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On Regular Graphs, V*

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Let Γ_3 be an infinite regular tree of valence 3. There exist subgroups B of Aut (Γ_3) which are 5-regular on Γ_3, i.e., sharply transitive on the set of 5-arcs of Γ_3. We prove that any two such subgroups are conjugate in Aut (Γ_3). The pair (Γ_3, B) is a universal 5-regular action in the sense that if (G, A) is a pair consisting of a cubical graph G and a 5-regular subgroup A of automorphisms of G then (G, A) can be "covered" by (Γ_3, B) in a certain natural way.

PRELIMINARIES

This paper can be read independently from the previous papers with the same title. The terminology is standard; we only recall that an s-arc ($s \geqslant 0$) in a graph G is a map S: $\{0, 1,..., s\} \to G$ (=the set of vertices of G) such that $S(i)$ is adjacent to $S(i+1)$ for $0 \leqslant i \leqslant s-1$ and $S(i) \neq S(i+2)$ for $0 \leqslant i \leqslant s-2$. If S is an s-arc then its *opposite s-arc* S' is defined by $S'(i) = S(s-i)$, $0 \leqslant i \leqslant s$.

In the whole paper G denotes a regular graph of valence 3 and A a subgroup of Aut(G) which is 5-regular. This means that the induced action of A on 5-arcs of G is regular, i.e., sharply transitive. If v_1,..., v_k are vertices of G then $A(v_1,..., v_k)$ denotes the subgroup of A consisting of all $\alpha \in A$ such that $\alpha(v_i) = v_i$ for $i = 1,..., k$. We say that $A(v_1,..., v_k)$ is *the fixer* in A of the set $\{v_1,..., v_k\}$.

Since (G, A) is 5-regular it is clear that the fixer of a vertex has order $3 \cdot 2^4 = 48$ and the fixer of an s-arc ($1 \leqslant s \leqslant 5$) has order 2^{5-s}. Moreover, according to Biggs [1, p. 126] these groups are unique up to isomorphism: If F_s is a fixer in A of an s-arc then

$$F_0 \cong S_4 \times C_2, \quad F_1 \cong D_4 \times C_2, \quad F_2 \cong C_2^3 \\ F_3 \cong C_2^2, \quad F_4 \cong C_2, \tag{1}$$

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where S_n, D_n, C_n are the symmetric group of degree n, the dihedral group of order $2n$ and the cyclic group of order n, respectively. This claim in Biggs is based on the paper [5] of W. J. Wong in which only finite primitive permutation groups are studied.

We shall not rely on this claim but will reprove it here in the course of our study of $A(v)$ and its action on vertices not far away from v. Note that we do not require the action of A on vertices of G to be primitive. We also allow G to be infinite. In fact, our main results are about the case when G is an infinite tree.

For some general results on the automorphism groups of trees the reader should consult a recent paper of J. Tits [3].

Amalgam of Two Vertex-Fixers

Lemma 1. *Let v be a vertex of G. Then*

(i) *there exists a unique non-trivial element $\tilde{v} \in A$ which fixes all vertices whose distance from v is $\leqslant 2$;*

(ii) *$\tilde{v}$ is an involution and belongs to the center of $A(v)$;*

(iii) *if $\alpha \in A$ and $\alpha(v) = w$ then $\alpha\tilde{v}\alpha^{-1} = \tilde{w}$;*

(iv) *if w is a vertex at distance 3 from v then $\tilde{v}(w) \neq w$;*

(v) *$\langle\tilde{v}\rangle$ is the fixer of any 4-arc S such that $S(2) = v$.*

Proof. Let $a, b, c, \ldots$ be the vertices of G as indicated on Fig. 1. Since A is 5-regular, the girth of G is $\geqslant 8$, [1, p. 113], and consequently all these vertices are distinct.

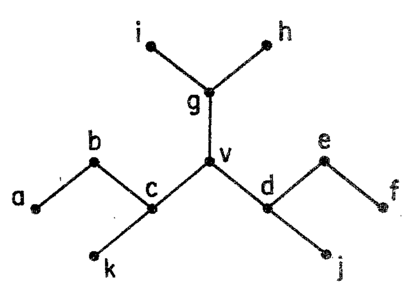

Figure 1

(i) The order of $A(b, v, e)$ is 2 and let α be its generator. Thus α is an involution. Since $\alpha(g) = g$ we have either $\alpha(h) = h$ or $\alpha(h) = i$. We shall show now that $\alpha(h) = i$ leads to a contradiction. Let π be the permutation representation of $A(v)$ on the vertices of G whose distance from v is $\leqslant 2$. If

$\beta \in \operatorname{Ker} \pi$ then $\beta \in A(b, v, e) = \langle \alpha \rangle$. Since $\alpha^2 = 1$ and $\alpha(h) = i$ by hypothesis it follows that $\beta = 1$, i.e., our permutation representation π is faithful. Let α' and α'' be the generators of $A(b, v, h)$ and $A(e, v, h)$, respectively. Then $\alpha'(e) = j$ and $\alpha''(b) = k$. Hence $\pi(\alpha)$, $\pi(\alpha')$, $\pi(\alpha'')$ are three pairwise disjoint transpositions and they generate an elementary abelian group of order 8. It follows that $A(c, v, d, g) = A(c, v, d)$ is elementary abelian of order 8. Choose $\beta \in A(b, v)$ so that $\beta(g) = d$ and $\beta(h) = e$. Then $\beta\alpha(h) = \beta(i) = j$, $\alpha\beta(h) = \alpha(e) = e$ and hence $\alpha\beta \neq \beta\alpha$. On the other hand $\alpha, \beta \in A(b, c, v)$ which is a conjugate of $A(c, v, d)$ and so $A(b, c, v)$ is elementary abelian of order 8. This is a contradiction.

We have proved that $\alpha(h) = h$, i.e., α fixes every vertex at distance $\leqslant 2$ from v. It is clear that α is the unique element of A with this property. From now on we shall write $\tilde{v}$ instead of this α.

(iii) Since $\tilde{v}$ fixes all vertices of G at distance $\leqslant 2$ from v it is clear that $\alpha\tilde{v}\alpha^{-1}$ fixes all vertices of G at distance $\leqslant 2$ from $w = \alpha(v)$. The uniqueness part of (i) implies that $\alpha\tilde{v}\alpha^{-1} = \tilde{w}$.

(ii) In the proof of (i) we have shown that $\tilde{v}$ is an involution. If $\alpha \in A(v)$ then (iii) gives $\alpha\tilde{v}\alpha^{-1} = \tilde{v}$, i.e., $\tilde{v}$ belongs to the center of $A(v)$.

(iv) This follows from $\tilde{v} \neq 1$ and 5-regularity of A.

(v) If S is a 4-arc and $S(2) = v$ then $\tilde{v}$ fixes S. Hence $\langle \tilde{v} \rangle$ must be the fixer of S since the latter has order 2.

LEMMA 2. *Using the notation of Fig.* 1 *we have*:

$$A(b, v, e) = \langle \tilde{v} \rangle, \qquad A(b, d) = \langle \tilde{c}, \tilde{v} \rangle,$$
$$A(c, d) = \langle \tilde{c}, \tilde{v}, \tilde{d} \rangle, \qquad A(c, v) = \langle \tilde{b}, \tilde{c}, \tilde{v}, \tilde{d} \rangle,$$
$$A(v) = \langle \tilde{b}, \tilde{c}, \tilde{v}, \tilde{d}, \tilde{e} \rangle.$$

Proof. We have $\tilde{c} \notin \langle \tilde{v} \rangle$, $\tilde{d} \notin \langle \tilde{c}, \tilde{v} \rangle$, $\tilde{b} \notin \langle \tilde{c}, \tilde{v}, \tilde{d} \rangle$, $\tilde{e} \notin \langle \tilde{b}, \tilde{c}, \tilde{v}, \tilde{d} \rangle$. The last statement is true because $\tilde{e}(c) = g$ and $\langle \tilde{b}, \tilde{c}, \tilde{v}, \tilde{d} \rangle \subset A(c)$. Since

$$A(b, v, e) \supset \langle \tilde{v} \rangle, \qquad A(b, d) \supset \langle \tilde{c}, \tilde{v} \rangle,$$
$$A(c, d) \supset \langle \tilde{c}, \tilde{v}, \tilde{d} \rangle, \qquad A(c, v) \supset \langle \tilde{b}, \tilde{c}, \tilde{v}, \tilde{d} \rangle,$$
$$A(v) \supset \langle \tilde{b}, \tilde{c}, \tilde{v}, \tilde{d}, \tilde{e} \rangle$$

and $|A(b, v, e)| = 2$, $|A(b, d)| = 4$, $|A(c, d)| = 8$, $|A(c, v)| = 16$, $|A(v)| = 48$ the assertion of the Lemma follows.

LEMMA 3. *Using the notation of Fig.* 1 *we have*

$$\tilde{c}\tilde{v} = \tilde{v}\tilde{c}, \qquad \tilde{c}\tilde{d} = \tilde{d}\tilde{c}, \qquad (\tilde{b}\tilde{d})^2 = \tilde{c}\tilde{v}, \qquad (\tilde{b}\tilde{e})^3 = 1.$$

Proof. The first two equalities follow from $\tilde{c}(v) = v$, $\tilde{c}(d) = d$ and Lemma 1(iii).

It is clear that $(\tilde{b}\tilde{d})^2 \in A(b, d)$. By Lemma 2, $A(b, d) = \langle \tilde{c}, \tilde{v} \rangle$. We have $\tilde{b}(d) = g$ and, say, $\tilde{b}(e) = h$. Then $(\tilde{b}\tilde{d})^2(e) = \tilde{b}\tilde{d}\tilde{b}(e) = \tilde{b}\tilde{d}(h) = \tilde{b}(i) = j$. Hence $(\tilde{b}\tilde{d})^2$ is neither 1 nor $\tilde{v}$. Similarly $(\tilde{b}\tilde{d})^2(a) \neq a$ and hence $(\tilde{b}\tilde{d})^2 \neq \tilde{c}$. But $A(b, d) = \langle \tilde{c}, \tilde{v} \rangle$ has only four elements $1, \tilde{c}, \tilde{v}, \tilde{c}\tilde{v}$. It follows then that $(\tilde{b}\tilde{d})^2 = \tilde{c}\tilde{v}$.

Since $\tilde{b}, \tilde{e} \in A(v)$ and $\tilde{b}\tilde{e}(c) = \tilde{b}(g) = d$ we must have $\tilde{b}\tilde{e}(b) = e$ or j. In both cases this vertex is fixed by $\tilde{e}$, i.e., $\tilde{e}\tilde{b}\tilde{e}(b) = \tilde{b}\tilde{e}(b)$. Consequently, we have $(\tilde{b}\tilde{e})^3(b) = b$. We claim now that $\tilde{b}\tilde{e}(b) = e$. Otherwise we would have $\tilde{b}\tilde{e}(b) = j$ and consequently $\tilde{j} = \tilde{b}\tilde{e}\tilde{b}\tilde{e}\tilde{b}$ by Lemma 1(iii). Then $e = \tilde{c}(j) = \tilde{c}\tilde{b}\tilde{e}(b) = \tilde{b}\tilde{c}\tilde{e}\tilde{c}(b) = \tilde{b}\tilde{j}(b)$ because $\tilde{c}(b) = b$ and $\tilde{c}\tilde{e}\tilde{c} = \tilde{j}$ by Lemma 1(iii). Replacing $\tilde{j}$ by $\tilde{b}\tilde{e}\tilde{b}\tilde{e}\tilde{b}$ in $\tilde{b}\tilde{j}(b) = e$ we get $\tilde{e}\tilde{b}\tilde{e}\tilde{b}(b) = e$, i.e., $\tilde{b}\tilde{e}(b) = e$ which contradicts $\tilde{b}\tilde{e}(b) = j$. Hence we have proved that $\tilde{b}\tilde{e}(b) = e$. It follows that $\tilde{e}\tilde{b}(e) = b$. Thus $\tilde{e}\tilde{b}(f)$ is a neighbour of b and consequently it is fixed by $\tilde{b}$, i.e., $\tilde{b}\tilde{e}\tilde{b}(f) = \tilde{e}\tilde{b}(f)$. It follows that $(\tilde{b}\tilde{e})^3(f) = f$. Hence $(\tilde{b}\tilde{e})^3 \in A(b, v, f)$ and by 5-regularity, $(\tilde{b}\tilde{e})^3 = 1$.

THEOREM 1. *Using notation of Fig.* 1 *we have*

(i) $A(c, v) = \langle \tilde{b}, \tilde{d} \rangle \times \langle \tilde{c} \rangle = \langle \tilde{b}, \tilde{d} \rangle \times \langle \tilde{v} \rangle$ *and* $\langle \tilde{b}, \tilde{d} \rangle \simeq D_4$;

(ii) $A(c) = \langle \tilde{a}, \tilde{b}\tilde{c}, \tilde{c}\tilde{v}, \tilde{d} \rangle \times \langle \tilde{c} \rangle$, $\langle \tilde{a}, \tilde{b}\tilde{c}, \tilde{c}\tilde{v}, \tilde{d} \rangle \simeq S_4$ *and* $\langle \tilde{a}, \tilde{d} \rangle \simeq D_3$;

(iii) $\langle \tilde{c} \rangle$ *is the center of* $A(c)$.

Proof. (i) Since $\tilde{b}, \tilde{d}$ are distinct involutions, the group $\langle \tilde{b}, \tilde{d} \rangle$ is dihedral. By Lemma 3, $(\tilde{b}\tilde{d})^2 = \tilde{c}\tilde{v}$, $\tilde{b}\tilde{d}$ has order 4 and hence $\langle \tilde{b}, \tilde{d} \rangle \simeq D_4$. The center of $A(c, v)$ is $\langle \tilde{c}, \tilde{v} \rangle$ and the center of $\langle \tilde{b}, \tilde{d} \rangle$ is $\tilde{c}\tilde{v}$. Thus we have the two direct decompositions stated above.

(ii) We claim that $\tilde{a}$ normalizes the four-group $\langle \tilde{b}\tilde{c}, \tilde{c}\tilde{v} \rangle$. Indeed, using Lemma 3,

$$\tilde{a}\tilde{b}\tilde{c}\tilde{a} = \tilde{b}\tilde{c}, \qquad \tilde{a}\tilde{c}\tilde{v}\tilde{a} = \tilde{c}(\tilde{a}\tilde{v})^2\, \tilde{v} = \tilde{c}\tilde{b}\tilde{c}\tilde{v}.$$

Similarly, $\tilde{d}$ normalizes $\langle \tilde{b}\tilde{c}, \tilde{c}\tilde{v} \rangle$. By Lemma 3, $\tilde{a}\tilde{d}$ has order 3 and hence $\langle \tilde{a}, \tilde{d} \rangle \simeq D_3$. Since $\langle \tilde{a}, \tilde{d} \rangle \cap \langle \tilde{b}\tilde{c}, \tilde{c}\tilde{v} \rangle = 1$ the group $\langle \tilde{a}, \tilde{b}\tilde{c}, \tilde{c}\tilde{v}, \tilde{d} \rangle$ is a semi-direct product and hence it is isomorphic to S_4. Since this subgroup together with $\tilde{c}$ generates $A(c)$ we must have $\tilde{c} \notin \langle \tilde{a}, \tilde{b}\tilde{c}, \tilde{c}\tilde{v}, \tilde{d} \rangle$. Therefore $A(c)$ is a direct product as stated in the theorem.

(iii) This is immediate from (ii).

An *amalgam* is an ordered pair of groups (X, Y) such that $X \cap Y$ is a subgroup in each of X and Y and the induced group structures on $X \cap Y$ from X and from Y coincide.

Two amalgams (X, Y) and (X', Y') are *isomorphic* if there is a map $f: X \cup Y \to X' \cup Y'$ such that $f(X) = X'$, $f(Y) = Y'$ and the restrictions

$$f_X: X \to X' \qquad \text{and} \qquad f_Y: Y \to Y'$$

are group isomorphisms. We shall say that such a map is *an isomorphism* of these amalgams.

A *special amalgam* is an amalgam (X, Y) which is equipped with a map $\phi: X \cup Y \to X \cup Y$ such that $\phi(X) = Y$, $\phi(Y) = X$ and the restrictions

$$\phi_X: X \to Y \qquad \text{and} \qquad \phi_Y: Y \to X$$

are group isomorphisms. In particular, if (X, Y, ϕ) is a special amalgam then $X \cong Y$.

Two special amalgams (X, Y, ϕ) and (X', Y', ϕ') are *isomorphic* if there exists an isomorphism of amalgams $f: (X, Y) \to (X', Y')$ such that $f \circ \phi = \phi' \circ f$.

Every 5-arc S in G determines a special amalgam (X, Y, ϕ) as follows. Let $S(i) = v_i$ $(0 \leqslant i \leqslant 5)$. Then we take $X = A(v_2)$, $Y = A(v_3)$. Note that $X \cap Y = A(v_2, v_3)$. Let $\alpha \in A$ be the unique automorphism such that $\alpha(v_i) = v_{5-i}$ $(0 \leqslant i \leqslant 5)$. Then $\alpha^2 = 1$ and we have $\alpha A(v_i)\alpha = A(v_{5-i})$ for $0 \leqslant i \leqslant 5$. In particular, we see that $\alpha(X \cap Y)\alpha = X \cap Y$. Let $\phi: X \cup Y \to X \cup Y$ be defined by $\phi(\beta) = \alpha \circ \beta \circ \alpha$. Then we have constructed a special amalgam (X, Y, ϕ). Note that $\alpha^2 =$ identity.

THEOREM 2. *The special amalgam defined above is unique up to isomorphism, i.e., it is independent of the choice of S and (G, A).*

Proof. Using the above notation we have

$$\begin{aligned} X &= A(v_2) = \langle \tilde{v}_0, \tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4 \rangle, \\ Y &= A(v_3) = \langle \tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4, \tilde{v}_5 \rangle, \\ X \cap Y &= A(v_2, v_3) = \langle \tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4 \rangle \end{aligned}$$

and

$$\phi(\tilde{v}_i) = \tilde{v}_{5-i} \qquad \text{for} \quad 0 \leqslant i \leqslant 5.$$

Now the assertion is valid because of Theorem 1. More precisely, if (G', A') is also 5-regular and S' is a 5-arc of G' with $v_i' = S'(i)$, $0 \leqslant i \leqslant 5$ then it suffices to define the isomorphism $f: (X, Y, \phi) \to (X', Y', \phi')$ by sending $\tilde{v}_i$ to $\tilde{v}_i'$ for $0 \leqslant i \leqslant 5$.

From now on we shall denote by (X, Y, ϕ) the special amalgam determined by a 5-arc in G. Explicitly, it is given by

$$X = \langle x_0, x_1, x_2, x_3, x_4 \rangle, \qquad Y = \langle x_1, x_2, x_3, x_4, x_5 \rangle$$

where the defining relations for X are

$$x_i^2 = 1, \qquad 0 \leqslant i \leqslant 4;$$
$$x_i x_j = x_j x_i, \qquad 0 \leqslant i \leqslant j \leqslant i+2, j \leqslant 4;$$
$$(x_0 x_3)^2 = x_1 x_2; \qquad (x_1 x_4)^2 = x_2 x_3; \qquad (x_0 x_4)^3 = 1;$$

the defining relations for Y are

$$x_i^2 = 1, \qquad 1 \leqslant i \leqslant 5;$$
$$x_i x_j = x_j x_i, \qquad 1 \leqslant i \leqslant j \leqslant i+2, j \leqslant 5;$$
$$(x_1 x_4)^2 = x_2 x_3; \qquad (x_2 x_5)^2 = x_3 x_4; \qquad (x_1 x_5)^3 = 1;$$
$$X \cap Y = \langle x_1, x_2, x_3, x_4 \rangle$$

and

$$\phi(x_i) = x_{5-i} \qquad \text{for} \quad 0 \leqslant i \leqslant 5.$$

THE UNIVERSAL 5-REGULAR ACTION

Let (X, Y, ϕ) be the special amalgam constructed in the previous section. We shall use the generators x_i, $0 \leqslant i \leqslant 5$ for X and Y and the defining relations given there.

Let H be the free product with amalgamation of X and Y with amalgamated subgroup $X \cap Y$. The map $\phi: X \cup Y \to X \cup Y$ can be extended in a unique way to an automorphism of H which we denote again by ϕ. It is clear that $\phi^2 = 1$. Let B be the semi-direct product of H and the cyclic group $C_2 = \langle y \rangle$ of order 2 where y acts on H as the automorphism ϕ. Thus we have $yzy = \phi(z)$ for $z \in H$. With the usual identifications we have that X and Y are subgroups of H and H is a normal subgroup of B.

Let Γ_3 be the graph whose vertex set is the set B/X of all left cosets aX, $a \in B$ and in which two vertices aX and bX are connected by an edge if and only if $a^{-1}b \in XyX$. Every $b \in B$ induces a bijection ϕ_b of B/X by left multiplication, i.e., $\phi_b(aX) = baX$. It is clear that ϕ_b is an automorphism of Γ_3 for each $b \in B$ and that the map $B \to \mathrm{Aut}(\Gamma_3)$ which sends b to ϕ_b is a group monomorphism. Hence we may consider B as a subgroup of $\mathrm{Aut}(\Gamma_3)$. It is clear that the action of B on Γ_3 is vertex-transitive.

THEOREM 3. *Γ_3 is a connected regular graph of valence* 3. *The group B is 5-regular on Γ_3.*

Proof. Since $yXy = Y$ it is clear that X and y generate B. This implies that Γ_3 is connected. Since B is vertex-transitive the graph Γ_3 is regular. The fixer in B of the vertex X is the subgroup X of B. The valence of the vertex X is equal to the number of left cosets of X contained in XyX. This number is the same as the index of $yXy \cap X = X \cap Y$ in X, which we know is 3. Thus Γ_3 is a regular connected graph of valence 3.

The three vertices adjacent to X are yX, $x_0 yX$ and $x_0x_4x_0 yX$. The element $x_0x_4 \in X$ fixes the vertex X and permutes cyclically the three vertices adjacent to X. The element y interchanges the adjacent vertices X and yX and hence B is 1-transitive on Γ_3.

The cosets

$$x_0 yX, \quad X, \quad yX, \quad x_5X, \quad yx_0x_5X, \quad (yx_0)^2 x_5X$$

are consecutive vertices of a 5-arc in Γ_3. The element x_3 fixes the first five of these vertices and moves the last one. This is proved by simple computations:

$$\begin{aligned}
x_3x_0 yX &= x_0(x_0x_3)^2 x_3 yX = x_0x_1x_2x_3 yX = x_0 yX,\\
x_3X = X, \qquad x_3 yX &= yx_2X = yX, \qquad x_3x_5X = x_5x_3X = x_5X,\\
x_3 yx_0x_5X &= yx_0x_2x_5X = yx_0x_5(x_5x_2)^2 X = yx_0x_5x_3x_4X = yx_0x_5X,\\
x_3(yx_0)^2 x_5X &= yx_0x_2 yx_0x_5X = yx_0 yx_3x_0x_5X = (yx_0)^2(x_0x_3)^2 x_5X\\
&= (yx_0)^2 x_1x_2x_5X = (yx_0)^2 x_1x_5(x_5x_2)^2 X\\
&= (yx_0)^2 x_1x_5x_3x_4X = (yx_0)^2 x_1x_5X \neq (yx_0)^2 x_5X.
\end{aligned}$$

The last inequation holds because $x_5x_1x_5 \notin X$. Indeed since $(x_1, x_5)^3 = 1$, $x_5x_1x_5 \in X$ implies $x_1x_5x_1 \in X$. This is impossible since $x_1 \in X$ but $x_5 \notin X$.

Since B is 1-transitive and we have found a 5-arc whose all vertices but the last are fixed by x_3, it follows that B is 5-transitive. Since the order of the fixer X of the vertex X is 48 it is clear that B must be 5-regular.

THEOREM 4. *Let (G, A) be a pair consisting of a connected regular graph G of valence 3 and a group A of automorphisms of G which is 5-regular. Then there exists a surjective group homomorphism $\psi\colon B \to A$ and a graph covering map $\theta\colon \Gamma_3 \to G$ such that the diagram*

$$\begin{array}{ccc}
\Gamma_3 & \xrightarrow{\;b\;} & \Gamma_3\\
\theta\big\downarrow & & \big\downarrow\theta\\
G & \xrightarrow[\psi(b)]{} & G
\end{array}$$

commutes for every $b \in B$.

Proof. Fix a 5-arc S in G and define the corresponding special amalgam. By Theorem 2 it is isomorphic to the special amalgam used to define the group H. In fact we shall assume that this special amalgam is identical with (X, Y, ϕ). By the universal property of generalized free products there exists a unique group homomorphism $\psi_0\colon H \to A$ which is identity on $X \cup Y$. Let $\alpha \in A$ be the unique automorphism such that $\alpha \circ S$ is the 5-arc opposite to S.

Then $\alpha^2 = 1$ and ϕ is the restriction of the map $\beta \to \alpha \circ \beta \circ \alpha$ to $X \cup Y$. It is easy to check that $\psi_0(yhy) = \alpha\psi_0(h)\alpha$ for all $h \in H$. Therefore there exists a unique group homomorphism $\psi: B \to A$ such that $\psi(y) = \alpha$ and ψ extends ψ_0. Since $A = \langle X, Y, \phi \rangle$ this homomorphism is surjective.

Let $S(2) = v$ and define a map $\theta_0: B \to G$ by $\theta_0(b) = \psi(b)(v)$. Since $X = A(v)$ we have for $x \in X$ that

$$\theta_0(bx) = \psi(bx)(v) = \psi(b)\,\psi(x)(v) = \psi(b)(x(v)) = \psi(b)(v) = \theta_0(b).$$

Thus θ_0 induces a map $\theta: \Gamma_3 \to G$ such that $\theta(aX) = \theta_0(a)$. Now we have

$$\begin{aligned}\theta(b(a(X)) &= \theta(baX)) = \theta_0(ba) = \psi(ba)(v)\\ &= \psi(b)\,\psi(a)(v) = \psi(b)(\theta_0(a)) = \psi(b)(\theta(aX)),\end{aligned}$$

i.e., the diagrams mentioned in the theorem are commutative.

Let aX and bX be two adjacent vertices of Γ_3. Then $a^{-1}b \in XyX$, i.e., $a^{-1}b = cyd$ for some $c, d \in X$. We have

$$\begin{aligned}\theta(bX) &= \psi(b)(v) = \psi(acyd)(v) = \psi(a)\,c\alpha d(v)\\ &= \psi(a)\,c\alpha(v) = \psi(a)\,c(w)\end{aligned}$$

where $w = S(3) = \alpha(v)$. Since v is adjacent to w and $c \in X$, v and $c(w)$ are adjacent. Consequently $\theta(aX) = \psi(a)(v)$ is adjacent to $\theta(bX)$.

Let $b'X$ be also adjacent to aX but $b'X \neq bX$. Then we can write $a^{-1}b' = c'yd'$ with $c', d' \in X$. We find that $\theta(b'X) = \psi(a)\,c'(w)$. We claim that $\theta(b'X) \neq \theta(bX)$, i.e., $c'(w) \neq c(w)$. Otherwise we would have $c^{-1}c' \in A(w) = Y$ and so $yc^{-1}c'y \in X$. Then $b^{-1}b' = (acyd)^{-1}(ac'yd') = d^{-1}yc^{-1}c'yd' \in X$ giving a contradiction $b'X = bX$.

Since both Γ_3 and G are regular of valence 3 the facts established above imply that θ is a covering map and the theorem is proved.

THEOREM 5. *Γ_3 is a tree.*

Proof. Let G be any regular 3-valent graph and A a 5-regular group of automorphism of G. Let T be an infinite regular 3-valent tree and $\pi: T \to G$ a covering map. By Theorem 3 of [2] the group A can be lifted to T. In particular, there exists a subgroup of Aut(T) which is 5-regular. The universal property of (Γ_3, B) shows that there is a covering $\theta: \Gamma_3 \to T$ and hence Γ_3 must be also a tree. This completes the proof of Theorem 5.

THEOREM 6. *Let A be any 5-regular subgroup of* Aut(Γ_3). *Then A and B are conjugate in* Aut(Γ_3).

Proof. We have two 5-regular pairs (Γ_3, B) and (Γ_3, A). By Theorem 4 there is a covering map $\theta: \Gamma_3 \to \Gamma_3$ and a surjective group homomorphism

$\psi: B \to A$ such that $\psi(b)\theta = \theta b$ for every $b \in B$. But θ must be an automorphism of Γ_3 and hence $\psi(b) = \theta b \theta^{-1}$. This shows that ψ is an isomorphism and that $\theta B \theta^{-1} = A$.

Remark. Similar results are valid for s-regular groups of automorphisms of cubical graphs when $s = 1, 2, 3, 4$. If $s = 2$ or 4 there exist *two* conjugacy classes of s-regular subgroups of Aut(Γ_3).

Of course, it is well-known that there are no s-regular groups of automorphisms of cubical graphs for $s > 5$. For a short and beautiful proof of this see the note [4] of R. Weiss.

An open question. A *doubly infinite arc* in Γ_3 is a map $S: Z \to \Gamma_3$ (=the set of vertices of Γ_3), where Z is the set of integers, such that $S(i)$ and $S(i+1)$ are adjacent and $S(i+1) \neq S(i-1)$ for all $i \in Z$. Every subgroup A of Aut(Γ_3) acts naturally on doubly infinite arcs of Γ_3: if $\alpha \in A$ and S is a doubly infinite arc in Γ_3 then α sends S to $\alpha \circ S$ which is again a doubly infinite arc. The question is the following: Does there exist a subgroup of Aut(Γ_3) which is sharply transitive on doubly infinite arcs of Γ_3?

References

1. N. Biggs, "Algbraic Graph Theory," Cambridge Univ. Press, London/New York, 1974.
2. D. Ž. Djoković, Automorphisms of graphs and coverings, *J. Combinatorial Theory Ser. B* **16** (1974), 243–247.
3. J. Tits, Sur le groupe des automorphisms d'un arbre, *in* "Essays on Topology and Related Topics," Springer-Verlag, 1970, Berlin/New York, 188–211.
4. R. M. Weiss, Uber s-reguläre Graphen, *J. Combinatorial Theory Ser. B* **16** (1974), 229–233.
5. W. J. Wong, Determination of a class of primitive permutation groups, *Math. Z.* **99** (1967), 235–246.