\mathbb{N}(\mathscr{C})| - |\mathbb{N}(\mathscr{K}_\nu)|$ among all classes $\mathscr{C}$ of pairs (x, y) such that $x \in \mathscr{K}_\lambda, y \in \mathscr{K}_\mu, xy \in \mathscr{K}_\nu$.*

Proof. The set of pairs

$$P = \{(x, y) \in \Gamma_\infty \times \Gamma_\infty \,|\, x \in \mathscr{K}_\lambda, y \in \mathscr{K}_\mu, xy \in \mathscr{K}_\nu\}$$

is apparently invariant under the simultaneous conjugation by Γ_∞, so is a union of conjugate classes.

We claim that the set of conjugate classes of such pairs in P is finite. Indeed, if $(x, y) \in P$, then $|\mathbb{N}(x, y)| \leqslant |\mathbb{N}(x)| + |\mathbb{N}(y)|$ while $|\mathbb{N}(x)|$ and $|\mathbb{N}(y)|$ are finite numbers depending only on λ and μ respectively. So by Lemma 2.3, each pair $(x, y) \in P$ is conjugate to some pair lying in $\Gamma_{|\mathbb{N}(x,y)|} \times \Gamma_{|\mathbb{N}(x,y)|}$, which is a finite set.

Let the conjugate classes of P be $\mathscr{C}_1, \mathscr{C}_2, \dots, \mathscr{C}_r$. Then, by Lemma 2.3, the number T of pairs in the intersection $P \cap (\Gamma_n \times \Gamma_n)$ is

$$T = \sum_{i=1}^{r} n(n-1)\cdots(n - |\mathbb{N}(\mathscr{C}_i)| + 1)/k(\mathscr{C}_i).$$

To obtain $a^{\nu}_{\lambda\mu}(n)$ for $n \geqslant \|\nu\| + \ell(\nu(c^0))$, we must divide T by the number of elements in $\mathscr{K}_\nu \cap \Gamma_n$, which is $n(n-1)\cdots(n - |\mathbb{N}(\mathscr{K}_\nu)| + 1)/k(\mathscr{K}_\nu)$, again by Lemma 2.3. Note that $|\mathbb{N}(\mathscr{K}_\nu)| = |\mathbb{N}(xy)| \leqslant |\mathbb{N}(x, y)| = |\mathbb{N}(\mathscr{C}_i)|$ for $(x, y) \in \mathscr{C}_i$. Thus,

$$a^{\nu}_{\lambda\mu}(n) = k(\mathscr{K}_\nu)\sum\nolimits_{i=1}^{r} (n - |\mathbb{N}(\mathscr{K}_\nu)|)(n - |\mathbb{N}(\mathscr{K}_\nu)| - 1)\cdots(n - |\mathbb{N}(\mathscr{C}_i)| + 1)/k(\mathscr{C}_i)$$

is a polynomial of degree equal to the maximal value among $|\mathbb{N}(\mathscr{C}_i)| - |\mathbb{N}(\mathscr{K}_\nu)|, 1 \leqslant i \leqslant r$. $\square$

Remark 2.10. Proposition 5.1 of [Wa4] (which is much more difficult to establish) can be used to derive the first part of Proposition 2.9 and an upper bound of the degree (but not the explicit formula for the degree).

Lemma 2.11. *Let $x = ((g_1, g_2, \dots), \sigma)$ be an element in Γ_∞, where $g_i \in \Gamma^{(i)}$ and $\sigma \in S_\infty$. If $\sigma(j) = i$ for $i \neq j$, then we can find a reduced expression of x which contains the element $(j \xrightarrow{g_i} i)$.*

Proof. One only needs to prove the lemma in the case when x consists of a single non-trivial cycle, i.e., the cycle containing i and j. We can write the cycle (by rotating cyclically if necessary) so that the letter i appears in the end of the cycle. In this case (with suitable change of letters), a reduced expression as required has been given in Example 2.6. $\square$

Proposition 2.12. *Let $x, y \in \Gamma_\infty$ be of modified types λ and μ, and let xy be of modified type ν. If $||\nu|| = ||\lambda|| + ||\mu||$, then $\mathbb{N}(x, y) = \mathbb{N}(xy)$.*

Proof. Write $x = ((g_1, g_2, \dots), \sigma)$, $y = ((h_1, h_2, \dots), \tau)$, and $xy = ((f_1, f_2, \dots), \sigma\tau)$, where $g_i, h_i, f_i \in \Gamma^{(i)}$. Clearly, $\mathbb{N}(xy) \subset \mathbb{N}(x, y)$. If i lies in $\mathbb{N}(x, y)$ but not in $\mathbb{N}(xy)$, then by definition of the set $\mathbb{N}(\cdot)$, we have $\sigma\tau(i) = i$ and $f_i = 1$. We have two possibilities: (a) $\tau(i) = j \neq i$, $\sigma(j) = i$, and $g_i h_j = 1$; or (b) $\sigma(i) = i$, $\tau(i) = i$, and $g_i h_i = 1$.

Let us first consider case (a). By Lemma 2.11, we can find a reduced expression of x which contains the element $(j \xrightarrow{g_i} i)$ and a reduced expression of y which contains the element $(i \xrightarrow{h_j} j)$. Note that the elements $(j \xrightarrow{g_i} i)$ and $(i \xrightarrow{h_j} j)$ are inverse to each other (and they are also equal) thanks to $g_i h_j = 1$. Therefore, writing xy as the product of these two specified reduced expressions, we can replace the part between (and which include) $(j \xrightarrow{g_i} i)$ and $(i \xrightarrow{h_j} j)$ by another expression involving two less elements (by conjugation). Thus, by Lemma 2.7, $||\nu|| < ||\lambda|| + ||\mu||$.

Let us now consider case (b). In fact, since i is fixed by both σ and τ, the element g_i (and thus h_i) is not equal to $1 \in \Gamma$ by the assumption that i lies in $\mathbb{N}(x, y)$. As a reduced expression can be constructed cycle by cycle, we can easily find a reduced expression for x (resp. y) which satisfies two requirements: (1) it contains the element g_i (resp. h_i) in $\Gamma^{(i)}$; (2) g_i (resp. h_i) commutes with all the other elements appearing in this reduced expression. Writing xy as the product of these two specified reduced expressions, we can remove simultaneously g_i and h_i since $g_i h_i = 1$. Thus, again by Lemma 2.7, $||\nu|| < ||\lambda|| + ||\mu||$. $\square$

2.5. The Farahat–Higman ring. Propositions 2.9 and 2.12 give us the following.

Theorem 2.13. *Let $\lambda, \mu, \nu \in \mathscr{P}(\Gamma_*)$. Then,*

(1) *there is a unique polynomial $f^\nu_{\lambda\mu}(x)$ such that $f^\nu_{\lambda\mu}(n) = a^\nu_{\lambda\mu}(n)$ for all positive integers n;*
(2) *the polynomial $f^\nu_{\lambda\mu}(x)$ is a constant if $||\nu|| = ||\lambda|| + ||\mu||$.*

Remark 2.14. The number $a^\nu_{\lambda\mu}(n)$ can take any value for $n < ||\nu|| + \ell(\nu(c^0))$ since $K_\nu(n) = 0$ by Remark 2.5. In the above theorem, it is understood that $a^\nu_{\lambda\mu}(n)$ for $n < ||\nu|| + \ell(\nu(c^0))$ is chosen appropriately (i.e. equal to the value of the polynomial $f^\nu_{\lambda\mu}(x)$ at $x = n$).

Remark 2.15. The inverse of Theorem 2.13(2) is not true even in the symmetric group case (i.e. when Γ is trivial), in contrast to the claim in [Mac, Example 24,

p. 131]. For example, consider the conjugacy classes $\mathscr{K}_\lambda, \mathscr{K}_\mu$ and $\mathscr{K}_\nu$ of modified types $\lambda = (1), \mu = (3)$ and $\nu = (1,1)$. Clearly, $||\nu|| < ||\lambda|| + ||\mu||$. For any $\sigma \in \mathscr{K}_\lambda, \tau \in \mathscr{K}_\mu$ such that $\sigma\tau \in \mathscr{K}_\nu$, it is easy to see that σ, τ and $\sigma\tau$ have to be of the form $\sigma = (i,k)$, $\tau = (i,j,k,l)$ and $\sigma\tau = (i,j)(k,l)$, where i,j,k,l are distinct positive integers. If follows that $\mathbb{N}(\sigma\tau) = \mathbb{N}(\sigma,\tau) = \{i,j,k,l\}$. Thus, by Proposition 2.9, the polynomial $f^\nu_{\lambda\mu}$ is a constant. One can further show that $f^\nu_{\lambda\mu}$ is actually 2 in this case, by showing that there is a single conjugate classes of such pairs (σ,τ) and then using the formula for $a^\nu_{\lambda\mu}(n)$ in the proof of Proposition 2.9.

Denote by R the subring of the polynomial ring $\mathbb{Q}[t]$ consisting of polynomials that take integer values at all integers. Thanks to Theorem 2.13, we can introduce a commutative R-ring $\mathscr{F}_\Gamma$ with R-basis (K_λ) indexed by $\lambda \in \mathscr{P}(\Gamma_*)$ and multiplication given by

$$K_\lambda K_\mu = \sum_\nu f^\nu_{\lambda\mu} K_\nu,$$

where the sum is over $\nu \in \mathscr{P}(\Gamma_*)$ such that $||\nu|| \leqslant ||\lambda|| + ||\mu||$, and $f^\nu_{\lambda\mu} \in R$ takes the value $a^\nu_{\lambda\mu}(n)$ at a positive integer n.

The R-ring $\mathscr{F}_\Gamma$ determines the class algebra $R_\mathbb{Z}(\Gamma_n)$ for each n by the natural surjective ring homomorphism $\mathscr{F}_\Gamma \to R_\mathbb{Z}(\Gamma_n)$ sending K_λ to $K_\lambda(n)$ and $f^\nu_{\lambda\mu}$ to $a^\nu_{\lambda\mu}(n)$.

The assignment of degree $||\lambda||$ to K_λ provides a filtration on the ring $\mathscr{F}_\Gamma$, thanks to Lemma 2.8. The associated graded ring, denoted by $\mathscr{G}_{R,\Gamma}$, has the multiplication

$$K_\lambda K_\mu = \sum_{||\nu|| = ||\lambda|| + ||\mu||} a^\nu_{\lambda\mu} K_\nu. \tag{2.2}$$

By abuse of notations, we have kept using the same symbols for elements in the ring $\mathscr{F}_\Gamma$ and the associated graded ring. By Theorem 2.13, the structure constants for $\mathscr{G}_{R,\Gamma}$ are *non-negative integers*. Thus, we have reached the following.

Theorem 2.16. *If $\mathscr{G}_\Gamma$ denotes the ring over $\mathbb{Z}$ with basis (K_λ) whose multiplication is given by* (2.2), *then the graded R-ring $\mathscr{G}_{R,\Gamma}$ associated to $\mathscr{F}_\Gamma$ is isomorphic to $R \otimes_\mathbb{Z} \mathscr{G}_\Gamma$.*

The ring $\mathscr{G}_\Gamma$ will be called the *Farahat–Higman ring* (or *FH-ring*) associated to Γ, as our results specialize in the symmetric group case to the ones obtained by FH.

We can also take the filtration for each n directly. Recall that those $K_\mu(n)$ indexed by $\mu \in \mathscr{P}(\Gamma_*)$ satisfying $||\mu|| + \ell(\mu(c^0)) \leqslant n$ form a basis for $R(\Gamma_n)$. The assignment of degree $||\mu||$ to $K_\mu(n)$ provides $R_\mathbb{Z}(\Gamma_n)$ a filtered ring structure, whose associated graded ring will be denoted by $\mathscr{G}_\Gamma(n)$. The next theorem is essentially a reformulation of Theorem 2.13 for Γ_n.

Theorem 2.17. *The graded $\mathbb{Z}$-ring $\mathscr{G}_\Gamma(n)$ has a multiplication given by*

$$K_\lambda(n)K_\mu(n) = \sum\nolimits_{||\nu||=||\lambda||+||\mu||} a^\nu_{\lambda\mu} K_\nu(n),$$

where the structure constants $a^\nu_{\lambda\mu}$ are non-negative integers independent of n.

Clearly, the ring $\mathscr{G}_\Gamma$ determines the family of rings $\{\mathscr{G}_\Gamma(n)\}_{n\geqslant 1}$, and vice versa. There is a natural surjective ring homomorphism $\mathscr{G}_\Gamma \to \mathscr{G}_\Gamma(n)$ for each n which is compatible with the surjective ring homomorphism $\mathscr{G}_\Gamma(n+1) \to \mathscr{G}_\Gamma(n)$ by restriction. When Γ is the trivial group and thus Γ_n reduces to the symmetric group S_n, we omit the subscript Γ from the notations for the rings $\mathscr{F}_\Gamma$, $\mathscr{G}_\Gamma$, $\mathscr{G}_\Gamma(n)$, etc.

Remark 2.18. In the symmetric group case (i.e. when Γ is trivial), the above results are due to [FH]. We have adopted however the elegant presentation of Macdonald (cf. [Mac, Example 24, p. 131]). Macdonald in addition found a symmetric function interpretation for the FH-ring $\mathscr{G} = \mathscr{G}_1$ (for $\Gamma = 1$). We expect that our FH-ring $\mathscr{G}_\Gamma$ also affords a natural symmetric function interpretation.

3. Computations in the Farahat–Higman ring

In this section, we will compute the formulas for the multiplication by a conjugacy class of single-cycle modified type in the FH-ring $\mathscr{G}_\Gamma$, and then derive various consequences.

3.1. Constraints from the filtration. Let $x, y \in \Gamma_\infty$, where we assume that y is of single-cycle modified type, i.e., y is of the form $y = ((g_{i_1}, \ldots, g_{i_r}), (i_1, \ldots, i_r))$. We will say that a cycle of x is *relevant* to y if it contains at least one of $i_1, \ldots, i_r$.

Lemma 3.1. *Let $y = ((h_{i_1}, \ldots, h_{i_r}), (i_1, \ldots, i_r))$ be of single-cycle modified type $(r > 2)$. Then, we can decompose y as a product $y_1 y_2$, where $y_1 = (f, (i_1, \ldots, i_{r-1}))$ and $y_2 = ((a^{-1}, a), (i_{r-1}, i_r)) \in \mathscr{K}_{(1)_{c^0}}$, for some $a \in \Gamma, f \in \Gamma^{r-1}$, such that $||y|| = ||y_1|| + ||y_2||$. For any such decomposition, the cycle-products of y and y_1 are conjugate in Γ.*

Proof. Fix y_2 by choosing $a = h_{i_r}$. Then one easily checks that $y_1 = y y_2^{-1}$ can be recast in the form in the lemma. The second statement follows from a straightforward computation. $\square$

The next key proposition is a purely wreath product phenomenon, which reduces to almost triviality in the symmetric group case. It indicates that the multiplication in the FH-ring is rigidly constrained by the filtration given by $||\cdot||$. Below $||\cdot||$ also applies to elements in S_∞ by regarding S_∞ as the natural subgroup of Γ_∞.

Proposition 3.2. *Let $y = ((h_{i_1}, \dots, h_{i_r}), \tau_r)$ be of single-cycle modified type, where $\tau_r = (i_1 i_2 \dots i_r)$. Let $x = (g, \sigma) \in \Gamma_\infty$ be such that $||x|| + ||y|| = ||xy||$. Then one of the following possibilities occurs:*

(1) $||\sigma\tau_r|| = ||\sigma|| + ||\tau_r||$. *In this case, there are r cycles of x relevant to y, i.e., each i_s $(s = 1, \dots, r)$ belongs to a distinct cycle in x; among the cycle of y and all r cycles of x relevant to y, at most one cycle-product is not* 1. *When all such cycle-products (resp. all but one, say p) are* 1, *then the relevant cycle-product in xy is also* 1 *(resp. belong to the conjugacy class of p).*
(2) $||\sigma\tau_r|| = ||\sigma|| + ||\tau_r|| - 2$. *In this case, there are $(r-1)$ cycles of x relevant to y; all the cycle-products of the relevant cycles in x are* 1 *and so is the cycle-product of y. Assume that i_s and i_t are contained in the same relevant cycle of x, then i_s and i_t are contained in distinct cycles of xy whose cycle-products lie in opposite conjugacy classes.*

Proof. Let us denote $n = |\mathbb{N}(x, y)|$, and regard x and y as elements in Γ_n. Since $||\cdot||$ is additive for disjoint cycles, we may assume that all the cycles of x are relevant to y without loss of generality. If we denote by k the number of (relevant) cycles appearing in x, then $k \leqslant r$.

Denote by Δ_x the number of cycles in x whose cycle-products are not 1, and by p_y the cycle-product of y as usual. It follows from the definitions that $||x|| = (n-k) + \Delta_x$, and $||y|| = (r-1) + \delta(p_y)$. Thus, we have

$$||x|| + ||y|| = n + (r-k) - 1 + \Delta_x + \delta(p_y), \tag{3.1}$$

$$||xy|| \leqslant n, \tag{3.2}$$

$$||x|| + ||y|| = ||xy||, \tag{3.3}$$

where (3.2) holds since $xy \in \Gamma_n$, and (3.3) by the assumption.

If follows from the consistency of (3.1)–(3.3) and $k \leqslant r$ that either (1) $k = r$ or (2) $k = r-1$.

Let us first consider case (1), i.e. $k = r$. There are r cycles of x relevant to y, i.e., each i_s $(s = 1, \dots, r)$ belongs to a distinct cycle in x. It follows that xy consists of a single n-cycle (see e.g. formula (3.4) below). In particular, $||\sigma\tau_r|| = n - 1 = ||\sigma|| + ||\tau_r||$. By the comparison of (3.1)–(3.3), we have two subcases: (1a) $||xy|| = n-1$, and $\Delta_x = \delta(p_y) = 0$; (1b) $||xy|| = n$, and then one of Δ_x and $\delta(p_y)$ is 0 while the other is 1.

In subcase (1a), the cycle of y and the r cycles of x all have cycle-product equal to 1. In subcase (1b), among the cycle of y and the r cycles of x, all cycle-products but one are equal to 1. Let us denote the non-identity cycle-product by p.

It remains to check the last statement in part (1) of the proposition. For $r = 1$ or 2, it is straightforward to check by hand. For $r > 2$, we can write y as a product $y_1 y_2$, where $y_1 = (h, (i_1, \dots, i_{r-1}))$ and $y_2 = ((a^{-1}, a), (i_{r-1}, i_r))$, by Lemma 3.1. Now, an obvious induction proves the last statement of part (1) of the proposition.

Let us now consider case (2), i.e. $k = r - 1$. It immediately follows that there are $(r-1)$ cycles of x relevant to y. Furthermore, $\sigma\tau_r$ consists of exactly two cycles (see e.g. formula (3.9) below). Thus, $||\sigma\tau_r|| = n-2$ and this coincides with $||\sigma|| + ||\tau_r|| - 2 = (n-k) + (r-1) - 2 = n-2$.

It follows again from the consistency of (3.1)–(3.3) that $\Delta_x = \delta(p_y) = 0$ (and $||xy|| = n$), i.e., the cycle of y and the r cycles of x all have cycle-product 1.

It remains to check the last statement in part (2) of the proposition. For $r = 2$, it is straightforward to check by hand.

Now let $r > 2$. There are exactly two i's among $i_1, \ldots, i_r$ which lie in an identical cycle of x. We can assume one of them is i_r by rotating cyclically $(i_1, \ldots, i_r)$ if necessary. Then, by Lemma 3.1, we can write $y = y_1 y_2$, where $y_1 = (f, (i_1, \ldots, i_{r-1}))$ and $y_2 = ((a^{-1}, a), (i_{r-1}, i_r))$ such that $||y|| = ||y_1|| + ||y_2||$. Together with the identity $||x|| + ||y|| = ||xy||$ and the sub-multiplicative property of $||\cdot||$ (cf. Lemma 2.8), we have $||x|| + ||y_1|| = ||xy_1||$ and $||xy_1|| + ||y_2|| = ||xy||$.

Since $i_1, \ldots, i_{r-1}$ lie in the $(r-1)$ distinct cycles of x, the product xy_1 is of single-cycle modified type. Thus, we are in the setup of (1) when considering the product x and y_1; and in the setup of (2) when considering the product (xy_1) and y_2. By applying (2) for $r = 2$ (which we have checked) to the product (xy_1) and y_2, we conclude that the cycle-product of (xy_1) is 1. Now, by applying (1) to the product x and y_1, we finish the proof of the part (2) of the proposition for general r. $\square$

3.2. Multiplication with a conjugacy class of single cycles. We are ready to compute the explicit formulas for multiplications in the FH-ring $\mathscr{G}_\Gamma$ with a conjugacy class of single-cycle modified type. Note that $K_{(0)_{c^0}} = K_\emptyset$ is the identity of the FH-ring $\mathscr{G}_\Gamma$.

Theorem 3.3. *Let $\lambda \in \mathscr{P}(\Gamma_*)$, $r \geqslant 0$, and $c \in \Gamma_* \backslash c^0$. Then, the product $K_\lambda K_{(r)_c}$ in the FH-ring $\mathscr{G}_\Gamma$ is*

$$K_\lambda K_{(r)_c} = \sum_{\mu} \frac{(m_{||\mu||+r}(\lambda(c)) + 1) \cdot (||\mu|| + r) \cdot (r-1)!}{(r - \ell(\mu))! \prod_{i \geqslant 1} m_i(\mu)!} K_{\lambda \cup (||\mu||+r)_c - \mu}$$

summed over $\mu = \mu(c^0) \subset \lambda(c^0)$ such that $\ell(\mu) \leqslant r$, where we denote $\mu = (1^{m_1(\mu)} 2^{m_2(\mu)} \ldots)$.

Proof. Let $y = ((h_{i_1}, \ldots, h_{i_r}), (i_1 \ldots i_r))$ be of type $(r)_c$, i.e. $h_{i_r} \cdots h_{i_1} \in c$. Let $x = (g, \sigma)$ be in the class $\mathscr{K}_\lambda$ such that $||xy|| = ||x|| + ||y||$. By Proposition 3.2(1), the numbers $i_1, \ldots, i_r$ belong to r distinct cycles of σ, which we denote by $(\ldots j_a i_a)$, $a = 1, \ldots, r$; the relevant cycle-products for x are all 1.

Note that

$$(\ldots j_1 i_1)(\ldots j_2 i_2) \cdots (\ldots j_r i_r) \cdot (i_1, \ldots i_r) = (\ldots j_1 i_1 \ldots j_2 i_2 \ldots j_r i_r). \tag{3.4}$$

Assume that $m_0(\mu)$ of the cycles of x relevant to y (i.e. those which we have used on the left-hand side of (3.4)) are 1-cycles, $m_1(\mu)$ 2-cycles, $m_2(\mu)$ 3-cycles, etc., of x.

Then, $\mu = \mu(c^0) = (1^{m_1(\mu)}2^{m_2(\mu)}\dots)$ is the modified type of the product of cycles in x relevant to y, and $m_0(\mu) = r - \ell(\mu)$. By Proposition 3.2(1), the cycle-product associated to the cycle $(\dots j_1 i_1 \dots j_2 i_2 \dots j_r i_r)$ of x lies in c. Then xy belongs to $\mathscr{K}_{\lambda \cup (||\mu||+r)_c - \mu}$. This implies that the conjugacy classes on the right-hand side of the formula in the theorem are the right ones.

Next, we determine the coefficient of $K_{\lambda \cup (||\mu||+r)_c - \mu}$. Fix $z \in \mathscr{K}_{\lambda \cup (||\mu||+r)_c - \mu}$ and $x_1 \in \mathscr{K}_\mu$. This coefficient is the number of ways a cycle of modified type $(||\mu|| + r)_c$ can be chosen from z, and bracketted to form a product of $m_0(\mu)$ cycles of modified type $(0)_{c^0}$, $m_1(\mu)$ cycles of modified type $(1)_{c^0}$, $m_2(\mu)$ cycles of modified type $(2)_{c^0}$, etc., and a cycle of modified type $(r)_c$. The cycle of modified type $(||\mu|| + r)_c$ can be chosen in $m_{||\mu||+r}(\lambda(c)) + 1$ ways. The number of brackettings, which no longer depends on λ but only on μ and $(r)_c$, is the coefficient $[K_{(||\mu||+r)_c}]K_\mu K_r(c)$. By Theorem 2.13, this coefficient is a constant, so we can calculate it as the coefficient $[K_{(n)_c}(n)]K_\mu(n)K_{(r)_c}(n)$, where $n = ||\mu|| + r$.

The number of elements u in $K_\mu(n)$ is, by (2.1), given by

$$\frac{|\Gamma|^n n!}{|\Gamma|^r 2^{m_1(\mu)} 3^{m_2(\mu)} \cdots (r - \ell(\mu))! \prod_{i \geqslant 1} m_i(\mu)!}. \tag{3.5}$$

To find a cycle y of modified type $(r)_c$ such that uy is an n-cycle (which will automatically be of modified type $(n)_c$ by Lemma 3.1), we need to choose a number from each of the r cycles of u as well as r elements from Γ, and then arrange them into a cycle of modified type $(r)_c$. The number of choices here is

$$2^{m_1(\mu)} 3^{m_2(\mu)} \cdots (r-1)! |\Gamma|^r / \zeta_c. \tag{3.6}$$

Finally, to obtain the required coefficient, we need to divide the product of (3.5) and (3.6) by the number of elements in $K_{(n)_c}(n)$ which is $|\Gamma|^n n!/n\zeta_c$. Remembering $n = ||\mu|| + r$, we have proved the theorem. $\square$

Given a partition $\mu = \mu(c^0) = (\mu_1, \mu_2, \dots, \mu_k)$ where $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_k \geqslant 1$ and $k = \ell(\mu)$. Assume $r \geqslant \ell(\mu)$. we denote by $\tilde{\mu}$ the partition $\tilde{\mu} = (\mu_1 + 1, \mu_2 + 1, \dots, \mu_k + 1, 1, \dots, 1)$ of $|\mu| + r$ which consists of r parts. There are $r!$ compositions (counted with multiplicities) obtained from rearranging the r parts of $\tilde{\mu}$. Given $s > 0$, μ and $r \geqslant \ell(\mu)$, we denote by $q(\mu, r, s)$ (resp. $p(\mu, r, s)$) the number of all compositions $(\tilde{\mu}_{k_1}, \tilde{\mu}_{k_2}, \dots, \tilde{\mu}_{k_r})$ associated to $\tilde{\mu}$, counted with (resp. without) multiplicities, such that s is not one of the r numbers $\tilde{\mu}_{k_1} + \tilde{\mu}_{k_2} + \cdots + \tilde{\mu}_{k_a} (a = 1, \dots, r)$. If we write $\mu = \mu(c^0) = (1^{m_1(\mu)}2^{m_2(\mu)}\dots)$, then $\tilde{\mu} = (1^{r-\ell(\mu)}2^{m_1(\mu)}3^{m_2(\mu)}\dots)$, and

$$q(\mu, r, s) = (r - \ell(\mu))! \prod_{i \geqslant 1} m_i(\mu)! \cdot p(\mu, r, s).$$

Clearly, $p(\mu, r, s)$ is zero unless $0 < s < |\mu| + r$. We mention that there is a symmetry: $p(\mu, r, s) = p(\mu, r, |\mu| + r - s)$, and $\sum_s q(\mu, r, s) = ||\mu|| \cdot r!$. But we do not need these properties in this paper.

Theorem 3.4. *Let* $\lambda \in \mathscr{P}(\Gamma_*)$ *and* $r \geqslant 0$. *Then, the product* $K_\lambda K_{(r)_{c^0}}$ *in the FH-ring* $\mathscr{G}_\Gamma$ *is*

$$\begin{aligned} K_\lambda K_{(r)_{c^0}} &= \sum_{I_1} \frac{(m_{||\mu||+r}(\lambda(c^0))+1)\cdot(||\mu||+r+1)\cdot r!}{(r+1-\ell(\mu))!\prod_{i\geqslant 1} m_i(\mu)!} K_{\lambda\cup(||\mu||+r)_{c^0}-\mu} \\ &+ \sum_{I_2} \frac{(m_{||\mu||+r+k}(\lambda(c))+1)\cdot(||\mu||+r+k)\cdot r!}{(r-\ell(\mu))!\prod_{i\geqslant 1} m_i(\mu)!} K_{\lambda\cup(||\mu||+r+k)_c-(\mu\cup(k)_c)} \\ &+ \sum_{I_3} s_1 s_2 \zeta_c p(\mu, r, s_1) m_{s_1}(\lambda(c))+1)(m_{s_2}(\lambda(\bar{c}))+1) K_{\lambda\cup(s_1)_c\cup(s_2)_{\bar{c}}-\mu} \\ &+ \sum_{I_4} s^2 \zeta_c p(\mu, r, s)(m_s(\lambda(c))+1)(m_s(\lambda(\bar{c}))+1) K_{\lambda\cup(s)_c\cup(s)_{\bar{c}}-\mu} \\ &+ \sum_{I_5} s^2 \zeta_c p(\mu, r, s)\binom{m_s(\lambda(c))+2}{2} K_{\lambda\cup(s)_c\cup(s)_c-\mu}, \end{aligned}$$

where $\mu = (1^{m_1(\mu)}2^{m_2(\mu)}\dots)$, $I_1 = \{\mu | \mu = \mu(c^0) \subset \lambda(c^0), \ell(\mu) \leqslant r+1\}$, $I_2 = \{(c, \mu, k) | c \neq c^0, \mu = \mu(c^0) \subset \lambda(c^0), \ell(\mu) \leqslant r, (k)_c \subset \lambda(c), k \geqslant 1\}$, $I_3 = \{(c, \mu, s_1, s_2) | c \neq c^0, \mu = \mu(c^0) \subset \lambda(c^0), \ell(\mu) \leqslant r, 0 < s_1 < s_2, s_1 + s_2 = ||\mu|| + r\}$, $I_4 = \{(\{c, \bar{c}\}, \mu, s) | c \neq \bar{c}, \mu = \mu(c^0) \subset \lambda(c^0), \ell(\mu) \leqslant r, 2s = ||\mu|| + r\}$, *and* $I_5 = \{(c, \mu, s) | c = \bar{c}, c \neq c^0, \mu = \mu(c^0) \subset \lambda(c^0), \ell(\mu) \leqslant r, 2s = ||\mu|| + r\}$. *Here* $\{c, \bar{c}\}$ *denotes an unordered pair of opposite conjugacy classes in* Γ_*.

Proof. Let $y = ((h_{i_1}, \dots, h_{i_r}, h_i), (i_1 \dots i_r i))$ be of type $(r)_{c^0}$, i.e. $h_i h_{i_r} \cdots h_{i_1} = 1$. Let $x = (g, \sigma)$ be in the class $\mathscr{K}_\lambda$ such that $||x|| + ||y|| = ||xy||$. Both cases of Proposition 3.2 occur when multiplying x and y. We will analyze them one by one. We keep in mind the strategy in the proof of Theorem 3.3.

(1) The numbers $i_1, \dots, i_r, i$ belong to $(r+1)$ distinct cycles of σ, which we denote by $(\dots ji)$ and $(\dots j_a i_a)$, $a = 1, \dots, r$. We divide this into two subcases according to Proposition 3.2(1):

(1a) all the relevant cycle-products for x are 1; (1b) all the relevant cycle-products but one (which lies in $c \in \Gamma_* \backslash c^0$) for x are 1.

In case (1a), we have the same computation as in the proof of Theorem 3.3, which gives us the first summand on the right-hand side of the formula in the theorem. The main difference we should keep in mind is that y is an $(r+1)$-cycle (not an r-cycle) since y is of *modified* type $(r)_{c^0}$. We omit the detail here.

In case (1b), we use from x the relevant cycles of modified type $\mu = \mu(c^0)$ (which is contained in $\lambda(c^0)$) as well as a cycle of modified type $(k)_c$ (which is contained in $\lambda(c)$). When multiplying with y, the same strategy as in the proof of Theorem 3.3 applies, as we sketch below.

The number of brackettings is the coefficient $[K_{(n)_c}(n)] K_{\mu\cup(k)_c}(n) K_{(r)_{c^0}}(n)$, where $n = ||\mu|| + r + k$.

The number of elements u in $K_{\mu\cup(k)_c}(n)$ is

$$\frac{|\Gamma|^n n!}{|\Gamma|^r \zeta_c k \cdot 2^{m_1(\mu)} 3^{m_2(\mu)} \cdots (r-\ell(\mu))! \prod_{i\geqslant 1} m_i(\mu)!}. \tag{3.7}$$

To find a cycle y of modified type $(r)_{c^0}$ such that uy is an n-cycle (which will automatically be of modified type $(n)_c$ by Lemma 3.1), we need to choose a number from each of the r relevant cycles of u (whose cycle-product is 1), a number from the cycle of modified type $(k)_c$, as well as $(r+1)$ elements from Γ, and then arrange into a cycle of modified type $(r)_{c^0}$. The number of choices of a cycle y of modified type $(r)_{c^0}$ such that uy is an n-cycle is

$$k \cdot 2^{m_1(\mu)} 3^{m_2(\mu)} \cdots r! |\Gamma|^r. \tag{3.8}$$

The product of (3.7) and (3.8) divided by the number of elements in $K_{(n)_c}(n)$ gives us the coefficient $[K_{(n)_c}(n)]K_{\mu\cup(k)_c}(n)K_{(r)_{c^0}}(n)$. Noting the number of elements in $K_{(n)_c}(n)$ is $|\Gamma|^n n!/n\zeta_c$ and remembering $n = ||\mu|| + r + k$, we obtain the coefficient in the second term on the right-hand side of the formula in the theorem.

(2). Assume that $i_t (t = 1, \ldots, r)$ belong to distinct cycles, say $(\ldots j_t i_t)$, of x, and we also assume that i_a $(a \leqslant r)$ and i lie in the same cycle of x. By Proposition 3.2(2), the cycle-product of y and all the relevant cycle-products for x are 1.

Note that

$$\begin{aligned}
&(\ldots j_1 i_1)\cdots(\ldots j_{a-1} i_{a-1})(\ldots ji \ldots j_a i_a)\cdots(\ldots j_r i_r)\cdot(i_1 \ldots i_r i) \\
&= (\ldots j_1 i_1 \ldots j_{a-1} i_{a-1} \ldots ji \ldots j_a i_a \ldots j_r i_r)(i_r i) \\
&= (\ldots j_1 i_1 \ldots j_{a-1} i_{a-1} \ldots ji)(\ldots j_a i_a \ldots j_r i_r).
\end{aligned} \tag{3.9}$$

The cycle-products in xy of the two cycles, say s_1-cycle and s_2-cycle, on the right-hand side of (3.9) belong to opposite conjugacy classes, by Proposition 3.2(2). We denote the modified type of the right-hand side of (3.9) to be $(s_1)_c \cup (s_2)_{\bar{c}}$. Let us assume that $m_0(\mu)$ of the cycles of x (relevant to y) are 1-cycles, $m_1(\mu)$ 2-cycles, $m_2(\mu)$ 3-cycles, etc. We denote by $\mu = \mu(c^0) = (1^{m_1(\mu)} 2^{m_2(\mu)} \ldots)$ the modified type of the product of cycles in x relevant to y, and hence $m_0(\mu) = r - \ell(\mu)$.

Thus, xy belongs to the conjugacy class $\mathscr{K}_{\lambda\cup(s_1)_c\cup(s_2)_{\bar{c}}-\mu}$ which give rise to the third, fourth, and fifth terms of the right-hand side of the formula in the theorem. So we see that the conjugacy classes appearing on the right-hand side of the formula in the theorem are indeed the right ones.

It remains to compute the coefficient $N := [K_{\lambda\cup(s_1)_c\cup(s_2)_{\bar{c}}-\mu}]K_\lambda K_{(r)_{c^0}}$, where $s_1 + s_2 = ||\mu|| + r$.

We denote by B the coefficient $[K_{(s_1)_c\cup(s_2)_{\bar{c}}}]K_\mu K_{(r)_{c^0}}$. Denote by A_1 the number of ways of cycles of modified type $(s_1)_c \cup (s_2)_{\bar{c}}$ can be chosen from a fixed element in $\mathscr{K}_{\lambda\cup(s_1)_c\cup(s_2)_{\bar{c}}-\mu}$. Clearly, we have $N = A_1 \cdot B$. Note that B does not depend on λ and only depends on μ, r and s_1. By Theorem 2.13, the number B can be computed as the coefficient $[K_{(s_1)_c \cup (s_2)_{\bar{c}}}(n)]K_\mu(n)K_{(r)_{c^0}}(n)$, where $n = ||\mu|| + r$.

We first easily have

$$A_1 = \begin{cases} \binom{m_s(\lambda(c)) + 2}{2} & \text{if } c = \bar{c} \text{ and } s_1 = s_2 = s, \\ (m_{s_1}(\lambda(c)) + 1)(m_{s_2}(\lambda(\bar{c})) + 1) & \text{otherwise.} \end{cases}$$

The number A_2 of elements u in $K_\mu(n)$ is, by (2.1), given by

$$A_2 = \frac{|\Gamma|^n n!}{|\Gamma|^r 2^{m_1(\mu)} 3^{m_2(\mu)} \cdots (r - \ell(\mu))! \prod_{i \geqslant 1} m_i(\mu)!}.$$

To find a cycle y of modified type $(r)_{c^0}$ such that uy is of modified type $(s_1)_c \cup (s_2)_{\bar{c}}$, we first choose a number from each of the r cycles of u and r elements of Γ, and then choose one additional new number, say i, from one of the r cycles. (Keep in mind that we have counted every element twice this way because we have made an *ordered* choice of two i's in the same cycle of x.) Then we arrange these $r + 1$ numbers and r elements of Γ into a cycle $y = ((h_{i_1}, \dots, h_{i_r}, h_i), (i_1 \dots i_r i))$ of modified type $(r)_{c^0}$ (we need $r + 1$ elements of Γ, however the ith element is uniquely determined to be $h_i = h_{i_1}^{-1} \dots h_{i_r}^{-1}$ since the cycle-product is 1). In this way, uy will be of the modified type of two cycles whose cycle-products belong to opposite conjugacy classes, by Proposition 3.2(2).

To ensure that uy is of modified type $(s_1)_c \cup (s_2)_{\bar{c}}$, we need to choose the number i such that the two cycles on the right-hand side of (3.9) consist of an s_1-cycle and an s_2-cycle. The number of choices of such i is $q(\mu, r, s)$ if $s_1 = s_2 = s$ and $2q(\mu, r, s_1)$ if $s_1 \neq s_2$. We can further write $y = y_1 y_2$ where $y_1 = ((h_{i_1} h_i, h_{i_2}, \dots, h_{i_r}), (i_1 i_2 \dots i_r))$ and $y_2 = ((h_i^{-1}, h_i), (i_r i))$. According to Proposition 3.2, uy_1 is of the modified type $(n-1)_{c^0}$, and $uy = (uy_1)y_2$ has cycle products lying in opposite conjugacy classes $\{c, \bar{c}\}$ if and only if $h_i = h_{i_1}^{-1} \dots h_{i_r}^{-1}$ lies in c or $\bar{c}$. Note that $\zeta_c = \zeta_{\bar{c}}$.

Taking all these considerations into account, we obtain the number A_3 of choices for y so that uy belongs to $K_{(s_1)_c \cup (s_2)_{\bar{c}}}$:

$$A_3 = \begin{cases} 2^{m_1(\mu)} 3^{m_2(\mu)} \cdots q(\mu, r; s) |\Gamma|^r / (2\zeta_c) & \text{if } c = \bar{c} \text{ and } s_1 = s_2 = s, \\ 2^{m_1(\mu)} 3^{m_2(\mu)} \cdots q(\mu, r; s_1) |\Gamma|^r / \zeta_c & \text{otherwise.} \end{cases}$$

The number D of elements in $K_{(s_1)_c \cup (s_2)_{\bar{c}}}(n)$ is, by (2.1), given by

$$D = \begin{cases} |\Gamma|^n n! / (2s^2 \zeta_c^2) & \text{if } c = \bar{c} \text{ and } s_1 = s_2 = s, \\ |\Gamma|^n n! / (s_1 s_2 \zeta_c^2) & \text{otherwise.} \end{cases}$$

Thus, $B = A_2 A_3 / D$, and the coefficient N is given by $N = A_1 B = A_1 A_2 A_3 / D$. Remembering $n = ||\mu|| + r$ and $q(\mu, r, s) = (r - \ell(\mu))! \prod_{i \geqslant 1} m_i(\mu)! \cdot p(\mu, r, s)$, we have established the theorem. $\square$

Example 3.5. (1) A specialization of Theorem 3.4 gives us

$$K_{(1)_{c^0}} K_{(1)_{c^0}} = 2K_{(1^2)_{c^0}} + 3K_{(2)_{c^0}} + \sum_{\{c,\bar{c}\}, c \neq c^0} \zeta_c K_{(1)_c \cup (1)_{\bar{c}}}.$$

(2) If we specialize Theorem 3.3 to $\lambda = (s)_{c^0}$, then we have two nontrivial terms (i.e. $\mu = \emptyset$ or $\mu = (s)_{c^0}$):

$$K_{(s)_{c^0}} K_{(r)_c} = K_{(s)_{c^0} \cup (r)_c} + (s+r) K_{(s+r)_c}, \quad c \neq c^0.$$

By changing r to s in Theorem 3.4 and then specializing to $\lambda = (r)_c$, we obtain the same answer (as expected).

We now introduce a partial ordering on $\mathscr{P}_n(\Gamma_*)$. Given $\lambda, \mu \in \mathscr{P}_n(\Gamma_*)$, $\mu \geqslant \lambda$ if $\mu(c^0)$ is strictly contained in $\lambda(c^0)$, or if $\mu(c) \geqslant \lambda(c)$ for each $c \in \Gamma_*$ in the case when $||\lambda(c)|| = ||\mu(c)||$ for all c. We can reformulate Theorem 3.3 as follows.

Theorem 3.6. *Let $c \neq c^0$, $\lambda \in \mathscr{P}(\Gamma_*)$, and $\tilde{\lambda} = \lambda - \lambda(c^0)$. Then*

(1) *if $||\lambda(c^0)|| + r = k$, then*

$$a_{\lambda (r)_c}^{\tilde{\lambda} \cup (k)_c} = \begin{cases} \dfrac{(m_k(\lambda(c)) + 1) \cdot k \cdot (r-1)!}{(r - \ell(\lambda(c^0)))! \prod_{i \geqslant 1} m_i(\lambda(c^0))!} & \text{if } \ell(\lambda(c^0)) \leqslant r, \\ 0 & \text{otherwise}. \end{cases}$$

(2) *if $||\lambda|| + r = ||v||$, and if we write $v = (v(c))_{c \in \Gamma_*}$ and the partition $v(c) = (v_1, v_2, \ldots)$, then*

$$a_{\lambda (r)_c}^{v} = \sum a_{\mu (r)_c}^{(v_i)_c}$$

summed over pairs (i, μ), where $\mu = \mu(c^0)$ and $\mu \cup v = \lambda \cup (v_i)_c$.

(3) *the coefficient $a_{\lambda (r)_c}^{v} = 0$ unless $v \geqslant \lambda \cup (r)_c$, and $a_{\lambda (r)_c}^{\lambda \cup (r)_c} > 0$.*

Proof. The coefficient in part (1) is the one on the right-hand side of the general formula in Theorem 3.3 corresponding to $\mu = \lambda(c^0)$. Part (2) is a simple rewriting of this general formula. The first half of part (3) follows from the definition of the partial order $\geqslant$. Note that $a_{\lambda (r)_c}^{\lambda \cup (r)_c}$ is the coefficient on the right-hand side of the general formula corresponding to $\mu = \emptyset$. $\square$

Remark 3.7. It is possible also to reformulate Theorem 3.4 in a form like Theorem 3.6. Let us simply mention that $a_{\lambda (r)_{c^0}}^{v} = 0$ unless $v \geqslant \lambda \cup (r)_{c^0}$, and $a_{\lambda (r)_{c^0}}^{\lambda \cup (r)_{c^0}} > 0$.

Remark 3.8. In the symmetric group case, the product in the FH-ring with a conjugacy class of single cycles was computed in [FH]. The wreath product case here is considerably more difficult. The reformulation in the form of Theorem 3.6 is similar to the presentation of Macdonald (cf. [Mac, Example 24, p. 132]). Some other structure constants have also been computed in the symmetric group case by Goulden–Jackson and others (cf. [GJ] and the references therein).

The structure constants $a^{\nu}_{\lambda\mu}$ for $\nu = \lambda \cup \mu$ are easily computed just as in the symmetric group case (cf. [FH, Lemma 3.10]).

Proposition 3.9. *Let* $\lambda = (r^{m_r(\lambda(c))})_{r \geqslant 1, c \in \Gamma_*}$ *and* $\mu = (r^{m_r(\mu(c))})_{r \geqslant 1, c \in \Gamma_*}$. *We have*

$$a^{\lambda \cup \mu}_{\lambda\mu} = \prod_{r \geqslant 1, c \in \Gamma_*} \binom{m_r(\lambda(c)) + m_r(\mu(c))}{m_r(\lambda(c))}.$$

Proof. Fix an element z in the conjugacy class $\mathscr{K}_{\lambda \cup \mu}$. The coefficient $a^{\lambda \cup \mu}_{\lambda\mu}$ is equal to the number of pairs (x, y) such that $xy = z$, where $x \in \mathscr{K}_\lambda$ and $y \in \mathscr{K}_\mu$, which is given by the formula in the proposition. $\square$

3.3. The structures of the Farahat–Higman ring

Theorem 3.10. *Let* $\lambda = (\lambda(c))_{c \in \Gamma_*}$ *be in* $\mathscr{P}(\Gamma_*)$, *where* $\lambda(c) = (\lambda_1(c), \lambda_2(c), \ldots)$. *Then the product* $\prod_{c \in \Gamma_*} K_{\lambda_1}(c) K_{\lambda_2}(c) \ldots$ *in the FH-ring* $\mathscr{G}_\Gamma$ *is of the form*

$$\prod_{c \in \Gamma_*} K_{\lambda_1}(c) K_{\lambda_2}(c) \ldots = \sum_{\mu \geqslant \lambda} d_{\lambda\mu} K_\mu$$

with $d_{\lambda\mu} \in \mathbb{Z}_+$ *and* $d_{\lambda\lambda} > 0$. *In particular*, $K_r(c)$, *where* $r \geqslant 1, c \in \Gamma_*$, *are algebraically independent elements of the FH-ring* $\mathscr{G}_\Gamma$, *and generate* $\mathbb{Q} \otimes_{\mathbb{Z}} \mathscr{G}_\Gamma$.

Proof. The first statement follows from Theorems 3.6(3) and Remark 3.7. The second statement holds since the matrix $(d_{\lambda\mu})$ is triangular and nonsingular. $\square$

Remark 3.11. Theorems 3.3 and 3.4 also provide a recursive way of computing the general structure constants $a^{\nu}_{\lambda\mu}$.

Denote by $\mathscr{I}_\Gamma(n)$ the subspace of $\mathscr{G}_\Gamma(n)$ which is spanned by those $K_\lambda(n)$ where $\lambda \neq \lambda(c^0)$. Similarly, we introduce the subspace $\mathscr{I}_\Gamma$ of $\mathscr{G}_\Gamma$. Recall that $\mathscr{G}(n)$ and $\mathscr{G}$ denote the rings $\mathscr{G}_\Gamma(n)$ and $\mathscr{G}_\Gamma$ when Γ is trivial.

Corollary 3.12. *The space* $\mathscr{I}_\Gamma(n)$ *is a graded ideal of the ring* $\mathscr{G}_\Gamma(n)$, *and the quotient ring* $\mathscr{G}_\Gamma(n)/\mathscr{I}_\Gamma(n)$ *is canonically isomorphic to the ring* $\mathscr{G}(n)$. *Similarly*, $\mathscr{I}_\Gamma$ *is a graded ideal of the ring* $\mathscr{G}_\Gamma$, *and the quotient ring* $\mathscr{G}_\Gamma/\mathscr{I}_\Gamma$ *is canonically isomorphic to the ring* $\mathscr{G}$.

Proof. It suffices to prove the statements for $\mathscr{G}_\Gamma$ as the same proof applies to $\mathscr{G}_\Gamma(n)$.

We proceed by induction on the length of elements of the set $\mathscr{P}(\Gamma_*)$. By observation from the formulas in Theorems 3.3 and 3.4, we see that $\mathscr{I}_\Gamma$ is closed under the multiplication by any conjugacy class K_ρ of single-cycle type (i.e. $\ell(\rho) = 1$).

For a given $\rho \in \mathscr{P}(\Gamma_*)$ with $\ell(\rho) > 1$, let us assume that $\mathscr{I}_\Gamma$ is closed under the multiplication by K_λ for all λ with $\ell(\lambda) < \ell(\rho)$.

First assume that some cycle-product of ρ is not 1, i.e., $\rho \neq \rho(c^0)$. Then we can write $\rho = \lambda \cup (r)_c$ for some $c \neq c^0$ and some λ with $\ell(\lambda) < \ell(\rho)$. By applying Theorem 3.3, we observe that all the conjugacy classes appearing in the product $K_\lambda K_{(r)_c}$ have length less than $\ell(\rho) = \ell(\lambda) + 1$ except for the term $K_{\lambda \cup (r)_c} = K_\rho$. By applying the induction assumption to $K_\lambda, K_{(r)_c}$, and all those conjugacy classes in the product $K_\lambda K_{(r)_c}$ except for K_ρ, we see that $\mathscr{I}_\Gamma$ is closed under the multiplication by K_ρ.

Now assume that $\rho = \rho(c^0)$. We can write $\rho = \lambda \cup (r)_{c^0}$ for some $\lambda = \lambda(c^0)$ with $\ell(\lambda) < \ell(\rho)$. By applying Theorem 3.4, we observe that all the conjugacy classes appearing in the first term on the right-hand side of the product $K_\lambda K_{(r)_{c^0}}$ have length less than $\ell(\rho) = \ell(\lambda) + 1$ except for the term $K_{\lambda \cup (r)_{c^0}} = K_\rho$; the second term is 0 since $\lambda = \lambda(c^0)$; we further observe that all the conjugacy classes appearing in the remaining three terms on the right-hand side of the product $K_\lambda K_{(r)_{c^0}}$ have length less than or equal to $\ell(\rho)$, and each of the corresponding modified types contains some cycle-product which is not 1. Thus, we can apply the induction assumption together with the case established in the previous paragraph. This proves that $\mathscr{I}_\Gamma$ is closed under the multiplication by K_ρ.

This finishes the proof that $\mathscr{I}_\Gamma$ is a graded ideal of the ring $\mathscr{G}_\Gamma$. It now follows from Theorem 3.10 and the explicit formula in Theorem 3.4 that the obvious map between the quotient ring $\mathscr{G}_\Gamma / \mathscr{I}_\Gamma$ and the ring $\mathscr{G}$ is a ring isomorphism. $\square$

Remark 3.13. Another proof of Corollary 3.12 goes as follows. Regarding S_n as a subgroup of Γ_n, we note that the intersection $\mathscr{K}_\lambda(n) \cap S_n$ of S_n with a conjugacy class $\mathscr{K}_\lambda(n)$ of Γ_n is a conjugacy class of S_n when $\lambda = \lambda(c^0)$, and is $\emptyset$ otherwise. Thus we have a surjective ring homomorphism $\varphi_n : R_{\mathbb{Z}}(\Gamma_n) \to R_{\mathbb{Z}}(S_n)$, by $K_\lambda(n) \mapsto K_\lambda(n) \cap S_n$ for $\lambda = \lambda(c^0)$ and $K_\lambda(n) \mapsto 0$ otherwise. The ring homomorphism φ_n is compatible with the filtrations and thus induces a surjective ring homomorphism $\phi_n : \mathscr{G}_\Gamma(n) \to \mathscr{G}(n)$. It follows that the kernel of ϕ_n coincides with $\mathscr{I}_\Gamma(n)$ and thus we have the ring isomorphism $\mathscr{G}_\Gamma(n)/\mathscr{I}_\Gamma(n) \cong \mathscr{G}(n)$.

One may look at Example 3.5, where the type of phenomena in Corollary 3.12 is manifest.

The next corollary can be established by the same type of induction argument as in the first proof of Corollary 3.12, which we omit here.

Corollary 3.14. *The $n|\Gamma_*|$ elements $K_{(r)_c}(n)$, where $0 \leqslant r < n$ for $c = c^0$ and $1 \leqslant r \leqslant n$ for $c \neq c^0$, form a set of ring generators of the ring $\mathbb{Q} \otimes_{\mathbb{Z}} \mathscr{G}_\Gamma(n)$.*

Remark 3.15. Corollary 3.14 implies that the same $n|\Gamma_*|$ elements also form a set of ring generators of the ring $R(\Gamma_n)$. This latter fact was established in [Wa4], Theorem 5.9(ii) in a totally different approach.

Definition 3.16. A conjugacy class $c \in \Gamma_*$ is called *real* (resp. *complex*) if $c = \bar{c}$ (resp. $c \neq \bar{c}$). Define $\Gamma_*^{\mathrm{cx}} = \{c \in \Gamma_* | c \neq \bar{c}\}$ and $\Gamma_*^{\mathrm{re}} = \{c \in \Gamma_* | c \neq c^0$ and $c = \bar{c}\}$.

Of course, we have $|\Gamma_*| = |\Gamma_*^{\mathrm{cx}}| + |\Gamma_*^{\mathrm{re}}| + 1$ and $|\Gamma_*^{\mathrm{cx}}|$ is always an even integer.

Theorem 3.17. *The ring $\mathbb{C} \otimes_{\mathbb{Z}} \mathscr{G}_\Gamma(n)$ for every n does not depend on the finite group Γ, but depends only on the two numbers $|\Gamma_*|$ and $|\Gamma_*^{\mathrm{re}}|$. That is, if G is another finite group such that $|\Gamma_*| = |G_*|$ and $|\Gamma_*^{\mathrm{re}}| = |G_*^{\mathrm{re}}|$, then the rings $\mathbb{C} \otimes_{\mathbb{Z}} \mathscr{G}_\Gamma(n)$ and $\mathbb{C} \otimes_{\mathbb{Z}} \mathscr{G}_G(n)$ are isomorphic for every n.*

Proof. We observe that the factor ζ_c in the formula in Theorem 3.4 can be annihilated by a rescaling $\tilde{K}_\lambda = \prod_{c \neq c^0} \zeta_c^{\ell(\lambda(c))/2} K_\lambda$ over $\mathbb{C}$. It follows from Theorems 3.3 and 3.4 that the ring $\mathbb{C} \otimes_{\mathbb{Z}} \mathscr{G}_\Gamma(n)$ with respect to the new basis $\tilde{K}_\lambda$ no longer depends on the group Γ.

Let G be another finite group such that $|\Gamma_*| = |G_*|$ and $|\Gamma_*^{\mathrm{re}}| = |G_*^{\mathrm{re}}|$. Fix a bijection φ from Γ_* to G_* such that its restriction to Γ_*^{re} is one-to-one and onto G_*^{re}. Then the obvious map by matching the rescaled conjugacy classes of Γ_n and those of G_n in terms of φ is a ring isomorphism by Theorems 3.3, 3.4 and Corollary 3.14. $\square$

The following conjecture concerns about the converse of Theorem 3.17.

Conjecture 3.18. *Let Γ and G be two finite groups. If the rings $\mathbb{C} \otimes_{\mathbb{Z}} \mathscr{G}_\Gamma(n)$ and $\mathbb{C} \otimes_{\mathbb{Z}} \mathscr{G}_G(n)$ are isomorphic for every n, then $|\Gamma_*^{\mathrm{re}}| = |G_*^{\mathrm{re}}|$ (in addition to the obvious identity $|\Gamma_*| = |G_*|$).*

3.4. Real conjugacy classes and the dual McKay correspondence. The classification of finite subgroups of $SL_2(\mathbb{C})$ is well known. The following is a complete list: cyclic groups $\mathbb{Z}_n$ of order n, the binary dihedral groups BT_{4n} of order $4n$ ($n \geqslant 3$), the binary tetrahedral group BT, the binary octahedral group BO, and the binary icosahedral group BI. It is well known (cf. e.g. [Cox,Slo]) that there is a bijection between finite subgroups Γ of $SL_2(\mathbb{C})$ (modulo conjugation) and Dynkin diagrams Δ of ADE types.

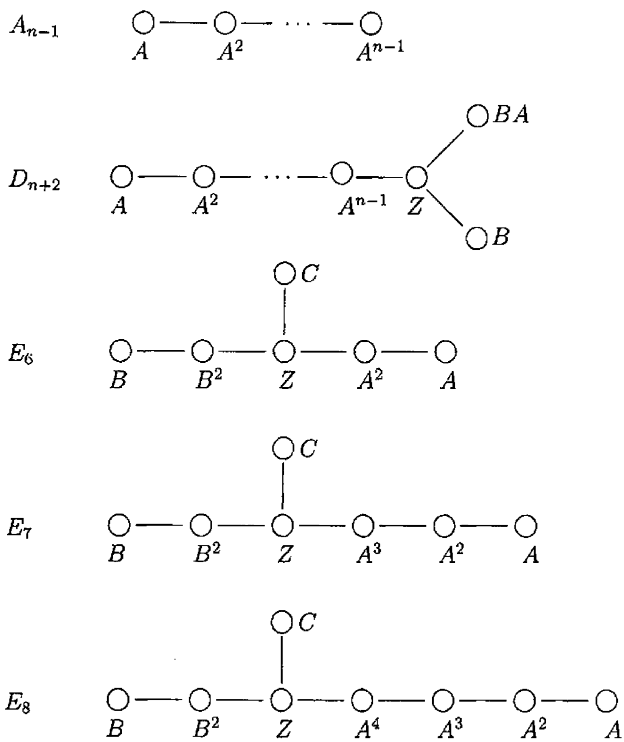
A nice presentation of these finite groups in terms of generators and relations can be found in [Cox]. In Table 1 below, we have set $Z = -I$, the negative of the identity matrix in $SL_2(\mathbb{C})$. So, $Z^2 = 1$. The cyclic group has one generator A, the binary dihedral group has two generators A, B, and each of the binary polyhedral groups has three generators A, B, C.

There is a dual McKay correspondence introduced by Ito–Reid [IR] which identify bijectively the non-trivial conjugacy classes of Γ and the vertices of the corresponding Dynkin diagram. Brylinski [Bry] later gave a more transparent and

Table 1

Δ	$\lvert\Gamma\rvert$	Γ	Relations
A_{n-1}	n	Cyclic $\mathbb{Z}_n$	$A^n = 1$
D_{n+2}	$4n$	Binary dihedral BD_{4n}	$A^n = B^2 = (AB)^2 = Z$
E_6	24	Binary tetrahedral BT	$A^3 = B^3 = C^2 = Z$
E_7	48	Binary octahedral BO	$A^4 = B^3 = C^2 = Z$
E_8	120	Binary icosahedral BI	$A^5 = B^3 = C^2 = Z$

canonical construction, by using Mumford's description [Mum] of Γ as the fundamental group of the complement of the exceptional divisor of the minimal resolution of $\mathbb{C}^2/\Gamma$. Following [Bry], we have attached the specific representatives of conjugacy classes to the vertices as follows.



Theorem 3.19. (1) *Via the dual McKay correspondence, the subset* Γ_*^{re} (*resp.* Γ_*^{cx}) *of conjugacy classes for* $\Gamma \leqslant SL_2(\mathbb{C})$ *corresponds to the set of vertices in the Dynkin diagrams which are fixed* (*resp. not fixed*) *by the diagram automorphism* τ^Γ. *Here* τ^Γ *is the automorphism which exchanges the two end-points of Dynkin diagrams for* $\Gamma = \mathbb{Z}_n$ $(n \geqslant 2)$, BD_{8k+4}, *and* BT, *and is the trivial automorphism otherwise.*

Table 2

Γ	$\mathbb{Z}_{2k+1}$	$\mathbb{Z}_{2k}$	BD_{8k}	BD_{8k+4}	BT	BO	BI
$\lvert\Gamma_*\rvert$	$2k+1$	$2k$	$2k+3$	$2k+4$	7	8	9
$\lvert\Gamma_*^{\mathrm{cx}}\rvert$	$2k$	$2k-2$	0	2	4	0	0
$\lvert\Gamma_*^{\mathrm{re}}\rvert$	0	1	$2k+2$	$2k+1$	2	7	8
$o(\tau^\Gamma)$	2	2	1	2	2	1	1

(2) *The set* Γ_*^{cx} *is given as follows*: $\{\langle A^i\rangle, 1\leqslant i<2k+1\}$ *for* $\mathbb{Z}_{2k+1}$ $(k>0)$; $\{\langle A^i\rangle, 1\leqslant i<2k, i\neq k\}$ *for* $\mathbb{Z}_{2k}$; $\{\langle B\rangle, \langle BA\rangle\}$ *for* BD_{8k+4} $(k>0)$; $\{\langle A\rangle, \langle B\rangle, \langle A^2\rangle, \langle B^2\rangle\}$ *for* BT; *the empty set* $\emptyset$ *for* BD_{8k}, BO *and* BI.

(3) *The cardinalities of* Γ_*, Γ_*^{re}, Γ_*^{cx}, *together with the order of* τ^Γ, *are listed in Table 2*.

Proof. Parts (1) and (3) easily follow from (2) by comparing with dual McKay correspondence described above. So let us prove (2) below. The cyclic group case is evident.

For the groups BD_{4n}, one can easily write down all $n+3$ conjugacy classes: $\{1\}$, $\{Z\}$, $\{A^{\pm i}\}$ $(1\leqslant i\leqslant n-1)$, $\{A^{2j}B, 0\leqslant j\leqslant n-1\}$, and $\{A^{2j-1}B, 0\leqslant j\leqslant n-1\}$. The results for BD_{4n} (depending on whether n is even or odd) follow now from this description together with the fact that $B^{-1}=A^nB$.

For the binary tetrahedral group BT, we use the explicit elements given in terms of quaternions in [Cox, (7.26), p. 77]. They are ± 1, $\pm i$, $\pm j$, $\pm k$, and $(\pm 1\pm i\pm j\pm k)/2$, where i,j,k stands for the quaternion generators. In terms of the notations in Table 1, we have $A=(1+i+j+k)/2$, $C=i$ and $B=(1+i-j+k)/2$ (cf. (7.22) in [Cox]). Then a direct calculation leads to the following list of 7 conjugacy classes of BT: $\{1\}, \{-1\}, K_0, K_1, K_2, -K_1$, and $-K_2$, where

$$K_0=\{\pm i, \pm j, \pm k\},$$

$$K_1=\{(1+i+j+k)/2, (1+i-j-k)/2, (1-i+j-k)/2, (1-i-j+k)/2\},$$

$$K_2=\{(1-i-j-k)/2, (1-i+j+k)/2, (1+i-j+k)/2, (1+i+j-k)/2\}$$

and $-K_i$ consists of the negatives of all elements in K_i, $i=1,2$. One verifies that $\langle A\rangle=K_1, \langle A^{-1}\rangle=\langle B\rangle=K_2$, and similarly, $\langle A^2\rangle=-K_2$ and $\langle A^{-2}\rangle=\langle B^2\rangle=-K_1$. Thus, we have $(BT)_*^{\mathrm{cx}}=\{\langle A\rangle, \langle B\rangle, \langle A^2\rangle, \langle B^2\rangle\}$.

For the binary icosahedral group BI, we note that the orders of the representatives A, A^2, A^3, A^4, B, B^2 and C of the conjugacy classes as marked in the E_8 Dynkin diagram are, respectively, $10, 5, 10, 5, 6, 3$ and 4. To show that $(BI)_*^{\mathrm{cx}}=\emptyset$, it remains to check that the elements of the same order do not lie in opposite conjugacy classes. Let us check that A is not conjugate to the inverse of A^3. If A were conjugate to A^{-3}, then A^3 is conjugate to $(A^{-3})^3=A$ (since $A^{10}=1$) which contradicts with the fact that A and A^3 are representatives of different conjugacy classes. Similarly, we see that A^2 is not conjugate to the inverse of A^4.

This argument also applies to the binary octahedral group BO. Alternatively, one can also verify the theorem for BO directly using its explicit realization in quaternions given by (7.26) and (7.35) in [Cox]. $\square$

Corollary 3.20. *The rings* $\mathbb{C}\otimes_{\mathbb{Z}}\mathcal{G}_{BD_{24}}(n)$ *and* $\mathbb{C}\otimes_{\mathbb{Z}}\mathcal{G}_{BI}(n)$ *are isomorphic for each* n.

Proof. Follows from Theorems 3.17 and 3.19. $\square$

The group Γ acts on $\mathbb{C}^2 - \{0\}$ freely and $\mathbb{C}^2/\Gamma$ has a simple singularity at the origin. Let us denote by $\mathbb{C}^2//\Gamma$ the minimal resolution of $\mathbb{C}^2/\Gamma$. The bijection between the finite subgroups of $SL_2(\mathbb{C})$ and the Dynkin diagrams of ADE types can also be seen through the geometry of minimal resolutions. A classical result of Du Val (cf. e.g. [Slo]) says that the set of exceptional curves in the minimal resolution $\mathbb{C}^2//\Gamma$ corresponds naturally to the set of vertices of the corresponding Dynkin diagram.

Remark 3.21. Combining with the dual McKay correspondence [IR,Bry], we may ask what is the geometric significance of the set of exceptional curves corresponding to the set Γ_*^{re}, or alternatively what is the geometric significance of the Dynkin diagram automorphisms τ^Γ in Theorem 3.19. It turns out that this question has the following beautiful answer which was suggested by Dolgachev to the author. Consider the automorphism τ_Γ of $\mathbb{C}^2//\Gamma$ which comes from the one which sends z to $-z$ if we realize the simple singularities as the hypersurface singularities in $\mathbb{C}^3$ in the standard way (cf. e.g. [Slo]): $x^n + y^2 + z^2 = 0$ for $\Gamma = \mathbb{Z}_n$; $x(y^2 - x^n) + z^2 = 0$ for $\Gamma = BD_{4n}$; $x^4 + y^3 + z^2 = 0$ for $\Gamma = BT$; $x^3 + xy^3 + z^2 = 0$ for $\Gamma = BO$; and $x^5 + y^3 + z^2 = 0$ for $\Gamma = BI$. Then τ_Γ induces an automorphism on the degree-2 homology group of $\mathbb{C}^2//\Gamma$ which fixes exactly the homology classes associated to the exceptional curves corresponding to Γ_*^{re}, i.e. this induced automorphism is exactly the τ^Γ in Theorem 3.19. In addition, τ_Γ is an involution of $\mathbb{C}^2/\Gamma$ such that τ_Γ and Γ generate a complex reflection subgroup $\hat{\Gamma}$ of $GL_2(\mathbb{C})$. The quotient $\mathbb{C}^2/\hat{\Gamma}$ is isomorphic to the affine plane.

On the other hand, it is brought to my attention by Brian Parshall that the diagram automorphism τ^Γ can be interpreted as minus the longest reflection in the corresponding Weyl group acting on the set of simple roots, if we attach a simple root to each of the vertices of the Dynkin diagram associates to Γ.

4. Connections with the cohomology rings of Hilbert schemes

4.1. Minimal resolutions and Hilbert schemes. Given a quasi-projective surface X, we denote by $X^{[n]}$ the Hilbert scheme of n points on X. An element in $X^{[n]}$ is represented by a length n zero-dimensional closed subscheme. A well-known theorem of Fogarty states that $X^{[n]}$ is nonsingular and the Hilbert–Chow morphism $\pi_n : X^{[n]} \to X^n/S_n$, which sends an element to its support, is a resolution of singularities (cf. [Na3]).

Connections between Hilbert schemes and wreath products were first pointed out in [Wa1]. In the special case when X is the minimal resolution $\mathbb{C}^2//\Gamma$ of a simple singularity $\mathbb{C}^2/\Gamma$ (for $\Gamma \leqslant SL_2(\mathbb{C})$), we have the following commutative diagram:

$$\begin{array}{ccc} (\mathbb{C}^2//\Gamma)^{[n]} & \rightarrow & (\mathbb{C}^2//\Gamma)^n/S_n \\ \downarrow & & \downarrow \\ \mathbb{C}^{2n}/\Gamma_n & \stackrel{\cong}{\leftarrow} & (\mathbb{C}^2/\Gamma)^n/S_n \end{array}$$

which defines a resolution of singularities $(\mathbb{C}^2//\Gamma)^{[n]} \to \mathbb{C}^{2n}/\Gamma_n$. It has thus been expected that all the geometric invariants of $(\mathbb{C}^2//\Gamma)^{[n]}$ can be described entirely by using the wreath product Γ_n.

4.2. Shift numbers for the wreath product orbifolds. Given a complex manifold M acted upon by a finite group G, Zaslow [Zas] introduced a shift number associated to $g \in G$ (depending on the connected components of the fixed-point set M^g). In the case when M is the affine space $\mathbb{C}^N$, also see Ito–Reid [IR]. For the orbifold $\mathbb{C}^{2n}/\Gamma_n$, the shift numbers have been computed in [WaZ]. Our observation is the following.

Lemma 4.1. *Let Γ be a finite subgroup of $SL_2(\mathbb{C})$. Let $g \in \Gamma_n$ be of modified type $\rho \in \mathscr{P}(\Gamma_*)$. The shift number for g in the orbifold $\mathbb{C}^{2n}/\Gamma_n$ coincides with $||\rho||$.*

Proof. The shift numbers for a general wreath product orbifold were calculated in [WaZ, formula (9), p. 162]. Let us pinpoint what the notations in [WaZ] mean, when we specialize to the affine orbifold $\mathbb{C}^{2n}/\Gamma_n$: $d = 2$ (which is the dimension of $\mathbb{C}^2$); the fixed-point set of an element $a \in \Gamma$ acting on $\mathbb{C}^2$ is a point (resp. the whole $\mathbb{C}^2$) and thus the shift number in $\mathbb{C}^2/\Gamma$ is $F^c = 1$ (resp. $F^c = 0$) if $a \neq 1$ (resp. $a = 1$); in particular, the fixed-point set is always connected which means $N_c = 1$. This shows that the formula (9) in [WaZ] agrees exactly with our definition of $||\rho||$. $\square$

Remark 4.2. A general wreath product Γ_n, associated to a finite subgroup of $\Gamma \leqslant SL_k(\mathbb{C})$, acts on $\mathbb{C}^{kn}$. One can still define the shift numbers in the sense of Zaslow. But they in general do not coincide with (a constant multiple of) the values of $|| \cdot ||$ except some rare cases. Such cases include V^n/S_n or V^n/Γ_n (where $\Gamma = \mathbb{Z}/2\mathbb{Z}$ acts as $\pm \mathrm{Id}_V$), for a vector space $V = \mathbb{C}^{2m}$, $m \geqslant 2$. It will be very interesting to analyze the structure of the graded algebra of $R_{\mathbb{Z}}(\Gamma_n)$ associated to this orbifold filtration for a general Γ.

4.3. A ring isomorphism. Lehn–Sorger [LS1] and independently Vasserot [Vas] gave a description of the cohomology ring $H^*(X^{[n]})$ for $X = \mathbb{C}^2$ in terms of the symmetric group S_n. (However, the reference [FH] has never featured in the literature on

Hilbert schemes.) In light of Lemma 4.1, a theorem of Etingof–Ginzburg [EG], when combined with the results of Nakajima, etc., leads to the following description of the cohomology ring $H^*((\mathbb{C}^2//\Gamma)^{[n]})$ with $\mathbb{C}$-coefficient (cf. the remarks in a footnote in the Introduction).

Theorem 4.3. *Let Γ be a finite subgroup of $SL_2(\mathbb{C})$. There exists a ring isomorphism between the cohomology ring $H^*((\mathbb{C}^2//\Gamma)^{[n]})$ with $\mathbb{C}$-coefficient and the ring $\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{G}_\Gamma(n)$.*

Combined with Theorem 4.3, the results in the previous sections provide finer structures on the cohomology ring of $(\mathbb{C}^2//\Gamma)^{[n]}$. Let us restrict ourselves to mention the following corollaries.

Corollary 4.4. *For each n, the cohomology rings of the Hilbert schemes of n points of the minimal resolution $\mathbb{C}^2//\Gamma$ are determined by the numbers $|\Gamma_*|$ and $|\Gamma_*^{\mathrm{re}}|$.*

Proof. Follows from Theorems 4.3 and 3.17. □

Remark 4.5. A theorem of [LQW2] says that the cohomology ring of $X^{[n]}$ associated to a *projective* surface X depends only on the cohomology ring of the surface X and the canonical class K_X. A natural and important question is what invariants on a quasi-projective surface X determine the cohomology ring of $X^{[n]}$ for all n. Corollary 4.4 can be regarded as a first step in this direction.

Corollary 4.6. *The cohomology rings of the Hilbert schemes of n points of the minimal resolutions for the binary group BD_{24} of order 24 and the binary Icosahedral group BI are isomorphic for all n.*

Proof. Follows from Theorem 4.3 and Corollary 3.20. □

Conjecture 3.18, if true, would imply that BD_{24} and BI are the only pair of finite subgroups of $SL_2(\mathbb{C})$ whose corresponding Hilbert schemes of n points have isomorphic cohomology ring structure for all n.

4.4. Implications on the cohomology rings of Hilbert schemes. In recent years, there has been much progress on the understanding of the cohomology rings of Hilbert schemes $X^{[n]}$ of n points on a (quasi-)projective surface X. Many results have been obtained for a general projective surface X using a vertex operator approach [Lehn,LQW1,LQW2,LQW3,LS2] (also cf. [Mar]) built on [Gro,Na2,Na3]. However, for X quasi-projective, the vertex operator techniques are not directly applicable, and little is known about the ring $H^*(X^{[n]})$, with the notable exceptions when X is the affine plane $\mathbb{C}^2$ or $\mathbb{C}^2//\Gamma$, cf. [LS1,Vas] and Theorem 4.3.

As suggested in [Wa3], the structures of the class algebras of wreath products Γ_n associated to an arbitrary finite group Γ (resp. the graded ring associated to some appropriate filtration) very much reflect those of the cohomology rings of Hilbert schemes $X^{[n]}$ of points on an arbitrary surface X which is projective (resp. quasi-projective). According to this philosophy, many of the results obtained on the rings $\mathscr{G}_\Gamma(n)$ (and $\mathscr{G}_\Gamma$) are expected to find their counterparts (some of which could be quite subtle) on the cohomology rings $H^*(X^{[n]})$ for a simply connected quasi-projective surface X. As an example for illustration, we will formulate a conjecture below.

Recall that a Heisenberg algebra was constructed [Gro,Na2,Na3] geometrically which acts on $\oplus_{n\geqslant 0} H^*(X^{[n]})$ irreducibly with the vacuum vector $|0\rangle = 1 \in H^*(X^{[0]})$. For the sake of notational simplicity, let us assume that X is simply connected and so it has no odd degree cohomology classes. For X either projective or quasi-projective, the half of the Heisenberg algebra, i.e. the creation operators, which are the only part we need below can always be modelled on the ordinary cohomology group of X (cf. [Lehn], Section 4.4; [Na3]), and they will be denoted by $\mathfrak{a}_{-n}(\alpha)$, where $n \in \mathbb{N}, \alpha \in H^*(X)$. Let us fix a linear basis B_2 of $H^2(X)$, so a linear basis S of $H^*(X)$ is given by $S = \{1_X\} \cup B_2$ for X quasi-projective (and with the extra class $\{[pt]\}$ for X projective), where 1_X denotes the unit. This leads to a natural linear basis for $H^*(X^{[n]})$ (cf. [Na3]), which we will refer to as the Heisenberg monomial basis.

Now let X be in addition quasi-projective. Just as the conjugacy classes of Γ_n, the Heisenberg monomial basis for $H^*(X^{[n]})$ can be also parametrized by modified types as follows. Given $\mu = (\mu(c))_{c\in S} \in \mathscr{P}(S)$, we denote by $\tilde{\mu} = (\tilde{\mu}(c))_{c\in S} \in \mathscr{P}(S)$ with $\tilde{\mu}(c) = \mu(c)$ for $c \in B_2$, and $\tilde{\mu}(1_X) = (r^{m_r(\tilde{\mu}(1_X))})_{r\geqslant 1}$, where $m_r(\tilde{\mu}(1_X)) = m_{r-1}(\mu(1_X))$ for $r\geqslant 2$ and $m_1(\tilde{\mu}(1_X)) = n - ||\mu|| - \ell(\mu)$. (Here we have denoted $\mu(1_X) = (r^{m_r(\mu(1_X))})_{r\geqslant 1}$).

We denote by

$$\mathfrak{h}_\mu(n) = \frac{1}{(n - ||\mu|| - \ell(\mu))!} \prod_{c\in S} \prod_{r\geqslant 1} \mathfrak{a}_{-r}(c)^{m_r(\tilde{\mu}(c))} |0\rangle$$

if $n \geqslant ||\mu|| + \ell(\mu)$, and 0 otherwise. Then, those nonzero $\mathfrak{h}_\mu(n)$'s, as μ runs over $\mathscr{P}_n(S)$, form a linear basis for $H^*(X^{[n]})$. (The $\mathfrak{h}_\mu(n)$ defined this way are different from the $\mathfrak{a}_\mu(n)$ used in [LQW2].) Recall further [Na3] that the cohomology degree for $\mathfrak{a}_{-k}(\gamma)$ is $2k - 2 + |\gamma|$. This implies that the cohomology degree of $\mathfrak{h}_\mu(n)$ is exactly $2||\mu||$, and therefore the cohomology grading and the grading given by $||\cdot||$ are compatible.

Remark 4.7. For each finite subgroup $\Gamma \leqslant SL_2(\mathbb{C})$, one associates the Cartan matrix of the corresponding Dynkin diagram. The inverse of a Cartan matrix already makes an appearance in the cup product of the cohomology class dual to the boundary divisor with itself in $H^*((\mathbb{C}^2//\Gamma)^{[n]})$ for $n \geqslant 2$. This might help to convince the reader that the cohomology ring of $(\mathbb{C}^2//\Gamma)^{[n]}$ can be very interesting even though the cohomology ring of a simply-connected quasi-projective surface X is trivial.

Conjecture 4.8 (The Constant Conjecture). *Let X be a quasi-projective surface. The structure constants of the ring $H^*(X^{[n]})$ with respect to the Heisenberg monomial basis*

$(h_\mu(n))$ are independent of n. Thus, we can construct a FH-type ring associated to X which encodes the cohomology rings $H^*(X^{[n]})$ for all n.

Remark 4.9. A notion of *Hilbert ring* $\mathfrak{H}_X$ was introduced in [LQW2] associated to a *projective* surface X which encodes the cohomology ring structures of $H^*(X^{[n]})$ for all n. This notion has a counterpart (which is called the *stable ring*) for the orbifold cohomology rings of symmetric products, and for the class algebras of wreath products. A central result which allowed one to introduce these notions is the n-independence of certain coefficients. However, the formulation of the n-independence in those constructions is different in a fundamental way from the one which we establish in this paper. The formulation of the Hilbert ring, etc. is very subtle as it uses *normalized* Heisenberg monomials for Hilbert schemes and symmetric products and *normalized* conjugacy classes for wreath products, which are not linearly independent for a fixed n.

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Note added. Conjecture 4.8 has been established in some cases in [LQW4].

References

[Bry] J.-L. Brylinski, A correspondence dual to McKay's, arXiv:alg-geom/9612003.
[Cox] H.S.M. Coxeter, Regular Complex Polytopes, Cambridge University Press, London, New York, 1974.
[EG] P. Etingof, V. Ginzburg, Symplectic reflection algebras, Calogero–Moser space, and deformed Harish–Chandra homomorphism, Invent. Math. 147 (2002) 243–348.
[FH] H. Farahat, G. Higman, The centres of symmetric group rings, Proc. Roy. Soc. (A) 250 (1959) 212–221.
[FJW] I. Frenkel, N. Jing, W. Wang, Vertex representations via finite groups and the McKay correspondence, Internat. Math. Res. Notices 4 (2000) 195–222.
[GJ] I. Goulden, D. Jackson, Symmetric functions and Macdonald's result for top connexion coefficients in the symmetric group, J. Algebra 166 (1994) 364–378.
[Gro] I. Grojnowski, Instantons and affine algebras I: the Hilbert scheme and vertex operators, Math. Res. Lett. 3 (1996) 275–291.
[IR] Y. Ito, M. Reid, The McKay correspondence for finite subgroups of $SL(3,\mathbb{C})$, Higher-dimensional Complex Varieties (Trento, 1994), 221-240, de Gruyter, Berlin, 1996, arXiv:alg-geom/9411010.
[Kuz] A. Kuznetsov, Quiver varieties and Hilbert schemes, preprint, arXiv:math.AG/0111092.

[Lehn] M. Lehn, Chern classes of tautological sheaves on Hilbert schemes of points on surfaces, Invent. Math. 136 (1999) 157–207.

[LS1] M. Lehn, C. Sorger, Symmetric groups and the cup product on the cohomology of Hilbert schemes, Duke Math. J. 110 (2001) 345–357.

[LS2] M. Lehn, C. Sorger, The cup product of the Hilbert scheme for $K3$ surfaces, Invent. Math. 152 (2003) 305–329.

[LQW1] W.-P. Li, Z. Qin, W. Wang, Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces, Math. Ann. 324 (1) (2002) 105–133.

[LQW2] W.-P. Li, Z. Qin, W. Wang, Stability of the cohomology rings of Hilbert schemes of points on surfaces, J. Reine Angew. Math. 554 (2003) 217–234.

[LQW3] W.-P. Li, Z. Qin, W. Wang, Hilbert schemes and $\mathscr{W}$ algebras, Internat. Math. Res. Notices 27 (2002) 1427–1456.

[LQW4] W.-P. Li, Z. Qin, W. Wang, Ideals of the cohomology rings of Hilbert schemes and their applications, Trans. Amer. Math. Soc., to appear, math.AG/0208070.

[Mac] I.G. Macdonald, Symmetric Functions and Hall Polynomials, 2nd Edition, Clarendon Press, Oxford, 1995.

[Mar] E. Markman, Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces, J. Reine Angew. Math. 544 (2002) 61–82.

[Mum] D. Mumford, The topology of normal singularities of an algebraic surface and a criterion for simplicity, Publ. Math. IHES 9 (1961) 23–64.

[Na1] H. Nakajima, Instantons on ALE spaces, quiver varieties, and Kac–Moody algebras, Duke Math. J. 76 (1994) 365–416.

[Na2] H. Nakajima, Heisenberg algebra and Hilbert schemes of points on projective surfaces, Ann. Math. 145 (1997) 379–388.

[Na3] H. Nakajima, Lectures on Hilbert Schemes of Points on Surfaces, in: University Lecture Series, Vol. 18, American Mathematical Society, Providence, RI, 1999.

[Ru] Y. Ruan, Stringy geometry and topology of orbifolds, Symposium in Honor of C.H. Clemens (Salt Lake City, UT, 2000), Contemporary Mathematics, Vol. 312, American Mathematical Society, Providence, RI, 2002, pp. 187–233.

[Slo] P. Slodowy, Platonic solids, Kleinian singularities, and Lie groups, in: I. Dolgachev (Ed.), Algebraic Geometry, Ann Arbor, Michigan, 1981, Lecture Notes in Mathematics, Vol. 1008, Springer, Berlin, 1983, pp. 102–138.

[Vas] E. Vasserot, Sur l'anneau de cohomologie du schéma de Hilbert de $\mathbb{C}^2$, C. R. Acad. Sci. Paris, Sér. I Math. 332 (2001) 7–12.

[Wa1] W. Wang, Equivariant K-theory, wreath products, and Heisenberg algebra, Duke Math. J. 103 (2000) 1–23.

[Wa2] W. Wang, Hilbert schemes, wreath products, and the McKay correspondence, preprint, math.AG/9912104.

[Wa3] W. Wang, Algebraic structures behind Hilbert schemes and wreath products, in: S. Berman et al. (Eds.), Recent Developments in Infinite-Dimensional Lie Algebras and Conformal Field Theory, Charlottesville, VA, 2000; Contemp. Math. 297 (2002) 271–295.

[Wa4] W. Wang, Vertex algebras and the class algebras of wreath products, Proc. London Math. Soc., to appear, arXiv:math.QA/0203004.

[WaZ] W. Wang, J. Zhou, Orbifold Hodge numbers of the wreath product orbifolds, J. Geom. Phys. 38 (2001) 153–170.

[Zas] E. Zaslow, Topological orbifold models and quantum cohomology rings, Commun. Math. Phys. 156 (1993) 301–331.

[Zel] A. Zelevinsky, Representations of finite classical groups. A Hopf Algebra Approach, in: Lecture Notes in Mathematics, Vol. 869, Springer, Berlin, New York, 1981.