On \( k \)-con-Critically \( n \)-Connected Graphs

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We prove that every \( n \)-connected graph \( G \) of sufficiently large order contains a connected graph \( H \) on four vertices such that \( G - V(H) \) is \((n - 3)\)-connected. This had been conjectured in Mader (High connectivity keeping sets in \( n \)-connected graphs, Combinatorica, to appear). Furthermore, we prove upper bounds for the order of all \( n \)-connected graphs of criticality 3, 4, and 5.

Key Words: critical connectivity; vulnerability of networks.

1. INTRODUCTION

It was proved in [6] that every \( n \)-connected finite graph \( G \) of sufficiently large order contains a vertex set \( S \) of prescribed cardinality such that \( G - S \) is \((n - 2)\)-connected. So the question arose, if this remains true, if we require the additional property that “\( S \) is connected”, i.e. that \( S \) spans a connected subgraph in \( G \). In [6], for every integer \( n \geq 18 \), infinitely many finite, \( n \)-connected graphs \( G \) and also infinite, \( n \)-connected graphs \( G \) were constructed, having the property that deleting any connected set of at least 3 vertices, the connectivity number decreases by at least 3. So the above-mentioned result does not remain true, if we add the condition “\( S \)-connected”, but a slightly modified result could hold. For the exact statement of this, let \(|G| := |V(G)|\) and let \( \kappa(G) \) denote the (vertex-) connectivity number of \( G \).

Conjecture 1.0. For all positive integers \( n \) and \( k \), there is a least non-negative integer \( h(n, k) \) such that every \( n \)-connected graph \( G \) with \(|G| > h(n, k)\) contains a connected set \( S \subseteq V(G) \) with \(|S| = k\) such that \( \kappa(G - S) \geq n - 3 \) holds.

Obviously, \( h(3, k) = k \) for \( k \geq 4 \) and \( h(3, k) = 0 \) for \( k \leq 3 \). The first non-trivial case of this conjecture, i.e. the existence of \( h(n, 4) \), is proved in this
paper. Whereas the above-mentioned examples show that at least for \( n \geq 18 \),
\( n - 3 \) would be best possible, stronger results might be true for small \( n \). So it
was conjectured in [7] that for every positive integer \( k \), every 3-connected
graph of sufficiently finite order contains a connected \( S \) with \( |S| = k \) such that
\( G - S \) is 2-connected. So far, this conjecture is verified only for \( k \leq 4 \) (see
[7, 3]).

Before we can state our results exactly, we need some concepts and
notation. The graphs considered here are undirected and do not have
multiple edges or loops. They may be finite or infinite, but the connectivity
number \( \kappa(G) \) is always finite. For the complete graph \( K_m \), we set \( \kappa(K_m) = m - 1 \) for every non-negative integer \( m \), and for non-complete graphs \( G \), we
have \( \kappa(G) = \min \{|T| : T \subseteq V(G) \text{ such that } G - T \text{ is disconnected}\} \). These
separating sets \( T \) with \( |T| = \kappa(G) \) are called smallest separating sets of \( G \), and
the set of all smallest separating sets of \( G \) is denoted by \( \mathcal{T}(G) \). Let \( \mathcal{G}(G) \) be
the set of all components of \( G \). For \( T \in \mathcal{T}(G) \), the union of at least one, but
not all \( C \in \mathcal{G}(G - T) \) is called a \( T \)-fragment of \( G \), or simply a fragment of \( G \).
If \( F \) is a \( T \)-fragment, then also \( \overline{F} := G - (F \cup T) \) is one. Of course, a
graph has a fragment, iff it is not complete. If the graph \( G \) has a finite
fragment, then a fragment of least vertex number is called an atom of \( G \). For
subgraphs \( F, H \subseteq G \) and \( S \subseteq V(G) \), let \( F \cap S := V(F) \cap S \), and \( F \cap H = \emptyset \)
means \( F \cap V(H) = \emptyset \). \( S \subseteq V(G) \) is called connected (in \( G \), if the subgraph
\( G(S) \) induced by \( S \) in \( G \) is connected. For \( H \subseteq G \), let \( G(H) := G(V(H)) \)
and \( N_G(H) := N_G(V(H)) := \{x \in G - V(H) : \text{there is a } y \in E(G) \} \), where \( x \in G \)
means \( x \in V(G) \) and \( [x, y] = [y, x] \) is the edge between \( x \) and \( y \). Furthermore, define \( \Delta(G) := \sup \{d_G(x) : x \in G\} \), where \( d_G(x) \) is the
degree of \( x \) in \( G \). In the notation \( N_G(X), d_G(x), \) etc. we suppress the
subscript, if the graph is clear from the context. Throughout, \( n \) and \( k \) are positive
integers. Let \( \mathbb{N}_m \) denote the set of positive integers \( i \leq m \) for a non-negative
integer \( m \); note \( \mathbb{N}_0 = \emptyset \).

A graph \( G \) is called \( W - (n,k) \)-critical or \( W - (n,k) \)-graph for \( W \subseteq V(G) \)
and positive integers \( n,k \), if \( W \cap F \neq \emptyset \) for every fragment \( F \) of \( G \) and
\( \kappa(G - W') = n - |W'| \) for every \( W' \subseteq W \) with \( |W'| \leq k \). A \( V(G) - (n,k) \)-critical
graph \( G \) is called \( (n,k) \)-critical or \( (n,k) \)-graph. For \( W' = \emptyset \), we get \( \kappa(G) = n \) for every \( W - (n,k) \)-graph \( G \). A complete \( W - (n,k) \)-graph is, therefore, isomorphic to \( K_{n+1} \), and \( K_{n+1} \) is \( W - (n,k) \)-critical
for every \( W \subseteq V(K_{n+1}) \) and every \( k \). If \( G \) is a non-complete \( W - (n,k) \)-
graph, then every \( W' \subseteq W \) with \( |W'| \leq k \) is contained in a \( T \in \mathcal{T}(G) \).
This implies \( |W| \geq k + 2 \), since \( F \cap W = \emptyset \) for all \( T \)-fragments \( F \), and hence
\( k \leq n \).

If we require \( W' \) to be connected in the definition of an \( (n,k) \)-graph, we
come to a concept which is of particular interest in our context. A graph \( G \) is
called \( k \)-con-critically \( n \)-connected, \( (n,k)_c \)-critical or \( (n,k)_c \)-graph for positive
integers \( n \) and \( k \), if \( \kappa(G - W) = n - |W| \) for every connected \( W \) in \( G \) with
Theorem 1.2 for the factor 6 was decreased to 4 in [2]. We get a slight improvement from it was shown in [6] that every \( n \) finite \( \left( n \right) \) graph \( (n, k) \)-critical for every \( k \), and \( k \leq n \) holds for every non-complete \( (n, k) \)-critical. Obviously, every \( (n, k) \)-graph is \( (n, k) \)-critical and every \( (n, 1) \)-graph is \( (n, 1) \)-critical. Every \( (n, k) \)-graph is \( (n, k') \)-critical for all \( 1 \leq k' \leq k \). If we will not specify the connectivity number of an \( (n, k) \)-critical graph or an \( (n, k) \)-critical graph, we speak of a \( k \)-critical graph or a \( k \)-con-critical graph, respectively.

Whereas in [4] it was proved that every \( (n, 3) \)-graph is finite and that for every \( n \), there is only a finite number of \( (n, 3) \)-graphs, for every \( n \geq 18 \), infinite \( (n, 3) \)-graphs and an infinite number of (non-isomorphic) finite \( (n, 3) \)-graphs have been constructed in [6]. On the other side, it was shown in [6] that every \( (n, 7) \)-graph is finite and that for every \( n \), there is only a finite number of \( (n, 7) \)-graphs, but it remained open what happens for \( k = 4, 5, 6 \). The closing of this gap is the main result of this paper.

**Theorem 1.1.** For every \( n \geq 4 \), \( g(n) := \max\{A(G) : G \text{ \( (n, 4) \)-critical graph}\} \) exists and is finite, and \( |G| \leq 2(n - 3)(g(n) - 1) + n \) holds for every \( (n, 4) \)-critical graph \( G \).

In particular, Theorem 1.1 says that \( h(n, 4) \) exists for all \( n \) and \( h(n, 4) \leq 2(n - 3)(g(n) - 1) + n \) for \( n \geq 4 \). (Obviously, \( h(3, 4) = 4 \) and \( h(n, 4) = 3 \) for \( n = 1, 2 \).

The next result generalizes and improves a result in [4].

**Theorem 1.2.** For every \( W - (n, 3) \)-critical graph with \( n \geq 2 \), \(|W| \leq (2n - 1)n \) holds.

It has been proved in [4] that \( |G| < 6n^2 \) holds for every \( (n, 3) \)-graph \( G \), and the factor 6 was decreased to 4 in [2]. We get a slight improvement from Theorem 1.2 for \( W = V(G) \).

**Corollary 1.3.** For every \( (n, 3) \)-graph \( G \not\cong K_2 \), \(|G| \leq (2n - 1)n \) holds.

The following upper bound for the order of an \( (n, 4) \)-graph follows easily from Theorems 1.1 and 1.2.

**Theorem 1.4.** For every \( (n, 4) \)-critical graph \( G \), \(|G| < 4n^3 \) holds.

For \( (n, 5) \)-critical graphs, we can improve this upper bound essentially.

**Theorem 1.5.** For every \( (n, 5) \)-critical graph \( G \) with \( n \geq 3 \), \(|G| \leq \frac{7}{8}(n - 1)n \) holds.
After some preliminary results in Section 2, we will prove Theorem 1.1 in Section 3. Modifying the proof of Theorem 1.1, we will prove Theorems 1.2–1.5 in Section 4. Section 5 gives examples of $k$-con-critical, but not $k$-critical graphs for every $k \geq 2$.

We add some further notations. A path $P : x_0, x_1, \ldots, x_n$ is called an $x_0, x_n$-path and the vertices $x \in P$ with $d_P(x) = 2$, i.e. $x_1, x_2, \ldots, x_{n-1}$, are the interior vertices of $P$. Two $x, y$-paths $P_1$ and $P_2$ are openly disjoint, if $P_1 \neq P_2$ and $P_1$ and $P_2$ have no interior vertex in common. For vertices $x \neq y$ in $G$, the maximal number of pairwise openly disjoint $x, y$-paths exists and is denoted by $\kappa(x, y; G)$. It is well known that $\kappa(G) = \min_{x \neq y} \kappa(x, y; G)$ holds for all graphs $G$ with $|G| \geq 2$ (see, for instance, [1, Chap. 3]). For $x \in G$ and $X \subseteq V(G - x)$, an $x, X$-fan of order $n$ consists of $x, x_i$-paths $P_i$ for $i \in \mathbb{N}_n$ with $P_i \cap X = \{x_i\}$ and $V(P_i) \cap V(P_j) = \{x\}$ for all $i \neq j$ from $\mathbb{N}_n$.

A fragment $F$ of a graph $G$ is called proper, if $|F| \leq |\bar{F}|$ in case $G$ finite and if $F$ is finite in case $G$ infinite. Let $\mathcal{F}(G)$ denotes the set of all proper fragments of $G$ and let $\mathcal{F}_m(G)$ consist of all maximal elements of $(\mathcal{F}(G), \subseteq)$. In a similar way, we call a fragment $F$ of a $W - (n, k)$-critical graph $G$ $W$-proper, if $|F \cap W| \leq |\bar{F} \cap W|$ in case $W$ finite and if $F \cap W$ is finite in case $W$ infinite. Then, $\mathcal{W}(G)$ denotes the set of all $W$-proper fragments of $G$ and again $\mathcal{W}_m(G)$ is the set of all maximal elements of $(\mathcal{W}(G), \subseteq)$.

2. PRELIMINARY RESULTS

In this section, we put together results which we need from previous papers, adapting them sometimes from $(n, k)$-critical graphs to $W - (n, k)$-critical ones. But also a few new results can be found.

**Lemma 2.1** (Cf. Mader [6, (1.1)]). Let $G$ be a graph of connectivity number $n$.

(a) Let $F_i$ be a $T_i$-fragment of $G$ for $i = 1, 2$ with $F_1 \cap F_2 \neq \emptyset$. Then, $|F_1 \cap T_2| \geq |\bar{F}_2 \cap T_1|$ holds. If also $\bar{F}_1 \cap \bar{F}_2 \neq \emptyset$, then even $|F_1 \cap T_2| = |\bar{F}_2 \cap T_1|$ holds and $F = \bar{F}_1 \cap \bar{F}_2$ is a $T$-fragment for $T := (T_2 - V(F_1)) \cup (T_1 \cap \bar{F}_2) = (T_1 - V(F_2)) \cup (T_2 \cap \bar{F}_1) = (T_1 \cap \bar{F}_2) \cup (T_1 \cap T_2) \cup (T_2 \cap \bar{F}_1)$ and $\bar{F} = G(F_1 \cup F_2)$.

(b) If $A$ is an atom of $G$ and if there is a $T \in \mathcal{F}(G)$ with $T \cap A \neq \emptyset$, then $V(A) \subseteq T$ and $|A| \leq n - \frac{|T \cap N(A)|}{2}$ hold.

We need also a further development of Lemma 2.1(a).

**Lemma 2.2.** Let $F_i$ be a $T_i$-fragment of the graph $G$ of finite connectivity number for $i = 0, 1, \ldots, k$. If $F := \bigcap_{i=0}^k F_i \neq \emptyset$ and $\bar{F}_0 \cap \bar{F}_j \neq \emptyset$ for $j \in \mathbb{N}_k$, then $F$ is a fragment of $G$ with $\bigcap_{i=0}^k T_i \subseteq N(F)$.
Proof. We use induction on $k$. The case $k = 1$ is contained in Lemma 2.1(a). Assume $k > 2$. Then $F_0' := F_0 \cap F_k$ is a $T_0'$-fragment with $T_0' \supseteq T_0 \cap T_k$ by induction hypothesis. Since $F_0' \supseteq \hat{F}_0$, also $F_0', F_1, \ldots, F_{k-1}$ satisfy the preassumptions, and the induction hypothesis completes the proof. 

Now we state some results from [6].

Lemma 2.3 (Mader [6, (3.9)]). Every $(n, 3)_c$-graph is locally finite.

Lemma 2.4 (Mader [6, (3.11)]). Let $z$ be a vertex of finite degree in the non-complete graph $G$ of connectivity number $n$. If $\kappa(G - \{z, x\}) = n - 2$ for every $x \in N_G(z)$, then there is a fragment $F$ of $G$ with $z \in N_G(F)$ and $|F| \leq \frac{n-1}{2}$.

Proposition 2.5 (Mader [6, (3.13)]). If $G$ is an $(n, k)_c$-graph with $k \geq 3$ and $k > \frac{n}{2}$, then $G$ is isomorphic to $K_{n+1}$.

Let us turn now to $W - (n, k)$-critical graphs. If $G$ is $(n, k)_c$-critical with $n, k \geq 2$ and $z \in G$, then, obviously, $G - z$ is $N_G(z) - (n - 1, k - 1)$-critical. So these graphs appear in a natural way. In Corollary 1 to Theorem 1 in [4], it was shown that there are no infinite $(n, 2)$-graphs. A corresponding result holds for $W - (n, 2)$-graphs.

Proposition 2.6. $W$ is finite for every $W - (n, 2)$-graph $G$.

A proof of this Proposition is easily drawn from the proof of Corollary 1 in [4].

Proposition 2.6 implies that for every fragment $F$ of a $W - (n, 2)$-graph $G$, we have $F \in W F(G)$ or $\hat{F} \in W F(G)$. Since there are infinite $(n, 1)$-graphs, Proposition 2.6 is not true for $k = 1$. Note that Proposition 2.6 does not say that every $W - (n, 2)$-graph is finite. This is not true as Proposition 3.10 in [6] shows, which says that there are infinite $(n, 3)_c$-graphs for every $n \geq 18$. But every infinite $(n, 3)_c$-graph $G$ delivers an infinite $N_G(z) - (n - 1, 2)$-graph $G - z$ for each $z \in G$ (by the way, so Proposition 2.6 implies Lemma 2.3). This does not change even for large $k$ as the following examples show.

Example 2.7. Let $G$ be a non-complete $W - (n, k)$-critical graph and let $S := \{S \subseteq W : |S| = k\}$. Suppose, we can assign to every $S \in S$ a $T_S \in F(G)$ with $T_S \supseteq S$ so that $V(G) - \bigcup_{S \in S} T_S$ is not empty. Then, we can blow up an $x \in G - \bigcup_{S \in S} T_S$ (hence $x \notin W$) to an infinite graph without destroying the $W - (n, k)$-criticality. Take, for instance, any $n$-connected graph $H$ with $H \cap G = \emptyset$ and add all edges (or $d_G(x)$ disjoint edges, if $|H| \geq d_G(x)$) between
$N_G(x)$ and $V(H)$ to $(G - x) \cup H$. Then, this new graph is also $W - (n,k)$-critical.

But $W - (n,k)$-graphs with the above property exist for every $k$. Let $k$ and an integer $m \geq k + 3$ be given. Let the graph $W_m(k+1)$ have the vertex set $\mathbb{N}_m^{k+1}$ of all $(k+1)$-tuples $(x^1,\ldots,x^{k+1})$ with $x^i \in \mathbb{N}_m$ for $i \in \mathbb{N}_{k+1}$ and let distinct vertices $(x^1,\ldots,x^{k+1})$ and $(y^1,\ldots,y^{k+1})$ be adjacent in $W_m(k+1)$ iff there is an $i \in \mathbb{N}_{k+1}$ with $x^i = y^i$. It was proved in [4] that $W_m(k+1)$ is regular of degree $m^{k+1} - (m - 1)^{k+1} - 1 =: w(m,k+1)$ and $(w(m,k+1),k+1)$-critical (cf. also Section 5). Let $z_i$ denote the vertex of $W_m(k+1)$ with all coordinates equal to $i$. Then, $G := W_m(k+1) - z_1$ is $N(z_1)-\{a\}$-critical, and we will show that it has the property of the last paragraph. For this, it is enough to prove that for every $S \subseteq N(z_1)$ with $|S| = k$, there is an $a_S \in N(z_1)$ with $N(a_S) \supseteq S$, but $z_m \notin N(a_S)$.

Let $x_i = (x^1_i,\ldots,x^{k+1}_i)$ for $i \in \mathbb{N}_k$ be distinct neighbours of $z_1$. Since $x_i \in N(z_1)$, there is a $a_i \in \mathbb{N}_{k+1}$ with $x^{a_i}_i = 1$. Denoting $N := \{i \in \mathbb{N}_k\}$, then $N' := \mathbb{N}_{k+1} - N$ and $N'$ both are not empty. For $i \in N'$, we define $a_i$ as the least element of $N_m := \{i \in \mathbb{N}_m\}$ with $a_i \not\in N$. Since $m \geq k + 3$, we have $a_i < m$. Defining $a' := a_i$ for $i \in N$, then $a := (a_1, a_2, \ldots, a^{k+1})$ is a common neighbour of $z_1, x_1, \ldots, x_k$, but $z_m \notin N(a)$.

These examples also show that for every $k$, a $W - (n,k)$-graph may have vertices of infinite degree, even in $W$.

In regard to this example, it will be important to know if a $W - (n,k)$-graph has finite fragments. For finite $W$, this follows by standard arguments.

**Lemma 2.8.** Every non-complete $W - (n,k)$-graph with finite $W$ has finite fragments.

**Proof.** Let $G$ be a non-complete $W - (n,k)$-graph with finite $W$. Since $G$ is non-complete and $W$ is finite, there is a $T$-fragment $F$ of $G$ such that $|F \cap W|$ is minimal. By definition of $W - (n,k)$-critical, there is a $w \in F \cap W$ and an $S \in \mathcal{T}(G)$ with $w \in S$. Since there is no fragment $F'$ of $G$ contained in $F - w$ by minimality of $|F \cap W|$, we can have neither

(i) $F \cap C \neq \emptyset$ and $\overline{F} \cap \overline{C} \neq \emptyset$ nor

(ii) $F \cap \overline{C} \neq \emptyset$ and $\overline{F} \cap C \neq \emptyset$

for an $S$-fragment $C$ of $G$ by Lemma 2.1(a). Hence, both in (i) and in (ii) at least one of the intersections in empty. But this implies that one of $V(F)$, $V(\overline{F})$ is contained in $S$ or one of $V(C), V(\overline{C})$ in $T$. Hence, at least one of $F, \overline{F}, C, \overline{C}$ is finite.

Lemma 2.8 is not true for infinite $W$ as simple $(n,1)$-critical graphs show. But Proposition 2.6 and Lemma 2.8 imply that every non-complete
$W - (n,2)$-graph has atoms. Now, we can prove an important tool for getting an upper bound for $W$, corresponding to Lemma 2 in [4].

**Lemma 2.9** (Mader [4, Lemma 2]). Let $G$ be a $W - (n,3)$-critical graph and let $F_0$ be a $T_0$-fragment of $G$. Assume that there are $c \in F_0 \cap W$ and $c' \in \partial F_0 \cap W$ so that $F \cap F_0 \neq \emptyset$ for every fragment $F$ of $G$ with $N(F) \supseteq \{c, c'\}$. Then $|\partial F_0| \leq \frac{3}{4} n - 1$ holds.

The proof of Lemma 2.9 can be adapted from Lemma 2 in [4] with the only modification that we have to use Proposition 2.6 and Lemma 2.8 for the existence of an atom of $G - \{c, c'\}$.

### 3. Proof of the Main Result

The proof of Theorem 1.1 is given in a series of lemmata.

**Lemma 3.1.** $\Delta(G) \leq \frac{2(3n-7)/2^{(n+1)/2 - 2}}{3n - 9}$ for every $(n, 4)_c$-critical graph $G$ with $n \geq 4$.

**Proof.** We may assume $G \not\cong K_{n+1}$, hence $n \geq 8$ by Proposition 2.5. Consider $x \in G$ and choose a $y \in N(x)$ with $d(y) = \min \{d(z) : z \in N(x)\}$. Consider $z \in N(x) - \{y\}$. Then, $\kappa(G - \{x, y, z, z'\}) = n - 4$ for every $z' \in N(x) - \{y, z\}$. Since $G$ is $(n, 4)_c$-critical, therefore, $G' := G - \{y, z\}$ has $\kappa(G') = n - 2$ and for each $z' \in N(x) - \{y, z\}$ there is a $T' \in \mathcal{T}(G')$ with $\{x, z'\} \subseteq T'$. Hence, there is a fragment $F$ of $G'$ with $|F| \leq \frac{n-3}{2}$ and $x \in N_G(F)$ by lemmata 2.3 and 2.4. Since there is a $z'' \in N_G(x) \cap F$, we get $d_G(y) \leq d_G(z'') \leq |F| - 1 + n \leq \frac{3n - 5}{2}$. Since $\{y, z\} \subseteq N_G(F)$, we have proved, that for every $z \in N(x) - \{y\}$, there is a $y, z$-path (through $F$) of length at most $\frac{n-1}{2}$ in $G - x$, where all interior vertices have degree at most $\frac{3n - 5}{2}$ in $G$. So we get $d_G(x) \leq d_G(y) + (d_G(y) - 1) \frac{3n - 7}{2} + \cdots + (d_G(y) - 1) \frac{3n - 5}{2} \leq \frac{(3n - 7)/2^{(n+1)/2 - 1}}{(3n - 9)/2}$. 

Hence, by this lemma, we can define an integer valued function $g(n) := \max \{\Delta(G) : G \ (n, 4)_c$-graph\} for all $n$ and have $n \leq g(n) \leq \frac{2(3n-7)/2^{(n+1)/2 - 2}}{3n - 9}$ for $n \geq 4$ and, obviously, $g(n) = n$ for $n \leq 3$.

**Remark 3.2.** In Section 4, we will even show $g(n) \leq (2n - 3)(n - 1)$ for $n \geq 4$.

**Lemma 3.3.** Let $G$ be an $(n, 4)_c$-critical graph and assume $F$ is a $T$-fragment of $G$ with $|F| > (n - 3)(\Delta(G) - 1)$. Denoting $\tilde{T} := \{t \in T : |N(t) \cap \bar{F}| \leq n - 3\}$, then $|N(x) \cap \tilde{T}| \geq 3$ for all $x \in \bar{F}$.
Proof. Suppose $U := \{x \in \tilde{F} : |N(x) \cap \tilde{T}| \leq 2\} \neq \emptyset$. Then $|\tilde{F}| \geq n - 2$ holds. If there is a $u \in U$ with $|N(u) \cap \tilde{T}| = 2$, say, $\{t, v\} = N(u) \cap \tilde{T}$, let $P$ be the path $v, u, t$ of length 2. If there is a $u \in U$ with $|N(u) \cap \tilde{T}| = 1$, say, $t \in N(u) \cap \tilde{T}$, then enlarge $u$, $[u, t]$, $t$ by an edge $[u, v]$ or $[t, v]$ with $v \in \tilde{F}$ to a path $P$ of length 2 in $G(\tilde{F} \cup T)$. If $N(u) \cap \tilde{T} = \emptyset$ for all $u \in U$, we can find a path $P$ of length 2 in $G(\tilde{F} \cup T)$ with $P \cup U \neq \emptyset$ and $P \cap T \neq \emptyset$, say, again $V(P) = \{u, t, v\}$ with $u \in U$ and $t \in T$.

For every $x \in F$, there is an $x, T$-fan $F_x$ of order $n$ in $G$, hence in $G - V(\tilde{F})$. Let $e_x$ denote the edge of $F_x$ incident to $t$. Since $|N(t) \cap \tilde{T}| \leq |\tilde{T}| - 1$ and $|\tilde{F}| \geq (n - 3)(|\tilde{T}| - 1)$, there are vertices $x_1, \ldots, x_{n-2}$ in $F$ with $e_{x_i} = e_{x_2} = \cdots = e_{x_{n-2}}$, say, $e_{x_i} = [t, t']$.

Since $G$ is $(n, 4)_c$-critical, there is a $t' \in T(G)$ with $\{u, t, v, t'\} \subseteq T'$. Since $\{u, t, v\} \cap F = \emptyset$, there is a $C \in \mathcal{C}(G - T')$ with $C \cap \{x_1, \ldots, x_{n-2}\} \neq \emptyset$, say, $x_1 \in C$. Since the $x_1, T$-fan $F_{x_1}$ contains $[t, t']$, we have $\kappa(x_1, y; G - \{t, t'\}) \geq n - 1$ for every $y \in \tilde{F}$, since we can combine $F_{x_1}$ with a $y, T$-fan of order $n$ to get a system of $n$ openly disjoint $x_1, y$-paths. So $\{t, t'\} \subseteq T'$ and $\kappa(x_1, y; G - \{t, t'\}) \geq n - 1$ imply $|T - V(C)| \cup (T' \cap F) > n$, hence $|T' \cap F| > |T \cap C| \geq 1$. So $|T' \cap \tilde{F}| \leq n - 3$ and by Lemma 2.1(a), $\tilde{C} \cap \tilde{F} = \emptyset$. But this implies $|N(s) \cap \tilde{F}| \leq n - 3$ for all $s \in T \cap \tilde{C}$, hence $T \cap \tilde{C} \subseteq \tilde{T}$. On the other side, $N(u) \cap \tilde{C} \neq \emptyset$, hence $N(u) \cap (\tilde{T} - T') \neq \emptyset$, in opposite to the choice of $u$, $v$, and $t$.

**Corollary 3.4.** If $F$ is a fragment of an $(n, 4)_c$-graph $G$, then $|F| \leq (n - 3)(|\tilde{T}| - 1)$ or $|\tilde{F}| \leq \frac{(n-3)n}{3}$ holds.

**Proof.** If $|F| > (n - 3)(|\tilde{T}| - 1)$, then $|N(x) \cap \tilde{T}| \geq 3$ for all $x \in \tilde{F}$ by Lemma 3.3, hence $|\tilde{F}| \leq \frac{(n-3)n}{3}$.

**Corollary 3.5.** The proper fragments of an $(n, 4)_c$-graph $G$ with $|G| > 2(n - 3)(|\tilde{T}| - 1) + n$ have the following properties:

(a) $|F| \leq \frac{(n-3)n}{3}$ for every $F \in \mathcal{F}(G)$.

(b) If $F_1 \cap F_2 \neq \emptyset$ for $F_1, F_2 \in \mathcal{F}(G)$, then also $G(F_1 \cup F_2) \in \mathcal{F}(G)$.

(c) For all $F_1 \neq F_2$ in $\mathcal{F}_m(G)$, $F_1 \cap F_2 = \emptyset$. Every $F \in \mathcal{F}(G)$ is contained in exactly one element of $\mathcal{F}_m(G)$.

**Proof.** For $F \in \mathcal{F}(G)$ we have $|\tilde{F}| \geq \frac{|G|-n}{2} > (n - 3)(|\tilde{T}| - 1)$, and (a) follows from Corollary 3.4.

Assume $F_1 \cap F_2 \neq \emptyset$ for a $T_1$-fragment $F_1 \in \mathcal{F}(G)$ and a $T_2$-fragment $F_2 \in \mathcal{F}(G)$. We may assume $F_1 \not\subseteq F_2$ and $F_2 \not\subseteq F_1$, hence $T_1 = T_2$ or $T_1 \cap F_2 \neq \emptyset$ and $T_2 \cap F_1 \neq \emptyset$. Therefore, by (a), $|F_1 \cup F_2 \cup T_1 \cup T_2| < (n - 3)n + \cdots$
2(n - 1) \leq 2(n - 3)(\Delta(G) - 1) + n < |G|, hence \( G - (V(F_1) \cup V(F_2) \cup T_1 \cup T_2) \neq \emptyset \). Then Lemma 2.1(a) implies that \( F := G(F_1 \cup F_2) \) is a fragment of \( G \).

Suppose \( F \) is not proper. Then, \( \bar{F} \) is proper and so \(|\bar{F}| \leq \frac{(n-3)n}{3} \) by (a), hence \(|\bar{G}| < \frac{3(n-3)n}{3} + n \), a contradiction for \( n \geq 4 \). This proves (b), and (c) follows immediately from (b), since every \( F \in \mathcal{F}(G) \) is contained in at least one \( \bar{F} \in \mathcal{F}_m(G) \) by (a).

We turn now to the proof of Theorem 1.1. In the following, we will show that there is no non-complete \((n,4)_c\)-graph \( G \) of order exceeding 

\[
2(n - 3)(\Delta(G) - 1) + n \quad \text{which proves Theorem 1.1, using Lemma 3.1. So for every } \quad \text{the } T-\text{fragment } \mathcal{F}(G) \text{ has the property } (F \cup T) \cap S_0 \neq \emptyset \quad \text{and } (F \cup T) \cap F_0 \neq \emptyset, \text{ then } F \cap \bar{F}_0 \neq \emptyset, F \cap F_0 = \emptyset, \quad \text{and } |T \cap F_0| = |T_0 \cap F|. \text{ Furthermore, } |F \cap \bar{F}_0| \geq |T \cap S_0| \text{ holds.}
\]

Proof. Since \( V(F_0) \subseteq N(T_0) \) by Lemma 3.3, we have \(|\bar{F}_0 - N(T_0)| = |G| - |T_0 \cup N(T_0)| > |G| - n - n\Delta(G) \geq (n - 6)\Delta(G) + 1 - 2(n - 3) \geq n - 7| \) \((n - 1) \geq n - 1\). Hence \(|S_0| \geq n - 1 \) or there is an \( x \in \bar{F}_0 - (N(T_0) \cup S_0) \). In the latter case, every path of an \( x, T_0 \)-fan of order \( n \) in \( G \) has an interior vertex in \( S_0 \). Hence, in this case even \(|S_0| \geq n \).

(\beta) If the \( T \)-fragment \( F \in \mathcal{F}(G) \) has the property \((F \cup T) \cap S_0 \neq \emptyset \) and \((F \cup T) \cap F_0 \neq \emptyset \), then \( F \cap \bar{F}_0 \neq \emptyset, F \cap F_0 = \emptyset, \bar{F} \cap F_0 \neq \emptyset \), and \(|T \cap F_0| = |T_0 \cap F| \). Furthermore, \(|F \cap \bar{F}_0| \geq |T \cap S_0| \text{ holds.}
\]

Proof. Since \( S_0 \cup N(S_0) \subseteq V(\bar{F}_0) \) by definition of \( S_0 \), \( F \cap \bar{F}_0 \neq \emptyset \) follows. Since \( F \cap F_0 = \emptyset \) by Corollary 3.5(b) and maximality of \( F_0 \), we get \( 1 \leq |T \cap F_0| \leq |T_0 \cap F| \) from assumption and Lemma 2.1(a). Since \( |T_0 \cap F| < |F| \) and \( F_0 \) is of largest order in \( \mathcal{F}(G) \), we conclude \( F_0 \cap \bar{F} \neq \emptyset \). Then \( |T \cap F_0| = |T_0 \cap F| \) follows by Lemma 2.1(a). Since \( F \cap T_0 \neq \emptyset \) and \( N(S_0) \cap T_0 = \emptyset, (T - S_0) \cup (F \cap N(S_0 \cap T)) \) separates \( G \). This implies \( |T \cap S_0| \leq |F \cap N(S_0 \cap T)| \leq |F \cap \bar{F}_0| \).

For every \( s \in S_0 \), there is an \( s, T_0 \)-path \( P_s \) of length 2; let \( t_s \) denote the vertex of \( P_s \cap T_0 \). Since \( G \) is \((n,4,4)_c\)-critical, for every \( x \in N(t_s) \cap F_0 \), there is an \( F \in \mathcal{F}(G) \) with \( N(F) \supseteq V(P_s) \cup \{x\} \). For every \( s \in S_0 \), let \( \mathcal{F}^s \) denote the set of maximal elements of \( \{F \in \mathcal{F}(G) : N(F) \supseteq V(P_s) \text{ and } N(F) \cap N(t_s) \cap F_0 \neq \emptyset\} \). Then for every \( s \in S_0 \), \( \mathcal{F}^s \) has the following properties.

(\gamma) \( F_1 \cap F_2 = \emptyset \) for all \( F_1 \neq F_2 \) from \( \mathcal{F}^s \), \( 2 \leq |\mathcal{F}^s| < \infty \), and \( \bigcup_{F \in \mathcal{F}^s} N(F), \supseteq V(F_0) \).
Proof. If $F_1 \cap F_2 \neq \emptyset$ for $F_1 \neq F_2$ from $\mathcal{F}^s$, then $F := G(F_1 \cup F_2) \in \mathcal{F}(G)$ by Corollary 3.5(b) and obviously $V(P_x) \subseteq N(F)$. Since $N(F_1) \cup N(F_2) \subseteq V(F) \cup N(F)$ and hence $F \cap F_0 = \emptyset$ by ($\beta$), we get $N(F) \cap N(t) \cap F_0 \neq \emptyset$, a contradiction to the maximality of $F_1$ and $F_2$. Hence $F_1 \cap F_2 = \emptyset$. Since $F \cap T_0 \neq \emptyset$ for all $F \in \mathcal{F}^s$, this implies $\mathcal{F}^s$ finite.

Every $F \in \mathcal{F}(G)$ with $N(F) \supseteq V(P_x)$ and $N(F) \cap N(t) \cap F_0 \neq \emptyset$ is contained in an $F' \in \mathcal{F}^s$. Since $F' \cap F_0 = \emptyset$ by ($\beta$), we have $N(F) \cap N(t) \cap F_0 \subseteq N(F')$. This implies $N(t) \cap F_0 \subseteq \bigcup_{F \in \mathcal{F}^s} N(F)$. By ($\beta$), we have $F \cap F_0 \neq \emptyset$ for all $F \in \mathcal{F}^s$. If $F_0 \cap \bigcap_{F \in \mathcal{F}^s} F \neq \emptyset$, we could apply Lemma 2.2 to $F_0$ and $F$ for $F \in \mathcal{F}^s$ and would get a fragment $F' := F_0 \cap \bigcap_{F \in \mathcal{F}^s} F$ of $G$ with $t \in N(F')$, a contraction to $N(t) \cap F_0 \subseteq \bigcup_{F \in \mathcal{F}^s} N(F)$. Hence $F_0 \cap \bigcap_{F \in \mathcal{F}^s} F = \emptyset$, hence $V(F_0) \subseteq \bigcup_{F \in \mathcal{F}^s} N(F)$, since $F_0 \cap F = \emptyset$ for $F \in \mathcal{F}^s$ by ($\beta$). Since $F \cap F_0 \neq \emptyset$ for $F \in \mathcal{F}^s$ by ($\beta$), $V(F_0) \subseteq \bigcup_{F \in \mathcal{F}^s} N(F)$ implies $|F'| \geq 2$.

From ($\gamma$), we easily deduce an improvement of Corollary 3.5(a).

($\delta$) $|F_0| \leq n$.

Proof. By ($\beta$), we have $|N(F) \cap F_0| = |F \cap T_0|$ for all $F \in \mathcal{F}^s$. Then, ($\gamma$) implies $|F_0| \leq \sum_{F \in \mathcal{F}^s} |N(F) \cap F_0| = \sum_{F \in \mathcal{F}^s} |F \cap T_0| \leq |T_0| = n$.

Every $F \in \bigcup_{s \in S_0} \mathcal{F}^s$ is contained in exactly one element of $\mathcal{F}_m$ by Corollary 3.5(c). We consider now this set $\mathcal{F}_0 := \{F \in \mathcal{F}_m : \text{there is an } s \in S_0 \text{ and an } F_s \in \mathcal{F}^s \text{ with } F_s \subseteq F\}$. By definition, for every $F \in \mathcal{F}_0$ there is an $s \in S_0$ with $V(P_s) \subseteq V(F) \cup N(F)$, and every $V(P_s)$ is contained in $V(F) \cup N(F)$ for an $F \in \mathcal{F}_0$. So $F \neq F_0$, hence $F \cap F_0 = \emptyset$ for all $F \in \mathcal{F}_0$ by Corollary 3.5(c). Since $F \cap T_0 \neq \emptyset$ for all $F \in \bigcup_{s \in S_0} \mathcal{F}^s$, $F \cap T_0 \neq \emptyset$ for all $F \in \mathcal{F}_0$, too, hence $|\mathcal{F}_0| \leq n$, since $F \cap F' = \emptyset$ for $F \neq F'$ from $\mathcal{F}_0$ by Corollary 3.5(c). If $F' \subseteq F$ for $F' \in \mathcal{F}^s$ and $F \in \mathcal{F}_0$, then $\emptyset \neq N(F') \cap F_0 \subseteq N(F)$, since $F \cap F_0 = \emptyset$. So every $F \in \mathcal{F}_0$ satisfies the preassumptions of ($\beta$) and $\bigcup_{F \in \mathcal{F}_0} N(F) \supseteq V(F_0)$ by ($\gamma$). We collect some properties of $\mathcal{F}_0$.

($\varepsilon$) If $F' \subseteq F$ for $F' \in \bigcup_{s \in S_0} \mathcal{F}^s$ and $F \in \mathcal{F}_0$, then $N(F') \cap F_0 \subseteq N(F)$ holds. Furthermore, we have $|\mathcal{F}_0| \geq 2$, $\bigcup_{F \in \mathcal{F}_0} N(F) \supseteq V(F_0)$, and for every $s \in S_0$ there is an $F \in \mathcal{F}_0$ with $s \in N(F)$.

Proof. It remains only to show $|\mathcal{F}_0| \geq 2$ and the last claim. Since $\mathcal{F} \cap F_0 \neq \emptyset$ for $F \in \mathcal{F}_0$ by ($\beta$), but $V(F_0) \subseteq \bigcup_{F \in \mathcal{F}_0} N(F)$, we must have $|\mathcal{F}_0| \geq 2$.

Consider $s \in S_0$ and $F'_1 \in \mathcal{F}^s$. There is an $F_1 \in \mathcal{F}_0$ with $F_1 \supseteq F'_1$, hence $N(F_1) \cap F_0 \supseteq N(F'_1) \cap F_0$ by the first claim of ($\varepsilon$). Since $N(F_1) \supseteq V(F_0)$ by ($\beta$), but $\bigcup_{F \in \mathcal{F}_0} N(F) \supseteq V(F_0) \subseteq \bigcup_{F \in \mathcal{F}_0} N(F)$ by the proved part of ($\varepsilon$) and ($\gamma$), there is an $F'_2 \in \mathcal{F}^s$ with $N(F'_2) \cap (V(F_0) - N(F_1)) \neq \emptyset$. Let $F_2$ denote the
element of $\mathcal{F}_0$ containing $F_1^i$. Then $N(F_1^i) \cap F_0 \subseteq N(F_2)$, hence $F_1 \neq F_2$ holds. 
Since $s \in N(F_1^i)$, hence $s \in F_1 \cup N(F_1)$ for $i = 1, 2$, but $F_1 \cap F_2 = \emptyset$ by Corollary 3.5(c), $s \in N(F_1)$ or $s \in N(F_2)$ follows. 

$(\zeta)$ If $N(F_1) \cap N(F_2) \cap F_0 \neq \emptyset$ holds for the distinct fragments $F_1, F_2 \in \mathcal{F}_0$, then $N(F_1) \cup N(F_2) \supseteq V(F_0)$. 

Proof. By (b), we have $|N(F_i) \cap F_0| = |T_0 \cap F_i|$ for $i = 1, 2$ and $F_1 \cap T_0$, $F_2 \cap T_0$ are disjoint by Corollary 3.5(c). If $V(F_0) - (N(F_1) \cup N(F_2)) \neq \emptyset$, then $T' := (T_0 - \sum_{i=1}^2 |T_0 \cap F_i| + \sum_{i=1}^2 |N(F_i) \cap F_0| = |T_0| = n$, since $N(F_1) \cap N(F_2) \cap F_0 \neq \emptyset$. This contradiction proves (\zeta).

We distinguish now two cases.

Case 1: There are $F_1 \neq F_2$ in $\mathcal{F}_0$ with $N(F_1) \cap N(F_2) \cap F_0 \neq \emptyset$.

Then $V(F_0) \subseteq N(F_1) \cup N(F_2)$ by (\zeta). Consider $F \in \mathcal{F}_0 - \{F_1, F_2\}$, if there is any. Then $N(F) \cap N(F_i) \cap F_0 \neq \emptyset$ for $i = 1$ or $i = 2$, say, for $i = 1$. Hence, $V(F_0) \subseteq N(F) \cup N(F_1)$ again by (\zeta). Since $N(F_2) \supseteq V(F_0) - N(F_1) \neq \emptyset$ by (b), this implies $N(F) \cap N(F_2) \cap F_0 \neq \emptyset$. If we apply this conclusion to the pair $F, F_1$, we recognize that $N(F) \cap N(F') \cap F_0 \neq \emptyset$ for all $F \neq F'$ from $\mathcal{F}_0$. Then $V(F_0) \subseteq N(F) \cup N(F')$ for all $F \neq F'$ from $\mathcal{F}_0$ by (\zeta).

We define $d_F := |N(F) \cap F_0|$ and $t_F := |N(F) \cap S_0|$ for $F \in \mathcal{F}_0$. Then, by the two last claims of (b), we have

(i) $d_F + t_F \leq |F|$ for every $F \in \mathcal{F}_0$.

Since $V(F_0) \subseteq N(F) \cup N(F')$ and $N(F) \cap N(F') \cap F_0 \neq \emptyset$ for all distinct $F, F' \in \mathcal{F}_0$, we conclude

(ii) $|F_0| < d_F + t_F$ for all $F \neq F'$ from $\mathcal{F}_0$.

Since for every $s \in S_0$ there is an $F \in \mathcal{F}_0$ with $s \in N(F)$ by (c), we get

(iii) $\sum_{F \in \mathcal{F}_0} t_F \geq |S_0|$.

Since $d_F = |F \cap T_0|$ by (b) and $F \cap F' = \emptyset$ for $F \neq F'$ from $\mathcal{F}_0$, we have

(iv) $\sum_{F \in \mathcal{F}_0} d_F \leq |T_0| = n$.

Since $|F| \leq |F_0|$ for $F \in \mathcal{F}_0$ by definition of $F_0$, (i) and (ii) imply

$$\sum_{F \in \mathcal{F}_0} (d_F + t_F) \leq |\mathcal{F}_0||F_0| \leq 2 \sum_{F \in \mathcal{F}_0} d_F - |\mathcal{F}_0|,$$

hence, $\sum_{F \in \mathcal{F}_0} t_F \leq \sum_{F \in \mathcal{F}_0} d_F - |\mathcal{F}_0|$. (Remember, $\mathcal{F}_0$ finite) Using (iii) and (iv), this implies $|S_0| \leq n - |\mathcal{F}_0|$, a contradiction to (z) and $|\mathcal{F}_0| \geq 2$ by (c). So case 1 cannot occur.

Case 2: $N(F_1) \cap N(F_2) \cap F_0 = \emptyset$ for all $F_1 \neq F_2$ from $\mathcal{F}_0$.

Then $N(F) \cap F_0$ ($F \in \mathcal{F}_0$) form a partition of $V(F_0)$.
Choose any $T$-fragment $F \in \mathcal{F}_0$ and $s \in S_0$. Then there is an $x \in T \cap F_0$. There is an $F' \in \mathcal{F}'$ with $x \in N(F')$ by (γ). Since $F$ is the only element of $\mathcal{F}_0$ with neighbour $x$, we must have $F' \subseteq F$ by definition of $\mathcal{F}_0$ and (ε). This implies $V(P_3) \subseteq V(F) \cup T$.

So we have shown $U := \bigcup_{s \in S_0} V(P_3) \subseteq V(F) \cup N(F)$ for every $F \in \mathcal{F}_0$. Since there are two distinct $T_j$-fragments $F_i$ ($i = 1, 2$) in $\mathcal{F}_0$ by (ε), $U \subseteq (V(F_1) \cup T_1) \cap (V(F_2) \cup T_2) = (F_1 \cup T_2) \cup (T_1 \cup T_2) \cup (T_1 \cup F_2) =: T$ holds by Corollary 3.5(c). But $|T| \leq n$ by Lemma 2.1(a), since $\tilde{F}_1 \cap \tilde{F}_2 \neq \emptyset$ by (δ) and the maximal order of $F_0$. On the other side, $|U| \geq |S_0| + 2 \geq n + 1$ by (z). This contradicts $|U| \leq |T|$, and Theorem 1.1 is proved.

4. PROOFS OF THE REMAINING RESULTS

We follow now the lines of the proof of Theorem 1.1 to give an upper bound for $|W|$ in $W - (n, 3)$-critical graphs, i.e. to prove Theorem 1.2. Since the considerations are very similar to those in the last section, but easier, we will be rather short and point out only the differences. First, we prove an analogue to Corollary 3.4.

**Lemma 4.1.** If $F$ is a fragment of a $W - (n,3)$-critical graph, then $|F \cap W| \leq n(n - 1)$ or $|\tilde{F}| \leq \frac{3}{4}n - 1$ holds.

**Proof.** Let us assume $|F \cap W| > n(n - 1)$ for a $T$-fragment $F$. Define $\tilde{T} := \{ t \in T : |N(t) \cap F \cap W| \leq n - 1 \}$. Since by assumption $|F \cap W| > n(n - 1)$, there is a $c \in F \cap W - N(\tilde{T}) \neq \emptyset$. By the concept of a $W - (n,3)$-critical graph, there is a $\tilde{c} \in \tilde{F} \cap W$. Then $c, \tilde{c}$ have the property described in Lemma 2.9 and this lemma implies $|\tilde{F}| \leq \frac{3}{4}n - 1$. $\blacksquare$

Let $F$ be a $W$-proper fragment of a $W - (n,3)$-graph $G$ with $|W| > (2n - 1)n$. Then $|F \cap W| \leq |\tilde{F} \cap W|$, hence $|\tilde{F} \cap W| > (n - 1)n$ holds, and Lemma 4.1 implies $|F| \leq \frac{3}{4}n - 1$. Hence $|F| \leq \frac{3}{4}n - 1$ holds for all $F \in W \mathcal{F}(G)$. Next, we state an analogue of Corollary 3.5.

**Lemma 4.2.** The $W$-proper fragments of a $W - (n,3)$-critical graph $G$ with $|W| > (2n - 1)n$ have the following properties.

(a) $|F| \leq \frac{3}{4}n - 1$ for every $F \in W \mathcal{F}(G)$.
(b) If $F_1 \cap F_2 \neq \emptyset$ for $F_1, F_2 \in W \mathcal{F}(G)$, then also $G(F_1 \cup F_2) \in W \mathcal{F}(G)$.
(c) For all $F_1 \neq F_2$ from $W \mathcal{F}_m(G)$, $F_1 \cap F_2 = \emptyset$ holds. Every $F \in W \mathcal{F}(G)$ is contained in exactly one element of $W \mathcal{F}_m(G)$.

**Proof.** (a) was shown in the preceding paragraph. Since (a) implies $|F_1 \cup F_2 \cup N(F_1) \cup N(F_2)| \leq 2 \cdot (\frac{3}{4}n - 1) + 2n \leq (2n - 1)n < |G|$, we get as in
Corollary 3.5(b) that $F \equiv G(F_1 \cup F_2)$ is a fragment of $G$, and even a $W$-proper one, since otherwise $|G| < 3 \cdot \left(\frac{3}{4}n - 1\right) + n$ by (a). The first claim of (c) follows from (b), which then implies the second, since every $F \in \mathcal{W} \mathcal{F}(G)$ is contained in at least one element of $\mathcal{W} \mathcal{F}_m(G)$ by (a).

For the proof of Theorem 1.2 we now assume on the contrary that there is a non-complete $W - (n, 3)$-critical graph $G$ with $|W| > (2n - 1)n$. Hence $n \geq 3$ holds.

We can choose $F_0 \in \mathcal{W} \mathcal{F}(G)$ of maximal order by Lemma 4.2(a). Define now $T_0 \equiv N(F_0)$, $T_0 \equiv \{t \in T_0 : |N(t) \cap \bar{F}_0| = n\}$, and $S_0 \equiv (\bar{F}_0 \cap W) - N(T_0)$. Then, we have

$$(\alpha') |S_0| \geq n + 1.$$ 

Proof. $|S_0| \geq |W| - |F_0| - |T_0| - n(n - 1) > (n - 1)n - \left(\frac{3}{4}n - 1\right) \geq n$ by assumption on $W$ and Lemma 4.2(a).

$F_0 \cap W$ is not empty by definition of a $W - (n, 3)$-graph and $S_0 \neq \emptyset$ by $(\alpha')$. Hence, $\mathcal{W} \mathcal{F}_0 \equiv \{F \in \mathcal{W} \mathcal{F}_m(G) : F' \subseteq F$ and $N(F') \cap S_0 \neq \emptyset$ and $N(F') \cap F_0 \neq \emptyset\}$ is also non-empty. For every $F \in \mathcal{W} \mathcal{F}_0$, we have $(F \cup N(F)) \cap S_0 \neq \emptyset$ and $(F \cup N(F)) \cap F_0 \neq \emptyset$. Therefore, $F \neq F_0$ for all $F \in \mathcal{W} \mathcal{F}_0$ and so $F \cap F_0 = \emptyset$ by Lemma 4.2(c), hence $N(F) \cap F_0 \neq \emptyset$ and $F \cap T_0 \neq \emptyset$ for all $F \in \mathcal{W} \mathcal{F}_0$. This implies $|\mathcal{W} \mathcal{F}_0| \leq n$, since $F_1 \cap F_2 = \emptyset$ for $F_1 \neq F_2$ from $\mathcal{W} \mathcal{F}_0$ by Lemma 4.2(c). For every pair $s \in S_0$ and $w \in W \cap F_0$, there is an $F' \in \mathcal{W} \mathcal{F}(G)$ with $\{s, w\} \subseteq N(F')$, since $G$ is $W - (n, 3)$-critical and non-complete. This implies $S_0 \cup (W \cap F_0) \subseteq \bigcup_{F \in \mathcal{W} \mathcal{F}_0} (V(F) \cup N(F))$. The vertices of $W - (S_0 \cup V(F_0))$ belong to $T_0$ or to $N(T_0) \cap \bar{F}_0$. For $t \in T_0 \cap F$ with $F \in \mathcal{W} \mathcal{F}_0$, we have $N(t) \subseteq V(F) \cup N(F)$. Therefore, we have at most $n - |\mathcal{W} \mathcal{F}_0|$ vertices of $T_0$, the neighbours of which are not covered by $\bigcup_{F \in \mathcal{W} \mathcal{F}_0} (V(F) \cup N(F))$. These considerations imply by Lemma 4.2(a) the inequality

$$(\beta') |W| \leq |\mathcal{W} \mathcal{F}_0| \left(\frac{3}{4}n - 1\right) + (n - |\mathcal{W} \mathcal{F}_0|)n,$$

since $|\{t\} \cup (N(t) \cap \bar{F}_0)| \leq n$ for $t \in T_0$. But $(\beta')$ implies $(2n - 1)n < n\left(\frac{3}{4}n - 1\right) + n^2$, a contradiction. Therefore, a non-complete $W - (n, 3)$-critical graph with $|W| > (2n - 1)n$ cannot exist.

If $G$ is $(n, 4)_c$-critical with $n \geq 4$ and $z \in G$, then $G - z$ is an $N_G(z) - (n - 1, 3)$-critical graph and Theorem 1.2 implies $|N_G(z)| \leq (2n - 3)(n - 1)$, hence $\Delta(G) \leq (2n - 3)(n - 1)$ and so $g(n) \leq (2n - 3)(n - 1)$ for $n \geq 4$. This together with Theorem 1.1 implies Theorem 1.4.

One can use the same method to give a better bound for $|W|$ in $W - (n, k)$-critical graphs for $k > 3$, but only the factor at $n^2$ is diminished. Since I do not believe that $n^2$ is the right order, I resign the statement.
CONJECTURE 4.3. There is a real number $c$ such that $|G| \leq cn^{3/2}$ holds for every $(n,3)$-critical graph $G$.

In [4] it was shown by examples (by the graphs $W_m(k)$ from Example 2.7 and another family of graphs) that this would be best possible.

Whereas we have bounded the order of an $(n,4)_c$-graph only by $cn^3$ in Theorem 1.4, we can improve this to $cn^2$ for $(n,5)_c$-graphs. Since the proof follows the same lines as the proofs for Theorems 1.1 and 1.2, we will sketch it only. First of all, we need a formulation of Lemma 2.9 for $(n,k)_c$-graphs.

**Lemma 4.4.** Let $F_0$ be a fragment of an $(n,k)_c$-critical graph $G$. Suppose there is a connected subgraph $C \subseteq G$ of order $k - 1$ with $C \cap F_0 \neq \emptyset$ and $C \cap \bar{F}_0 \neq \emptyset$ such that $F \cap F_0 \neq \emptyset$ for every fragment $F$ of $G$ with $N(F) \supseteq V(C)$. Then $|F_0| \leq \frac{3(n-1)}{4} - 1$ holds.

**Proof.** Since our assumptions imply $k \geq 4$, $G$ is finite by Theorem 1.1 and has an atom. If $A$ is an atom of $G - V(C)$, then we can enlarge $C$ to a connected subgraph $C' \subseteq G$ with $|C'| = k$ and $C' \cap A \neq \emptyset$, and there is a $T \in \mathcal{T}(G)$ with $T \supseteq V(C')$, since $G$ is $(n,k)_c$-critical. Then, the transfer of the proof of Lemma 2 in [4] is obvious and left to the reader.

The following result corresponds to Corollary 3.4 and Lemma 4.1.

**Lemma 4.5.** Let $F$ be a $T$-fragment of an $(n,5)_c$-critical graph $G$. Then $|F| \leq \frac{(n-2)n}{3}$ or $|\bar{F}| \leq \frac{3(n-1)}{4} - 1$ holds.

**Proof.** Suppose $|F| > \frac{(n-2)n}{3}$. Denoting $\hat{T} := \{t \in T : |N(t) \cap F| \leq n - 2\}$, then $U := \{x \in F : |N(x) \cap \hat{T}| \leq 2\}$ is not empty. Now we choose a path $P$ of length 2 as at the beginning of the proof of Lemma 3.3, which we enlarge by an edge $[t,t']$ between $t \in P \cap T$ and an $t' \in N(t) \cap \bar{F} \neq \emptyset$. Then, application of Lemma 4.4 to $C := P \cup [t,t']$ gives $|\bar{F}| \leq \frac{3(n-1)}{4} - 1$.

This implies $|F| \leq \frac{3(n-1)}{4} - 1$ for all the proper fragments $F$ of an $(n,5)_c$-critical graph $G$ with $|G| > 2\frac{(n-2)n}{3} + n$, and the properties stated in Corollary 3.5(b) and (c) for elements of $\mathcal{F}(G)$ and $\mathcal{F}_m(G)$ hold in the same way.

Suppose now for the proof of Theorem 1.5 that there is a non-complete $(n,5)_c$-critical graph $G$ with $|G| > \frac{2}{5}(n - 1)n(\geq 2\frac{(n-2)n}{3} + n)$. Choose again $F_0 \in \mathcal{F}(G)$ of maximal order and define $T_0 := N(F_0)$ and $S_0 := (N(N(T_0)) \cap \bar{F}_0)$ as in the proof of Theorem 1.1. If $V(\bar{F}_0) - (N(T_0) \cup N(N(T_0))) \neq \emptyset$, then
we get $|S_0| \geq n$ (see (z) in the proof of Theorem 1.1) and deduce a contradiction in the same way as in the proof of Theorem 1.1. Therefore, we have $V(\tilde{F}_0) \subseteq N(T_0) \cup N(N(T_0))$.

Define now $\mathcal{F}_0 := \{F \in \mathcal{F}(G) : \text{there is an } F' \in \mathcal{F}(G) \text{ with } F' \subseteq F, N(F') \cap F_0 \neq \emptyset, \text{ and } N(F') \cap \tilde{F}_0 \neq \emptyset\}$. Again, $F \cap T_0 \neq \emptyset$ for all $F \in \mathcal{F}_0$ and the elements of $\mathcal{F}_0 \cup \{F_0\}$ are disjoint, hence $|\mathcal{F}_0| \leq n$. Since also $V(F_0) \subseteq N(T_0)$ (as $|F_0| < n$), $V(G)$ can be covered by paths $P$ of length 3 with $P \cap F_0 \neq \emptyset$ and $P \cap \tilde{F}_0 \neq \emptyset$. This implies $V(G) \subseteq \bigcup_{F \in \mathcal{F}_0} (V(F) \cup N(F))$, since $G$ is $(n,4)_c$-critical. If $V(P) \subseteq V(F) \cup N(F)$ for an $F \in \mathcal{F}_0$, then $V(P) \cap F_0 \subseteq N(F)$, and we can add an edge between $V(P) \cap F_0 \neq \emptyset$ and $\tilde{F}$ to $P$, so getting $P'$. Since $G$ is even $(n,5)_c$-critical, there is an $F' \neq F$ in $\mathcal{F}_0$ with $V(P') \subseteq V(F') \cup N(F')$. Hence, $V(G)$ is covered at least twice by $\bigcup_{F \in \mathcal{F}_0} (V(F) \cup N(F))$. But this implies $2|G| \leq n\left(\frac{3(n-1)}{4} - 1 + n\right) = \frac{7}{4}(n-1)n$, a contradiction.

**Remark 4.6.** One can also use Lemma 4.4 for the following slight improvement of Theorem 1.4.

(4.6.1) For every $(n,4)_c$-critical graph $G$ with $n \geq 4$, $|G| \leq (n-1)(n-2)(n-3) + n$ holds.

The main tool in the proof is the following modification of Lemma 3.3 and Corollary 3.4.

(4.6.2) If $F$ is a $T$-fragment of an $(n,4)_c$-critical graph $G$ satisfying $|F| > \frac{1}{2}(n-1)(n-2)(n-3)$, then $V(\tilde{F}) \subseteq N(T)$ and $|\tilde{F}| \leq \frac{(n-3)n}{3}$ hold.

**Proof of (4.6.2).** We may assume $|\tilde{F}| > \frac{3(n-1)}{4} - 1$. Then Lemma 4.4 implies that for every path $P$ of length 2 with $P \cap F \neq \emptyset$ and $P \cap \tilde{F} \neq \emptyset$, there is a fragment $F'$ of $G - V(P)$ with $F' \cap F = \emptyset$. From this it easily follows that for every $x \in N(T) \cap F$, there are at least two neighbours $t \in T$ with $|N(t) \cap F| \leq n - 2$. But this implies $|N(t) \cap F| \leq \frac{(n-1)(n-2)}{2}$ for all $t \in T$. Then, one easily checks that Lemma 3.3 remains true for $F$, if we replace there $A(G) - 1$ with $\frac{(n-1)(n-2)}{2}$. Then Lemma 3.3 implies $V(\tilde{F}) \subseteq N(T)$ and $|\tilde{F}| \leq \frac{(n-3)n}{3}$.

Using (4.6.2) instead of Lemma 4.5, (4.6.1) can be proved in a similar way as Theorem 1.5.

The upper bound for $(n,4)_c$-graphs in Theorem 1.4 is by a factor $n$ larger than that for $(n,3)$-graphs in Theorem 1.3, but I do not believe that this reflects the facts. It is not at all obvious to find $(n,k)_c$-graphs for large $k$, which are not $k$-critical. For $k = 2, 3$, we have constructed a lot of such graphs in [6], but for $k \geq 4$ we have not pointed out one so far. We will close this gap in the next section.
5. EXAMPLES OF $k$-CON-CRITICAL, NOT $k$-CRITICAL GRAPHS

In this section, for every $k \geq 4$, we will give examples for $k$-con-critical graphs which are not $k$-critical. For this, we will study the graphs $W_m(k)$ introduced in Example 2.7 more in detail.

In [4, p. 146], it was shown that $N(z) \in \mathcal{F}(W_m(k))$ for every $z \in W_m(k)$ and all $m \geq 2$, $k \geq 2$, but we had not proved that every $T \in \mathcal{F}(W_m(k))$ has this form. We will supply this now. First a notation. For $j \in \mathbb{N}_m$, let $z_j$ denote again the vertex $(z_1^j, \ldots, z_k^j)$ of $W_m(k)$ with $z_j^i = j$ for all $i \in \mathbb{N}_k$. Let $\mathcal{S}_n$ denote the symmetric group on $\mathbb{N}_m$. Note that the application of any $\pi \in \mathcal{S}_m$ to the $i$th coordinate in $W_m(k)$ is an automorphism. For $i, j \in \mathbb{N}_m$, let $f_{i,j}$ denote the automorphism which interchanges the numbers $i$ and $j$ in every coordinate.

**Lemma 5.1.** If $T \in \mathcal{F}(W_m(k))$ with $m \geq 2$, $k \geq 2$, then there is a $z \in W_m(k)$ with $T = N(z)$.

**Proof.** Consider $T \in \mathcal{F}(W_m(k))$ and let $c_1$, $c_2$ be in distinct components of $W_m(k) - T$. We may assume that $c_i = z_i$ for $i = 1, 2$. Define $D := \{x = (x_1, \ldots, x_k) \in W_m(k)\}$: there are $i, j \in \mathbb{N}_k$ with $x_i = 1$ and $x_j = 2$ and $Z_2 = \{x = (x_1, \ldots, x_k) \in W_m(k) - z_2: \text{there is an } i \in \mathbb{N}_k \text{ with } x_i = l \} - D$ for $l = 1, 2$. Then, obviously, $D = N(z_1) \cap N(z_2) \subseteq T$ and $f_{1,2}|Z_1$ is a bijection of $Z_1$ on $Z_2$. For every $x \in Z_1$, we have $\{x, f_{1,2}(x)\} \cap T \neq \emptyset$ and hence $|\{x, f_{1,2}(x)\} \cap T| = 1$ and $T \subseteq D \cup Z_1 \cup Z_2$, since $N(z_i) = D \cup Z_i$ for $i = 1, 2$ and $|T| = \kappa(W_m(k)) = d(z_i)$. (This latter was proved in [4], but we have yet shown $|T| \geq d(z_i) \geq \kappa(W_m(k))$, hence equality.) Let us suppose, there are $x_i = (x_i^1, \ldots, x_i^k) \in Z_1 - T$ for $i = 1$ and 2. Then $x_2 \neq f_{1,2}(x_1)$. Hence, there is a $\kappa \in \mathbb{N}_k$ with $|\{x_1^\kappa, x_2^\kappa\} \cap \{1, 2\}| = 1$, say, $\kappa = 1$ and $x_1^1 = 1$, $x_2^1 = 3$. Replace every coordinate 1 in $x_1$ with 3, so getting $y$. Then $y \in N(x_1) \cap N(x_2)$, hence $y \in T$, a contradiction to the fact $T \subseteq D \cup Z_1 \cup Z_2$. This contradiction shows $Z_1 \subseteq T$ or $Z_2 \subseteq T$, hence $T = N(z_1)$ or $T = N(z_2)$, as claimed. \n
Lemma 5.1 implies immediately

**Corollary 5.2.** The graphs $W_m(k)$ for $m > k \geq 2$ are $k$-critical, but not $(k + 1)$-critical.

**Proof.** It is known from [4, p. 146] that $W_m(k)$ is $k$-critical. On the other hand, $z_1, z_2, \ldots, z_{k+1}$ have no common neighbour, hence $\kappa(W_m(k) - \{z_1, \ldots, z_{k+1}\}) \geq w(m, k) - k$ by Lemma 5.1. \n
Corollary 5.2 is not true without a condition for $m$, as the graphs $W_2(k)$ show. These arise from $K_{2^n}$ by deletion of the edges of an 1-factor and are well-known $(2^{k-1} - 1)$-critical graphs (cf. [5, Conjecture 1a]).
PROPOSITION 5.3. The graphs $W_m(k)$ for $m > k \geq 4$ are $(k + 2)$-con-critical, but not $(k + 3)$-con-critical.

Proof. Define $z := (z^1, \ldots, z^k)$ with $z^i = i$ for $i \in \mathbb{N}_k$ and $z' := (z^1, \ldots, z^k)$ with $z^i = k$ for $i \in \mathbb{N}_{k-1}$ and $z^k = k + 1$. Then $Z := \{z_i : i \in \mathbb{N}_{k+1}\} \cup \{z, z'\}$ is connected in $W_m(k)$, but there is no $y$ with $N(y) \supseteq Z$. Hence $W_m(k)$ is not $(k + 3)$-con-critical by Lemma 5.1.

Consider now a connected set $X$ of $k + 2$ vertices $x_j = (x_j^1, \ldots, x_j^k)$ for $j \in \mathbb{N}_{k+2}$ in $G := W_m(k)$. Define $p := \max_{i \in \mathbb{N}_k} \max\{|J| : J \subseteq \mathbb{N}_{k+2} \text{ with } x_j^i = x_j^{i'} \text{ for all } j, j' \in J\}$. In the following cases (a) and (b), we assume that the maximum is attained for $i = k$ and $x_k^1 = 1$ holds for $j = k + 3 - p, \ldots, k + 2$ and we set $y^k := 1$. Since $X$ is connected, we have $p \geq 2$. We distinguish three cases.

(a) $p = 4$.

Set $y^i := x_j^i$ for $i \in \mathbb{N}_{k+2}$. Since $k + 2 - p \leq k - 2$ in case (a), $y^{k-1}$ is not defined so far. If $M := \mathbb{N}_m - \{x_1^{k-1} : i = k + 3 - p, \ldots, k + 2\} \neq \emptyset$, then we choose $y^{k-1} \in M$ and $y^i \in \mathbb{N}_m$ arbitrarily for $i = k + 3 - p, \ldots, k - 2$. If $M = \emptyset$, then $p \geq m \geq k + 1$ and we can choose $y^i \in \mathbb{N}_m$ for $i = k + 3 - p, \ldots, k - 1$ so that $(y^1, y^2, \ldots, y^k) \not\in X$, since then $k + 3 - p \leq 2 < k - 1$. Then, we have in any case $N(y) \supseteq X$ for $y := (y^1, y^2, \ldots, y^k)$.

(b) $p = 3$.

Let us first assume that there are an $i \in \mathbb{N}_{k-1}$ and $j_1 < j_2 \in \mathbb{N}_{k-1}$ with $x_{j_1}^i = x_{j_2}^i$, say, $i = k - 1$ and $\{j_1, j_2\} = \{k - 2, k - 1\}$. Set $y^{k-1} := x_{k-1}^{k-1}$, $y^i := x_j^i$ for $i \in \mathbb{N}_{k-3}$, and choose $y^{k-2} \in \mathbb{N}_m - \{x_1^{k-2} : i = k, k + 1, k + 2\} \neq \emptyset$. Then, $y := (y^1, \ldots, y^k)$ has the property $N(y) \supseteq X$.

So we may assume $|\{x_j^i : j \in \mathbb{N}_{k-1}\}| = k - 1$ for every $i \in \mathbb{N}_{k-1}$. Then, we can choose $\pi \in \mathcal{S}_{k-1}$ so that $y := (x_{\pi(1)}^1, x_{\pi(2)}^2, \ldots, x_{\pi(k-1)}^{k-1}, y^k) \not\in \{x_k, x_{k+1}, x_{k+2}\}$, since $(k - 1)! > 3$. But then $N(y) \supseteq X$ holds.

(c) $p = 2$.

First, we assume that the following situation appears:

(S) There are $i_1 \neq i_2 \in \mathbb{N}_k$ and four distinct $j_1, j_2, j_3, j_4 \in \mathbb{N}_{k+2}$ such that $x_{j_1}^{i_1} = x_{j_2}^{i_1}$ and $x_{j_3}^{i_3} = x_{j_4}^{i_3}$ hold.

We may assume $y^{k-1} := x_{k-1}^{k-1} = x_k^{k-1}$ and $y^k := x_{k+1}^k = x_{k+2}^k$. Then for $y := (y^1, \ldots, y^k)$ with $y^i := x_j^i$ for $i \in \mathbb{N}_{k-2}$, we have $N(y) \supseteq X$, since $p = 2$ implies that $x_{k+1}, x_{k+2}$ are the only vertices of $X$ with $k$th coordinate equal to $y^k$ and correspondingly for $x_{k-1}, x_k$. So we assume in the following that (S) does not appear.

If $N(x_{k+2}) \supseteq X - \{x_k\}$, then there are an $i \in \mathbb{N}_k$ and $j_1 \neq j_2 \in \mathbb{N}_{k+1}$ with $x_{j_1}^i = x_{j_2}^i = x_{k+2}^i$ by the pigeon-hole principle. But this is a contradiction to $p = 2$. So for every $x_j \in X$ there is a $j' \in \mathbb{N}_{k+2} - \{j\}$ with $x_j^i \neq x_{j'}^i$ for all $i \in \mathbb{N}_k$. So there are vertices of distance 2 in $G(X)$, say, $x_1, x_3$ with common neighbour $x_2$. We may assume $x_1^1 = x_2^1$ and $x_3^2 = x_2^3$. 

If $G(X) - \{x_1, x_2, x_3\}$ has an edge, say, $[x_4, x_5] \in E(G)$, then there is an $i \in \mathbb{N}_k$ with $x_i^j = x_j^i$. But $i \neq 1$ or $i \neq 2$, and we have situation (S) for 1, $i$ and $x_1, x_2, x_4, x_5$ or for 2, $i$ and $x_2, x_3, x_4, x_5$. So $G(X) - \{x_1, x_2, x_3\}$ is independent.

Since $X - \{x_2\} \not\subseteq N(x_2)$ and $G(X)$ connected, but $G(X) - \{x_1, x_2, x_3\}$ independent, there must be an $x \in X - \{x_1, x_2, x_3\} \neq \emptyset$ with $\{x_1, x_3\} \neq \emptyset$, say, $[x_1, x_4] \in E(G)$ and $x_1^j = x_4^i$. Since (S) does not occur, we must have $i = 2$. Since $p = 2$, this implies that there can be no further $x \in X - \{x_1, x_2, x_3, x_4\}$ with $x_1 \in N(x)$. Since $|X| > 4$, we conclude $N(x_3) \cap \{x_2, x_3\} \neq \emptyset$. If $[x_2, x_5] \in E(G)$, say, $x_2^i = x_5^j$, then $i \neq 1, 2$, since $p = 2$, and (S) occurs with $2, i$ and $x_1, x_4, x_2, x_5$. So we have $[x_3, x_5] \in E(G)$, say, $x_3^i = x_5^i$. Then, we conclude $i = 1$ as above. But then (S) occurs for 1,2 and $x_3, x_5, x_1, x_4$.

$W_m(3)$ is not 4-con-critical for each $m \geq 2$, as the connected set $\{(1, 1, 1), (1, 2, 2), (2, 1, 2), (2, 2, 1)\}$ shows, using Lemma 5.1. In the same way, the connected set $\{(1, 1), (1, 2), (2, 2)\}$ shows that $W_m(2)$ is not 3-con-critical. Therefore, for the construction of 4-con-critical graphs which are not 4-critical, we must proceed in a different way. In the case of a given set of $k + 1$ vertices in $W_m(k)$, we can improve Proposition 5.3.

**Proposition 5.4.** Let $X$ be a connected set of $k$ vertices in $W_m(k)$ with $m > k \geq 4$. Then for every $z \in W_m(k)$, there is a $y \in W_m(k) - (X \cup \{z\})$ with $N(y) \supseteq X \cup \{z\}$.

**Proof.** For $X \cup \{z\}$ connected, the assertion is obvious from Proposition 5.3 and Lemma 5.1. So we may assume that $z$ has no coordinate in common with any of the $x \in X$. But there are $x_1 \neq x_2$ in $X$ which have a coordinate in common, say, the $k$th coordinate $y^k$. Choosing from every element of the $(k - 1)$-set $(X - \{x_1, x_2\}) \cup \{z\}$ one coordinate $y^i$ for $i \in \mathbb{N}_{k-1}$, we get $y := (y^1, \ldots, y^k)$ with $N(y) \supseteq X \cup \{z\}$.

Since the automorphism group of $W_m(k)$ is transitive, Corollary 5.2 implies that $W_m(k) - z$ is not $k$-critical for any $z \in W_m(k)$. But by Proposition 5.4, $W_m(k) - z$ is $k$-con-critical for all $m > k \geq 4$. So we get also examples for the still missing case $k = 4$.

In [6, Proposition 3.14], it was proved that the diameter of an $(n, 7)_c$-critical graph is at most 4. This implies that an $(n, k)_c$-graph is at least $(1 + \lceil \frac{k-1}{4} \rceil)$-critical for $k \geq 7$. But I believe that there must be a much better lower bound for the criticality of an $(n, k)_c$-critical graph.

**Note added in proof.** As noticed by an unknown referee, the following sharper form of Proposition 5.4 even holds: Let $X$ be a set of $k + 1$ vertices containing at least two adjacent ones in $W_m(k)$ with $m > k \geq 4$. Then, there is a $y \in W_m(k) - X$ with $N(y) \supseteq X$. 
For a proof, consider an edge \([x_1, x_2]\) of \(W_m(k)\) with \(x_1, x_2 \in X\). Then, there is a \(z \in W_m(k)\) with \(N(z) \supseteq X - \{x_1\}\) by Corollary 5.2 and Lemma 5.1. Hence, \(X' := X \cup \{z\}\) is connected and we can apply Proposition 5.3 and Lemma 5.1. 

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