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Hamiltonicity and Combinatorial Polyhedra

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We say that a polyhedron with 0–1 valued vertices is *combinatorial* if the midpoint of the line joining any pair of nonadjacent vertices is the midpoint of the line joining another pair of vertices. We show that the class of combinatorial polyhedra includes such well-known classes of polyhedra as matching polyhedra, matroid basis polyhedra, node packing or stable set polyhedra and permutation polyhedra. We show the graph of a combinatorial polyhedron is always either a hypercube (i.e., isomorphic to the convex hull of a k-dimension unit cube) or else is hamilton connected (every pair of nodes is the set of terminal nodes of a hamilton path). This implies several earlier results concerning special cases of combinatorial polyhedra.

1. INTRODUCTION

The graph $G(P)$ of a polyhedron P is the graph whose nodes are the vertices of the polyhedron and which has an edge joining each pair of nodes for which the corresponding vertices of the polyhedron are adjacent, that is, joined by an edge of the polyhedron. Such graphs have been studied since the beginnings of graph theory; in 1857 Sir William Hamilton introduced his "tour of the world" game which consisted of constructing a closed tour passing exactly once through each vertex of the dodecahedron. Since then, great effort has been expended in developing necessary conditions and

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sufficient conditions for the existence of such tours in various classes of graphs. Fittingly, graphs which possess such tours are called "hamiltonian."

The graphs of three dimensional polyhedra are the three connected planar graphs. In 1880 Tait conjectured that every cubic three connected planar graph was hamiltonian and showed that this would provide a proof of the four color theorem. (It is not difficult to see that this would also have implied that the graph of every three dimensional polyhedron is hamiltonian.) Tutte [13], however, provided a counterexample to the Tait conjecture in 1947. (See Capobianco and Moluzzo [4, p. 165].)

Since then, various results have been proved showing that the graphs of certain classes of polyhedra are hamiltonian. For example, Balinski and Russakoff [1] proved that the graph of an assignment polytope (the convex hull of the $n!$ $n \times n$ permutation matrices) is hamiltonian.

Brualdi and Gibson [3] studied the graph of the convex hull of the perfect matchings of a bipartite graph and showed that these graphs are hamilton connected, unless the graph is a hypercube. (These terms are all defined in Section 2.)

Holzmann and Harary [9] showed that the graph of a matroid basis polytope (the convex hull of the incidence vectors of the bases of a matroid) is uniformly hamiltonian, provided that it contains at least two cycles. This was a generalization of earlier work of Cummings [7] and Shank [12], proving a similar result for tree graphs.

Our main result provides a unification and extension of these results. We show that for a certain class of polyhedra, whose vertices are 0–1 valued vectors, the graphs of these polyhedra are either hamilton connected or hypercubes. This class includes the matching polyhedra and matroid polyhedra already mentioned, variations on these polyhedra as well as stable set polyhedra and permutation polyhedra. We say that a polyhedron is *combinatorial* if its satisfies the following two properties:

(1) all its vertices are 0–1 valued;

(2) if vertices x and y are not adjacent, then there exist two other vertices u and v such that $x + y = u + v$.

The second condition can be rephrased to be: if two vertices are nonadjacent then the midpoint of the line joining them is the midpoint of the line joining two other vertices. This condition may appear rather unusual, but in fact it is satisfied by all the examples cited previously. Section 3 is concerned with establishing various classes of polyhedra which satisfy this property. In Section 2 we develop the theory of polyhedra with 0–1 valued vertices and combinatorial polyhedra and prove the main theorem, namely, that the graph of a combinatorial polyhedron is hamilton connected or a hypercube. In the final section, we present concluding remarks.

2. Polyhedra with 0–1 Valued Vertices

Let E be a finite set and let $\{0, 1\}^E$ denote the set of all 0–1 vectors indexed by E. Let $X \subseteq \{0, 1\}^E$ and let conv(X) denote the convex hull of X, where these vectors are considered as elements of $\mathbb{R}^E$. It is well known that

(2.1) for any $X \subseteq \{0, 1\}^E$ the vertices of conv(X) are precisely the members of X.

We let $G(X)$ denote the graph whose nodes are the members of X and which has an edge joining two nodes if and only if the corresponding vertices are adjacent on conv(X). Since two vertices of a polyhedron are adjacent if and only if they are the vertices of a one dimensional face, we see that

(2.2) $u, v \in X$ are adjacent nodes of $G(X)$ if and only if, for every λ satisfying $0 < \lambda < 1$, the point $\lambda u + (1 - \lambda)v$ cannot be expressed as a convex combination of members of $X - \{u, v\}$.

For any $u, v \in X$, we let $A(u, v) \subseteq E$ denote the set of coordinates wherein u and v agree in value and we let $D(u, v) \subseteq E$ denote the set of coordinates wherein they disagree. We let $\bar{X}(u, v)$ be the set of members of X which agree with u and v in all coordinates of $A(u, v)$. Trivially, $u, v \in \bar{X}(u, v)$ and (2.2) can be strengthened to

(2.3) $u, v \in X$ are adjacent nodes of $G(X)$ if and only if there does not exist λ satisfying $0 < \lambda < 1$ such that the point $\lambda u + (1 - \lambda)v$ is a convex combination of members of $\bar{X}(u, v) - \{u, v\}$.

(This is essentially Proposition 2.1 of Hausmann and Korte [8].) In other words, when checking adjacency it is sufficient to consider only members of X which agree with u and v for all those coordinates where they themselves have the same value. Therefore, if $D(u, v)$ is a minimal member of $\{D(u, x): x \in X - \{u\}\}$, we have $\bar{X}(u, v) - \{u, v\} = \varnothing$ and so

(2.4) if $D(u, v)$ is a minimal member of $\{D(u, x): x \in X - \{u\}\}$ or of $\{D(x, v): x \in X - \{v\}\}$ then u and v are adjacent.

It is not difficult to construct examples that show that the converse of (2.4) is false.

A *hypercube* is a graph isomorphic to the graph of the convex hull of $\{0, 1\}^E$ for some set E. If $|E| = d$, then we say that the hypercube is of *dimension d*. See Fig. 1. (Brualdi and Gibson [3] call this graph a *d*-box.) It is easily verified that a d dimension hypercube is constructed by taking two copies of a $d - 1$ dimension hypercube and then joining the two corresponding copies of each node. It follows that hypercubes are bipartite with the same number of nodes in each part, when the dimension is at least 1. It is an easy exercise to show that

(2.5) if G is a hypercube, then there exists a hamilton path joining two distinct nodes if and only if they belong to opposite parts of the graph.

For any $S \subseteq E$, for any $x \in X$, we let $x[S] \equiv (x_j : j \in S)$ and we let $X[S] \equiv \{x[S]: x \in X\}$. An important notion when studying 0–1 polyhedra is

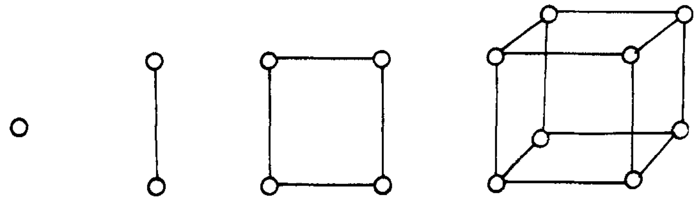

FIG. 1. Hypercubes of dimension d.

that of separability. We say that $S \subseteq E$ is a *separator* of X if and only if for every $x' \in X[S]$, for every $x'' \in X[E-S]$, the *concatenation* x of x' and x'', defined by

$$\begin{aligned} x_j &\equiv x'_j : j \in S \\ &\equiv x''_j : j \in E - S \end{aligned}$$

belongs to X. In other words, the values of the coordinate positions corresponding to S and $E-S$ are independent of each other. If a separator S satisfies $\varnothing \neq S \neq E$ then we say that S is a *proper separator*. We say that S is *nonseparable* if there exists no proper separator. Otherwise, X is said to be *separable*. A *component* of X is a minimal nonempty separator of X (which is E if X is nonseparable). Clearly, the components of X provide a partition of E.

Let S be a component of X and let $r \equiv |X[S]|$. We say that S is an *r-valued component*. If S is a 1-valued component, then $|S| = 1$. If S is a 2-valued component then $X[S]$ consists of two complementary vectors. That is, $X[S] = \{u, v\}$, where $u_i = 0$ if and only if $v_i = 1$, for $i \in S$. For any positive integer k, we say that S is *$\geqslant k$-valued* provided that $r \geqslant k$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. Following the terminology of Berge [2], we say that $G = (V, E)$ is the *cartesian sum* of G_1 and G_2 if $V = V_1 \times V_2$ and nodes (v_1, v_2) and $(w_1, w_2) \in V$ are adjacent in G if and only if either $v_1 = w_1$ and v_2 is adjacent to w_2 in G_2, or $v_2 = w_2$ and v_1 is adjacent to w_1 in G_1. Since this is the only form of graph product that we use, we write in this case, $G = G_1 \times G_2$. It is easily verified that this product is associative under isomorphism so if $G_1, G_2, \ldots, G_k$ are graphs we can write $G = \prod_{i=1}^k G_i$, to denote their cartesian sum without ambiguity. Moreover, we note that $G_1 \times G_2$ and $G_2 \times G_1$ are isomorphic graphs, so this product is commutative under isomorphism.

LEMMA 2.1. *If $S_1, S_2, \ldots, S_k$ are the $\geqslant 2$-valued components of X, then $G(X)$ is isomorphic to $\prod_{i=1}^k G(X[S_i])$.*

Proof. We show that if S is any separator of X, then $G(X)$ is isomorphic to $G(X[S]) \times G(X[E-S])$ which implies the result. If either $|X[S]| = 1$ or $|X[E-S]| = 1$ then the result is trivial. Otherwise, let $u, v \in X$. If $A(u, v)$ contains S, resp. $E-S$, then u and v are adjacent if and only if $u[E-S]$ and $v[E-S]$, resp. $u[S]$ and $v[S]$, are adjacent in $G(X[E-S])$ resp. $G(X[S])$. If u and v disagree in coordinate positions of both S and $E-S$ then if w is the concatenation of $u[S]$ and $v[E-S]$ and x is the concatenation of $v[S]$ and $u[E-S]$ then u, v, w and x are all distinct and $0.5u + 0.5v = 0.5w + 0.5x$ so u and v are not adjacent. ∎

A graph G is *hamilton connected* (Berge [2]) if every pair of distinct nodes is joined by a hamilton path. There are two, admittedly trivial, hamilton connected graphs which do not contain hamilton cycles, namely K_1 and K_2, the complete graphs on one and two nodes. In all other cases, this property implies strong hamiltonicity (G is connected and every edge belongs to a hamilton cycle) which of course implies hamiltonicity. Holtzmann and Harary [9] use a property called *uniform hamiltonicity* which includes, in addition to the property of being strongly hamiltonian, the property that every edge does *not* belong to some hamilton cycle. It is clear that there exist graphs which are uniformly hamiltonian but not hamilton connected—consider the complete bipartite graph $K_{n,n}$ for $n \geqslant 3$. Recently, Adrian Bondy observed that if an eleventh node is joined to three nodes of a pentagon of the Petersen graph, which do not form a consecutive subsequence of the pentagon, then this graph is hamilton connected, but has an edge belonging to every hamilton cycle, so is not uniformly hamiltonian. Thus these two properties, though closely related, are independent.

In general, bipartite graphs present certain difficulties when dealing with hamiltonicity of polyhedra. For example, if they contain more than two nodes, then they are never hamilton connected, because if there is to be a hamilton path joining nodes in opposite parts, then the two parts must be of the same cardinality. In this case there cannot exist a hamilton path joining two nodes in the same part. Fortunately, these difficulties can be minimized for the case of combinatorial polyhedra. We will show (Theorem 2.9) that the only bipartite graphs of combinatorial polyhedra and hypercubes, and in view of (2.5), these are "almost" hamilton connected. Brualdi and Gibson [3] proved the foolowing lemma which, when combined with Lemma 2.1, will enable us to restrict our attention to nonseparable sets.

LEMMA 2.2. *Let $G_1, G_2, \ldots, G_k$ be hamilton connected graphs. If $|V(G_i)| \geqslant 3$ for some i, then $G = \prod_{i=1}^{k} G_i$ is hamilton connected. If $|V(G_i)| \leqslant 2$ for all $i = 1, 2, \ldots, k$, then G is a hypercube.*

Let $X \subseteq \{0, 1\}^E$ and let $e \in E$. We let

$$X_e^0 \equiv \{x \in X: x_e = 0\},$$
$$X_e^1 \equiv \{x \in X: x_e = 1\}.$$

This defines a partition of X and unless $\{e\}$ is a 1-valued component of X, this is a proper partition.

We can obtain the following variant of (2.4).

PROPOSITION 2.3. *Let $e \in E$ and let $x \in X_e^0$ and $w \in X_e^1$. If $D(x, w)$ is a minimal member of $\{D(x, u): u \in X_e^1\}$ then x and w are adjacent in $G(X)$.*

Proof. Suppose that x and w are nonadjacent. Then there exists λ satisfying $0 < \lambda < 1$ such that the point $\lambda x + (1-\lambda)w$ is a convex combination of members of $X - \{x, w\}$, all of which must agree with x and w on $A(x, w)$. At least one of these, say, $\bar{w}$, must belong to X_e^1 since the eth coordinate position of $\lambda x + (1-\lambda)w$ is positive. Therefore $D(x, \bar{w}) \subset D(x, w)$ since $\bar{w} \neq w$, a contradiction. ∎

We remark at this point, that in Proposition 2.3 and in fact, in all results concerning the sets X_e^0 and X_e^1, these two sets can be interchanged. In general, we will avoid belabouring this point.

An immediate consequence of Proposition 2.3 is that every member of X_e^0 is adjacent with at least one member of X_e^1, unless this latter set is empty, and of course, conversely. In fact, we have the following:

LEMMA 2.4. *Every $v \in X_e^0$ is adjacent to at least one $w \in X_e^1$ provided that this set is nonempty. Moreove if v is adjacent to exactly one $w \in X_e^1$ then $v[E-S] = w[E-S]$, where S is the set of coordinates of 1-valued components of X_e^1.*

Proof. If $|X_e^1| = 1$ then $S = E$ and the result is immediate. Otherwise, suppose $|X_e^1| > 1$ and $v[E-S] \neq w[E-S]$. Let $j \in E-S$ be such that $v_j \neq w_j$. If $x_j \neq v_j$ for all $x \in X_e^1$, then everything in X_e^1 is constant in the jth coordinate position so $j \in S$, a contradiction. Therefore there exists $\bar{w} \in X_e^1$ such that $\bar{w}_j = v_j$. Then $A(v, \bar{w}) \nsubseteq A(v, w)$ so if we let w^* be a member ofX_e^1 such that $A(v, w^*) \supseteq A(v, \bar{w})$ and $A(v, w^*)$ is maximal, we will have $w^* \neq w$ and by Proposition 2.3, w^* and v are adjacent. ∎

Finally, we observe that when we "split" X by means of an element e, the adjacencies within $G(X_e^0)$ and $G(X_e^1)$ are the same as within $G(X)$.

LEMMA 2.5. *Elements $v, w \in X_e^0$ are adjacent in $G(X_e^0)$ if and only if they are adjacent in $G(X)$.*

Proof. This is an immediate consequence of (2.3). ∎

A *universal node* of a graph is a node that is adjacent to every other node. We say that G is a *pyramid* if its contains a universal node. We now have the following corollary of Lemma 2.4.

COROLLARY 2.6. *If there exists* $e \in E$ *such that* $|X_e^0| = 1$ *or* $|X_e^1| = 1$ *then* $G(X)$ *is a pyramid.*

Moreover, it is easy to verify the following:

PROPOSITION 2.7. *If* v *is a universal node of* G *and* $G - v$ *is hamilton connected or a hypercube, then* G *is hamilton connected.*

In the following section we will show that many polyhedra of well-known combinatorial objects have a relatively simple "nonadjacency" situation. If vertices x and y are nonadjacent then there exist two different vertices u and v such that $1/2x + 1/2y = 1/2u + 1/2v$. (Often this amounts to the fact that when the vertices corresponding to two combinatorial "objects" are nonadjacent, there exist two different "objects" with the same union and intersection.) In other words, the midpoint of the line segment joining two nonadjacent vertices is the midpoint of the line joining two different vertices. This prompts the following definition: We say $X \subseteq \{0, 1\}^E$ is a *combinatorial set* if whenever x and y are nonadjacent on $G(X)$ there exists $u, v \in X - \{x, y\}$ such that $x + y = u + v$. In this case we say that $\mathrm{conv}(X)$ is a *combinatorial polyhedron* and $G(X)$ is a *combinatorial graph.* The following important lemma shows that if $|X| \geqslant 3$ and X is nonseparable, for a *combinatorial set* X, then $G(X)$ is nonbipartite. At present we do not know of any counterexamples to this assertion for noncombinatorial sets, and we conjecture that the result remains true, with this hypothesis removed.

LEMMA 2.8. *Let* $X \subseteq \{0, 1\}^E$ *be a combinatorial set. If* $|X| \geqslant 3$ *and* X *is nonseparable then* $G(X)$ *is nonbipartite.*

Proof. We prove by induction on $|X|$. If $|X| = 3$ then $G(X)$ is a triangle and the result is immediate. Suppose that it is true whenever $|X| < k$ and we have $|X| = k$. Choose $e \in E$. Since X is nonseparable. $X_e^0 \neq \varnothing \neq X_e^1$. If either X_e^0 or X_e^1 had a $\geqslant$ 3-valued component, then by induction the graph of this component would be nonbipartite so by Lemma 2.1, $G(X_e^0)$ or $G(X_e^1)$ would be nonbipartite. Then by Lemma 2.5, $G(X)$ would be nonbipartite. Therefore we can assume that both X_e^0 and X_e^1 consist of 1-valued and 2-valued components. If either $|X_e^0| = 1$ or $|X_e^1| = 1$ then, by Corollary 2.6, $G(X)$ is a pyramid and the result is immediate. Therefore we can assume that each of X_e^0 and X_e^1 contains at least one 2-valued component.

Since X is nonseparable, there exists $w \in X_e^0$, say, such that $w[E - \{e\}] \notin X_e^1[E - \{e\}]$. (Otherwise, $\{e\}$ would be a 2-valued component of X.) By Lemma 2.4 there exists $x \in X_e^1$ adjacent to w. Let S be a 2-valued component of X_e^1 and let $\bar{x}$ be the vector obtained from x by taking the other possibility for the coordinate positions indexed by S. It follows from (2.4) that x and $\bar{x}$ are adjacent. Suppose that $\bar{x}$ and w are not adjacent. Then, since X is combinatorial, there exist $u, v \in X - \{w, \bar{x}\}$ such that $w + \bar{x} = u + v$. Moreover, exactly one of u, v, say u, must belong to X_e^1. Let $\bar{u}$ be obtained from u by taking the other possibility for the coordinate positions in S. Then $w + x = \bar{u} + v$ and $\bar{u}$, $v \in X - \{w, x\}$ so w and x are not adjacent, a contradiction. Therefore $\bar{x}$ and w are adjacent and w, $\bar{x}$, x are the nodes of a triangle of $G(X)$ and the result follows. ∎

We can now obtain the following:

THEOREM 2.9. *If $G(X)$ is bipartite for a combinatorial set X, then $G(X)$ is a hypercube.*

Proof. If any component were $\geqslant$3-valued then, by Lemmas 2.8 and 2.1, $G(X)$ would be nonbipartite. Therefore every component is 1-valued or 2-valued and so by Lemma 2.1, $G(X)$ is the cartesian sum of a number of K_2's and K_1's. By Lemma 2.2, therefore, G is a hypercube. ∎

For hypercubes, there is a very simple adjacency criterion: If $G(X)$ is a hypercube, where X is a combinatorial set, then nodes $x, y \in X$ are adjacent in $G(X)$ if and only if $\{j \in E: x_j \neq y_j\}$ is a 2-valued component of X. Moreover, it follows immediately from Lemma 2.1 that $|X| = 2^n$, where n is the number of 2-valued components.

We now prove the main result of this paper.

THEOREM 2.10. *Let G be the graph of a combinatorial* 0–1 *polyhedron. Then G is either a hypercube or else is hamilton connected.*

Proof. Let $X \subseteq \{0, 1\}^E$ be a combinatorial set and let $G = G(X)$. We prove by induction on $|X|$, the number of nodes of G. If $|X| \leqslant 3$ then the result is immediate, so suppose $|X| = k \geqslant 4$ and that the result is true for all smaller values of $|X|$. We can assume that X has no 1-valued components (i.e., there are no $j \in E$ such that x_j is constant for all $x \in X$) as these can be eliminated without changing G.

If X is separable then it is easily verified that for each component S, $X[S]$ is a combinatorial set and $|X[S]| < |X|$. Therefore, using induction and Lemmas 2.1 and 2.2, G is hamilton connected or a hypercube depending on whether or not there exists a $\geqslant$3-valued component. Therefore, we assume that X is nonseparable. We will show that every pair of distinct nodes x, y is joined by a hamilton path. Let x and y be distinct and let $e \in E$ be such that $x_e \neq y_e$. Then $X_e^0 \neq \varnothing \neq X_e^1$. If $|X_e^0| = 1$ or $|X_e^1| = 1$ then G is a pyramid, by Corollary 2.6. By induction, whichever of X_e^1 or X_e^0 contains three or more elements must be hamilton connected or a hypercube. Therefore by Proposition 2.7, G is hamilton connected. So we assume $|X_e^0| \geqslant 2$, $|X_e^1| \geqslant 2$. Each of these sets contains one of x, y; suppose $x \in X_e^0$, $y \in X_e^1$. We now distinguish three cases:

Case 1. *Both* $G(X_e^0)$ *and* $G(X_e^1)$ *are hamilton connected.* If we can find $u \in X_e^0 - \{x\}$ and $v \in X_e^1 - \{y\}$ such that u and v are adjacent we are done, for we know there exist hamilton paths Π_0 in $G(X_e^0)$ from x to u and Π_1 in $G(X_e^1)$ from v to y. Then the concatenation of Π_0, the edge joining u, v, and Π_1 yields the desired hamilton path from x to y. (See Fig. 2.) So suppose that y is the only node of X_e^1 adjacent to a node of $X_e^0 - \{x\}$, and hence, x is the only node of X_e^0 adjacent to a node of $X_e^1 - \{y\}$. We will show that in this case

$$|X_e^0| = 2 \qquad \text{and} \qquad |X_e^1| = 2. \tag{2.6}$$

By Lemma 2.4, we must have $u[E-S] = y[E-S]$ for every $u \in X_e^0 - \{x\}$, where S is the set of coordinates of 1-valued components of X_e^1. Let $u \in X_e^0 - \{x\}$ and let $w \in X_e^1 - \{y\}$. Since u and w are nonadjacent, and X is combinatorial, there exist $\bar{u} \in X_e^0 - \{u\}$ and $\bar{w} \in X_e^1 - \{w\}$ such that $u + w = \bar{u} + \bar{w}$. Suppose $\bar{u} \neq x$. Then $\bar{u}[E-S] = y[E-S] = u[E-S]$, so $w[E-S] = \bar{w}[E-S]$. Since S was the set of coordinates of 1-valued components of X_e^1, $w[S] = \bar{w}[S]$ and therefore $w = \bar{w}$, a contradiction. Therefore $\bar{u} = x$, and similarly, $\bar{w} = y$. But that means for a fixed $u \in X_e^0 - \{x\}$, every $w \in X_e^1 - \{y\}$ must satisfy $w = x + y - u$. But this means $|X_e^1 - \{y\}| = 1$ and similarly $|X_e^0 - \{x\}| = 1$ so (2.6) is established.

But this leads immediately to a contradiction. Since, by hypothesis, $|X| \geqslant 4$ and X is nonseparable, it follows from Lemma 2.8 that $G(X)$ is nonbipartite. Therefore, if $X_e^0 = \{x, u\}$ and $X_e^1 = \{y, v\}$ we must have x and y adjacent. But since we assumed u and v to be nonadjacent, and X is combinatorial, we have $u + v = x + y$, contradicting x and y being adjacent.

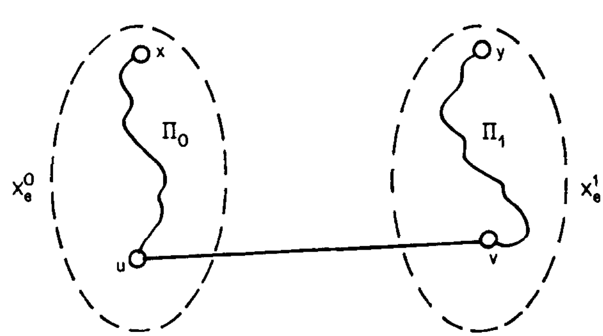

FIGURE 2

Thus we can always find adjacent $u \in X_e^0 - \{x\}$ and $v \in X_e^1 - \{y\}$ so the first case is complete.

Case 2. *Only one of* $G(X_e^0)$ *and* $G(X_e^1)$, *say*, $G(X_e^0)$, *is hamilton connected.* Then by induction, $G(X_e^1)$ is a hypercube of dimension at least 2, so $|X_e^1| \geqslant 4$. Let W be the part of $G(X_e^1)$ that does not contain y. If any $v \in W$ is adjacent to $u \in X_e^0 - \{x\}$, then, using (2.5) we can proceed exactly as in the previous case. Therefore, suppose that x is the only node of X_e^0 adjacent to any $v \in W$. Then, by Lemma 2.4, $v[E-S] = x[E-S]$ for all $v \in W$, where S is the set of coordinates of 1-valued components of X_e^0. Suppose there existed $v' \in X_e^1 - \{W\}$ such that $v'[E-S] \neq x[E-S]$. Since v' is adjacent to some member of W, one of the 2-valued components of X_e^1 must contain an element of $E - S$. Since $|X_e^1| \geqslant 4$ there is another 2-valued component and let $\bar{v}'$ be obtained from v' by changing the value on this component. Then $\bar{v}'$ and v' are adjacent, so $\bar{v}' \in W$. But $\bar{v}'[E-S] \neq x[E-S]$, a contradiction. Therefore $v[E-S] = x[E-S]$ for all $v \in X_e^1$. Now let $w \in X_e^0 - \{x\}$ and $v \in W$. By hypothesis they are not adjacent so since X is combinatorial, there exist $\bar{w} \in X_e^0 - \{w\}$ and $\bar{v} \in X_e^1 - \{v\}$ such that $v + w = \bar{v} + \bar{w}$. We have just seen that $\bar{v}[E-S] = v[E-S]$. But since S is the set of 1-valued components of X_e^0, $w[S] = \bar{w}[S]$ and therefore $v[S] = \bar{v}[S]$. But this means $v = \bar{v}$, a contradiction. Therefore there must exist adjacent $u \in X_e^0 - \{x\}$ and $v \in W$, so we are finished with this case.

Case 3. *Neither* $G(X_e^0)$ *nor* $G(X_e^1)$ *is hamilton connected.* Then $|X_e^0| \geqslant 4$, $|X_e^1| \geqslant 4$ and $G(X_e^0)$ and $G(X_e^1)$ are hypercubes, by induction. Let Y^0 and Z^0 be the parts of $G(X_e^0)$ and Y^1 and Z^1 be the parts of $G(X_e^1)$, where $x \in Y^0$, $y \in Z^1$. Our objective is to establish

(2.7) there exist an edge l of $G(X)$ joining a node u of Z^0 to a node v of Y^1.

For then the result will follow easily from (2.5) by concatenating a hamilton path in $G(X_e^0)$ from x to u, the edge l, and a hamilton path in $G(X_e^1)$ from v to y. Let S^i be the set of coordinates of 1-valued components of X_e^i for $i = 0, 1$. First we observe that if $S^0 \cup S^1 = E$ then every $u \in X_e^0$ is adjacent to every $v \in X_e^1$. For if such a u and v were not adjacent, since X is combinatorial, there would exist $\bar{u} \in X_e^0 - \{u\}$ and $\bar{v} \in X_e^1 - \{v\}$ such that $u + v = \bar{u} + \bar{v}$. But then $u[S^0] = \bar{u}[S^0]$ and consequently $v[S^0] = \bar{v}[S^0]$. But since we also have $v[S^1] = \bar{v}[S^1]$ we would have $v = \bar{v}$, a contradiction. Therefore $E - (S^0 \cup S^1) \neq \varnothing$.

Since $|X| \geqslant 3$ and X is nonseparable, if (2.7) is not satisfied there must exist $s \in Y^0$ and $t \in Z^1$ which are adjacent in $G(X)$, for otherwise $G(X)$ would be bipartite, contradictory to Lemma 2.8. Let $j \in E - (S^0 \cup S^1)$, let C_0 be the (2-valued) component of X_e^0 that contains j and let C_1 be the (2-valued) component of X_e^1 that contains j. Let $\bar{s}$ and $\bar{t}$ be obtained from s and

t by switching the values indexed by coordinates in C_0 and C_1, respectively. Then $\bar{s} \in Z^0$ and $\bar{t} \in Y^1$, since s and $\bar{s}$ are adjacent, as are t and $\bar{t}$. If $\bar{s}$ and $\bar{t}$ are nonadjacent, then there exist $u \in X_e^0 - \{\bar{s}\}$ and $v \in X_e^1 - \{\bar{t}\}$ such that $\bar{s} + \bar{t} = u + v$. Note that this implies $u_k \neq \bar{s}_k$ if and only if $v_k \neq \bar{t}_k$ for $k \in E$, so if we let K be the set of indices where $u_k \neq \bar{s}_k$ we have $K \subseteq E - (S_0 \cup S_1)$. Further, K must be the union of some set of 2-valued components of X_e^0 and of X_e^1. If $C_0 \not\subseteq K$, then $C_0 \cap K = \varnothing$ and $C_1 \cap K = \varnothing$ and so $u[C_0] = \bar{s}[C_0]$ and $v[C_1] = \bar{t}[C_1]$. Hence, if we let $\bar{u}$ and $\bar{v}$ be obtained from u and v by taking the opposite choice for the components indexed by C_0 and C_1, respectively, we have $\bar{u} + \bar{v} = s + t$ and $\bar{u} \neq s$, $\bar{v} \neq t$, contradicting the adjacency of s and t. Therefore, we must have $C_0 \subseteq K$ and $C_1 \subseteq K$.

Next, we observe that for any $h \in E$, if we have $\bar{s}_h = \bar{t}_h$, then we must have $u_h = v_h = \bar{s}_h = \bar{t}_h$ so $h \notin K$. Therefore, for each $k \in K$ we have $\bar{s}_k + \bar{t}_k = u_k + v_k = 1$. Since K is the union of 2-valued components of X_e^0, we can obtain u' from u by reversing the values coresponding to coordinates in these components and similarly obtain v' from v. Then $u' + v' = u + v$ and since $C_0, C_1 \subseteq K$, we have $u'[C_0] \neq \bar{s}[C_0]$ and $v'[C_1] \neq \bar{t}[C_1]$. Therefore $u' + v' = \bar{s} + \bar{t}$ and so, letting $\bar{u}'$ and $\bar{v}'$ be obtained from u' and v' by switching the values corresponding to coordinates in C_0 and C_1, respectively, we have $\bar{u}' + \bar{v}' = s + t$. But since s and t are adjacent, this means that $\bar{u}' = s$ and $\bar{v}' = t$. But then $K = C_0 = C_1$ and the two possible values for $u[K]$ for $u \in X_e^0$ and $v[K]$ for $v \in X_e^1$ are identical. Thus, K is a component of X, which contradicts the nonseparability of X. This final contradiction completes the proof of the theorem. ∎

3. Some Applications

In this section we show that the polyhedra of many well-known combinatorial problems are, in fact, "combinatorial polyhedra" as defined in the previous section. Therefore, it follows from Theorem 2.10 that their graphs are either hypercubes or are hamilton connected. In many of these cases, the proof that the polyhedra are combinatorial already appears in the literature, usually imbedded in the justification of an adjacency criterion. Generally we have included the proof that it is combinatorial, first for completeness and second to illustrate that often this is a very easily verified property.

3.1. *Matchings*

A *matching* of a graph G is a set M of edges such that every node of G is incident with at most one member of M. A matching M is *perfect* if every node is incident with exactly one member of M. The symmetric difference of two matchings M_1 and M_2 consists of a number of node disjoint even cycles and simple paths in general, or simply even cycles if M_1 and M_2 are perfect. The (*perfect*) *matching polyhedron* ($PM(G)$) $M(G)$ is the convex hull of the incidence vectors of the (perfect) matchings of G. Chvátal [6] proved that vertices v_1 and v_2 of $M(G)$ or $PM(G)$ are adjacent if and only if the symmetric difference of the corresponding matchings induces a connected subgraph of G.

THEOREM 3.1.1. *Let X be the incidence vectors of the perfect matchings of G. Then X is a combinatorial set.*

Proof. If v_1 and v_2 are nonadjacent then the symmetric difference of the corresponding perfect matchings M_1 and M_2 contains two disjoint alternating cycles C and $\bar{C}$. Let x_1 and x_2 be the incidence vectors of the perfect matchings $M_1 \Delta C$ and $M_2 \Delta C$. Then $x_1 + x_2 = v_1 + v_2$ and $\{x_1, x_2\} \cap \{v_1, v_2\} = \varnothing$ since x_1 disagrees with v_1 on C and with v_2 on $\bar{C}$ and since x_2 disagrees with v_2 on C and with v_1 on $\bar{C}$. ∎

COROLLARY 3.1.2. *The graph of $PM(G)$ is either a hypercube or hamilton connected.*

For the case of G bipartite, this corollary was proved by Brualdi and Gibson [3].

THEOREM 3.1.3. *Let X be the set of incidence vectors of all matchings of G. Then X is a combinatorial set.*

Proof. The proof is identical to that of Theorem 3.1.1, except that C and $\bar{C}$ can now be simple paths or alternating cycles. ∎

COROLLARY 3.1.4. *The graph of $M(G)$ is either a hypercube or is hamilton connected.*

3.2. *Stable Sets*

A *stable set* S of a graph G is a set of nodes such that no two are adjacent in G. The *stable set polyhedron* $S(G)$ is the convex hull of the set of incidence vectors of stable sets of G. Chvátal [6] showed that vertices v_1 and v_2 of $S(G)$ are adjacent if and only if the subgraph G' of G induced by the symmetric difference of the stable sets corresponding to v_1 and v_2 is connected.

THEOREM 3.2.1. *Let X be the set of incidence vectors of all stable sets of G. Then X is a combinatorial set.*

Proof. If v_1 and v_2 are nonadjacent then the symmetric difference of S_1 and S_2, the stable sets corresponding to v_1 and v_2, has at least two components. Let C be the nodeset of a component. Then the incidence vectors x_1, x_2 of $S_1 \Delta C$ and $S_2 \Delta C$ are easily seen to be the incidence vectors of stable sets distinct from S_1, S_2 and satisfying $x_1 + x_2 = v_1 + v_2$. ∎

COROLLARY 3.2.2. *The graph of $S(G)$ is either a hypercube or is hamilton connected.*

In fact, Chvátal proves his adjacence criterion for matchings by observing that vertices v_1 and v_2 of $M(G)$ are adjacent if and only of the corresponding vertices $\bar{v}_1$ and $\bar{v}_2$ of $S(L(G))$ are adjacent, where $L(G)$ is the line graph of G. Thus, in fact, we can view Theorem 3.1.3 as a corollary of Theorem 3.2.1.

3.3. *Matroids*

Let $M = (E, \mathscr{I})$ be a matroid where E is the underlying set and $\mathscr{I}$ is the family of independent sets. Let $\mathscr{B}$ be the set of bases (maximal independent sets) of M. Let $B(M)$ be the convex hull of the incidence vectors of the bases of M and let $I(M)$ be the convex hull of the incidence vectors of all members of $\mathscr{I}$. The graph $G(B(M))$, the so-called *matroid basis graph*, has been studied in the literature (see Maurer [10, 11]), as has the special case when M is the forest matroid of a graph (Cummings [7], Shank [12]). As mentioned in the Introduction, several results have been proven concerning the hamiltonicity of these graphs. The strongest, to our knowledge, is that of Holzmann and Harary [9] who show that $G(B(M))$ is *uniformly hamiltonian*, that is, for every edge j there exists a hamilton cycle containing j (provided that $G(B(M))$ has at least one cycle) and there exist another hamilton cycle not containing j (provided that $G(B(M))$ has at least two cycles).

Two vertices v_1, v_2 of $B(M)$ are adjacent, if and only if $|B_1 \Delta B_2| = 2$, where B_1 and B_2 are the two bases of M corresponding to v_1 and v_2. (It is difficult to know to whom this characterization should be attributed. It was known to Jack Edmonds in the early 1970s. It appears in print in Hausmann and Korte [8].)

THEOREM 3.3.1. *Let X be the set of incidence vectors of all bases of a matroid M. Then X is a combinatorial set.*

Proof (Hausmann and Korte [8]). Let v_1 and v_2 be nonadjacent vertices of $B(M)$ corresponding to bases B_1 and B_2. Then $|B_1 \Delta B_2| > 2$. Let $e \in B_2 - B_1$. By the matroid basis exchange axiom there exists $f \in B_1 - B_2$ such that $\bar{B}_1 \equiv B_1 \cup \{e\} - \{f\}$ and $\bar{B}_2 \equiv B_2 \cup \{f\} - \{e\}$ are bases of M. Then, if we let x_1 and x_2 be the incidence vectors of $\bar{B}_1$ and $\bar{B}_2$, respectively, we have $x_1 + x_2 = v_1 + v_2$. Moreover, since $|\bar{B}_1 \Delta B_1| = |\bar{B}_2 \Delta B_2| = 2$ and since $|\bar{B}_1 \Delta \bar{B}_2| \geqslant 2$, the vectors $\{x_1, x_2, v_1, v_2\}$ are pairwise different. ∎

COROLLARY 3.3.2. *The graph of $B(M)$ is either a hypercube or else is hamilton connected.*

This corollary is stronger than the "positive" half of the Holzmann-Harary theorem, in that if a graph with a cycle is either a hypercube or hamilton connected then every edge belongs to a hamilton cycle, but the converse is not true. However, it does not imply the "negative" part of this result.

To the best of our knowledge, Hausmann and Korte [8] were the first to consider the polyhedron $I(M)$, the convex hull of the incidence vectors of all independent sets on a matroid. They establish the following adjacency criterion: the vertices of $I(M)$ corresponding to independent sets I_1 and I_2 of a matroid M are adjacent if and only if

(i) $|I_1 \Delta I_2| = 1$ or

(ii) $|I_1 \Delta I_2| = 2$ and $I_1 \cup I_2 \notin \mathscr{I}$.

From our point of view, the interesting part is that they show that if v_1 and v_2 are nonadjacent, then there exist two other vertices x_1, x_2 such that $v_1 + v_2 = x_1 + x_2$. (See [8, p. 118, proof of Theorem 1.2] for details.) Thus they establish:

THEOREM 3.3.3. *Let X be the set of incidence vectors of independent sets of a matroid M. Then X is a combinatorial set.*

COROLLARY 3.3.4. *The graph of $I(M)$ is either a hypercube or is hamilton connected for a matroid M.*

3.4. *Permutation Polyhedra*

Let $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n)$ be a permutation of $\{1, 2, 3, ..., n\}$. Let E^σ be the $n \times n$ matrix with a 1 in the position (i, j) if i precedes j in σ and 0 otherwise. The *permutation polytope* Π_n is defined to be the convex hull of the set of matrices E^σ for all permutations σ. (This should not be confused with the *assignment polytope of order* n: the convex hull of all $n \times n$ permutation matrices. The assignment polytope is the special case of the perfect matching polyhedron, where G is a complete bipartite graph with n nodes on each side. Thus its membership in the class of combinatorial polyhedra and consequent hamiltonicity result follows from Theorem 3.1.1.)

Young [14] showed that the graph of Π_n is hamiltonian. In fact, as observed by Young, this result is well known in computer science, but in a different form. It follows directly from the fact that if permutations σ and τ

differ only by the interchange of an adjacent pair of elements, then the matrices E^σ and E^τ are easily seen to be adjacent on Π_n. Several algorithms are known for generating the complete set of permutations of an n element set, where each is obtained from the previous permutation by the transposition of an adjacent pair of elements.

When proving an adjacency criterion for Π_n, Young shows that if E^σ and E^τ are nonadjacent, then there exist different $E^{\sigma'}$ and $E^{\tau'}$ such that $E^\sigma + E^\tau = E^{\sigma'} + E^{\tau'}$. (See [14, p. 122-133].) Therefore we have:

THEOREM 3.4.1. *If X is the set of matrices E^σ for all permutations σ of an n element set, then X is a combinatorial set.*

We have the following corollary, which is slightly stronger than usual.

COROLLARY 3.4.2. *The graph of Π_n is hamilton connected.*

Proof. For $n = 1$ or 2 this is trivial. If $n \geqslant 3$ then $|\{E^\sigma\colon \sigma \text{ permutation of } \{1, 2,..., n\}\}| = n!$ cannot be a power of 2 so the graph cannot be a hypercube. The result follows from Theorem 2.10. ∎

4. CONCLUDING REMARKS

We have shown that it is easily verified that the polyhedra of many well-known combinatorial problems satisfy the two conditions required of a "combinatorial polyhedron": The vertices are 0–1 valued and the midpoint of the line joining any two nonadjacent vertices is the midpoint of the line joining another pair of vertices. The two main results proved in this paper are:

(1) If G is a bipartite combinatorial graph, then G is a hypercube (Theorem 2.9).

(2) If G is the graph of a combinatorial polyhedron, then G is a hypercube or is hamilton connected (Theorem 2.10).

The two alternatives of Theorem 2.10 are almost mutually exclusive; the only hamilton connected hypercubes are the graphs K_1 and K_2. It should be noted that a theorem such as Theorem 2.10 cannot be proved by simply considering the degree sequence of the graph of a combinatorial polyhedron and then applying a theorem of the form of Dirac. (See Berge [2].) This theorem and its subsequent strengthenings (Chvátal [5]) have the following general form. The nodes of a n-node graph are sorted by decreasing degree and then it is proved that if the sum of the degrees of the ith node and the $n - i$th node is at least n, then the graph is hamiltonian. For consider the case of Theorem 3.3.1 and Corollary 3.3.2. Let E be an m element set and let $\mathscr{B}$ be the set of all $k = \lfloor m/2 \rfloor$ element subsets of E. Then $\mathscr{B}$ is the family of bases of a (rather trivial) matroid on E and $|\mathscr{B}| = 2^{m-1}$. But for each $B \in \mathscr{B}$, the number of neighbours of the corresponding vertex of the matroid basis polyhedron is $k \cdot (m-k) \leqslant m^2/4$. Thus all degrees are constant and the ratio between this constant degree and the number of vertices of the polyhedron tends to zero as n tends to infinity.

At present, we know of no example of a 0–1 polyhedron which violates Theorem 2.9 or Theorem 2.10 with the "combinatorial" hypothesis removed. Certainly there exist non-hamiltonian polyhedral graphs (for example, Tutte's counter example to the Tait conjecture) but the problem seems to be that if such a graph is embedded in $\mathbb{R}^n$ in such a way that all the nodes have 0–1 coordinates, and so that all adjacencies are maintained, then we cannot avoid producing enough other adjacencies that the graph of the polyhedron becomes hamiltonian. Thus an outstanding open question is: To what extent can Theorems 2.9 and 2.10 be generalized?

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