We define, in each finite group $G$, some subgroups of Frattini-type in relation with a saturated formation and with a set of primes and study their properties, especially their influence in the structure of $G$.

1. Introduction

All groups considered are finite. Since the classification of the finite simple groups, one of the main goals in finite group theory has been to obtain a good understanding of the subgroup structure of finite groups. One approach is to study the maximal subgroups. The knowledge of the maximal subgroups of a finite group often yields a wealth of information about the group itself. For instance, one can read off all its primitive permutation representations if its maximal subgroups are known.

Gaschütz in [7] and Bechtell in [1] study the intersection of all self-normalizing maximal subgroups of a finite group. This subgroup is denoted by $L(G)$. There, they proved that $L(G)$ is a nilpotent group. According to our definitions, a maximal subgroup $M$ of a group $G$ is self-normalizing in $G$ if and only if $M$ is $\mathcal{N}$-abnormal in $G$ (here, $\mathcal{N}$ denotes the class of all nilpotent groups). Thus, $L(G) = \bigcap \{ M \leq G \mid M$ is $\mathcal{N}$-abnormal maximal subgroup of $G \}$.

Inspired by Gaschütz's result, we ask ourselves if an analogous result holds for any saturated formation $\mathcal{F}$\(^1\). In this paper, a satisfactory answer is obtained for subgroup-closed saturated formations containing the class of nilpotent groups (Corollary 3.4).

Following this idea, we also study other generalizations of the Frattini subgroup of a finite group and investigate their influence in the structure of the group.

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\(^{1}\) The reader is assumed to be familiar with the theory of Schunck classes and saturated formations of finite groups and their projectors and covering subgroups. The relevant definitions, notations and results can be found in [4-6].
We introduce here three subgroups, $CL_{\mathcal{F}}(G)$, $L_{\mathcal{F}}(G, \pi)$ and $CL_{\mathcal{F}}(G, \pi)$, related with a saturated formation $\mathcal{F}$ and with a set of primes $\pi$ and study their properties. We prove that $CL_{\mathcal{F}}(G)$ has a nilpotent $\mathcal{F}$-residual. Moreover if $\pi = \{p\}$, $L_{\mathcal{F}}(G, p)$ is an $\mathcal{F}_p$-group and $CL_{\mathcal{F}}(G, p)$ is an $\mathcal{F}\mathcal{F}$-group under certain hypothesis about its $\mathcal{F}$-residual. When $\mathcal{F}$ is the class of nilpotent groups or the class of supersoluble groups, we obtain some well-known results.

The $\mathcal{F}\mathcal{F}$-groups (groups which are extension of a soluble group by an $\mathcal{F}$-group) are very important in the theory of saturated formations.

It is well known that the definitions of $\mathcal{F}$-projector and $\mathcal{F}$-covering subgroup make perfectly good sense for finite (non-necessarily soluble) groups and $\mathcal{F}$-projectors exist in all finite groups – yet they need not be $\mathcal{F}$-covering subgroups and, even worse, $\mathcal{F}$-covering subgroups do not exist in general.

However, in $\mathcal{F}\mathcal{F}$-groups the $\mathcal{F}$-projectors enjoy the classical properties of their analogues in soluble groups: they form a conjugacy class of $\mathcal{F}$-covering subgroups (cf. [4-6]).

2. Notation and preliminaries

First recall that a primitive group is a group $G$ such that for some maximal subgroup $U$ of $G$, $U_G = 1$ (where $U_G$ is the intersection of all $G$-conjucates of $U$, i.e., the unique largest normal subgroup of $G$ contained in $U$).

A primitive group is of one of the following types:

1. $\text{Soc}(G)$, the socle of $G$, is an abelian minimal normal subgroup of $G$, complemented by $U$.
2. $\text{Soc}(G)$ is a non-abelian minimal normal subgroup of $G$.
3. $\text{Soc}(G)$ is the direct product of the two minimal normal subgroups of $G$ which are both non-abelian and complemented by $U$.

We will denote by $\mathcal{P}$ the class of all primitive groups and by $\mathcal{P}_i$, $i \in \{1,2,3\}$ the class of all primitive groups of type $i$.

Let $M$ be a maximal subgroup of a group $G$. Then the group $X = G/M_G$ is a primitive group; we say that $M$ is of type $i$ if $X \in \mathcal{P}_i$ ($1 \leq i \leq 3$).

For basic properties of primitive groups, the reader is referred to [5,6].

Let $\mathcal{F}$ be a saturated formation. A maximal subgroup $M$ of a group $G$ is said $\mathcal{F}$-normal in $G$, if the primitive group $G/M_G$ lies in $\mathcal{F}$ and called $\mathcal{F}$-abnormal otherwise.

A subgroup-closed saturated formation is a saturated formation $\mathcal{F}$, such that if $G$ is an $\mathcal{F}$-group, every subgroup of $G$ is an $\mathcal{F}$-group.

Let $\pi$ be a set of primes and $H \leq G$. Denote by $|G:H|_\pi$ the $\pi$-part of $|G:H|$.

The rest of the notation is standard and can be found mainly in [9].

2.1. Definitions. Given a group $G$, a saturated formation $\mathcal{F}$ and a set of primes $\pi$, we consider the following families of maximal subgroups of $G$:
\[ \Phi(G) = \{ M \mid M \text{ is an } \mathcal{F}\text{-abnormal maximal subgroup of } G \}. \]
\[ \Phi(G, \pi) = \{ M \mid M \text{ is an } \mathcal{F}\text{-abnormal maximal subgroup of } G \text{ and } |G:M|_\pi = 1 \}. \]
\[ \mathcal{C}\Phi(G) = \{ M \mid M \text{ is an } \mathcal{F}\text{-abnormal maximal subgroup of } G \text{ and } |G:M| \text{ is composite} \}. \]
\[ \mathcal{C}\Phi(G, \pi) = \{ M \mid M \text{ is an } \mathcal{F}\text{-abnormal maximal subgroup of } G, |G:M|_\pi = 1 \}. \]

Now we define the corresponding characteristic subgroups:
\[ L\Phi(G) = \bigcap \{ M \mid M \in \Phi(G) \} \text{ if } \Phi(G) \text{ is non-empty, otherwise } L\Phi(G) = G. \]
\[ L\Phi(G, \pi) = \bigcap \{ M \mid M \in \Phi(G, \pi) \} \text{ if } \Phi(G, \pi) \text{ is non-empty, otherwise } L\Phi(G, \pi) = G. \]
\[ \mathcal{C}L\Phi(G) = \bigcap \{ M \mid M \in \mathcal{C}\Phi(G) \} \text{ if } \mathcal{C}\Phi(G) \text{ is non-empty, otherwise } \mathcal{C}L\Phi(G) = G. \]
\[ \mathcal{C}L\Phi(G, \pi) = \bigcap \{ M \mid M \in \mathcal{C}\Phi(G, \pi) \} \text{ if } \mathcal{C}\Phi(G, \pi) \text{ is non-empty, otherwise } \mathcal{C}L\Phi(G, \pi) = G. \]

It is clear that \( \Phi(G) \) is contained in all of them. In fact, \( \Phi(G) \subseteq L\Phi(G) \cap \mathcal{C}L\Phi(G) \).

Denoting by \( S(G) \) whatever of the above defined four subgroups, it is easy to prove that \( S(G)K/K \leq S(G/K) \) for every \( K \trianglelefteq G \), and if \( K \trianglelefteq S(G) \), \( S(G/K) = S(G)K/K \).

For \( \pi = \{ p \} \), we simply denote \( L\Phi(G, \pi) = L\Phi(G, p) \), etc. Notice that if \( p \) is a prime not dividing the order of \( G \), then \( L\Phi(G) = L\Phi(G, p) \).

### 3. The subgroups \( L\Phi(G, p) \) and \( \mathcal{C}L\Phi(G) \)

In this section, \( \mathcal{F} \) will denote a subgroup-closed saturated formation containing \( \mathcal{N} \), the class of nilpotent groups.

Bhattacharya and Mukherjee, in [9], study the subgroup \( \Phi_p(\mathcal{G}) = \bigcap \{ M \mid M \text{ is a maximal subgroup of } G, |G:M|_p = 1 \}. \) There, they prove that \( \Phi_p(G) \) has a normal Sylow \( p \)-subgroup \( P \), such that \( \Phi_p(G)/P \) is a nilpotent group. This is to say that \( \Phi_p(G) \in \mathcal{P}_p\mathcal{N} \) or, equivalently, the \( \mathcal{N} \)-residual of \( \Phi_p(G) \) is a \( p \)-group.

Here, we prove that the subgroup \( L\Phi(G, p) \) lies in \( \mathcal{P}_p\mathcal{F} \) and study the intersection of two of these subgroups corresponding to two distinct primes. Moreover, we prove that the \( \mathcal{F} \)-residual of \( \mathcal{C}L\Phi(G) \) is nilpotent.

#### 3.1. Lemma. If a group \( G \in \mathcal{F} \) contains a normal \( \mathcal{F} \)-projector, then \( G \) is an \( \mathcal{F} \)-group.

**Proof.** Suppose the result is false and let \( G \) be a minimal counterexample. Let \( D \) be an \( \mathcal{F} \)-projector of \( G \) such that \( D \ntrianglelefteq G \). Then \( D \) is a proper subgroup of \( G \). Let \( M \) be a maximal subgroup of \( G \) such that \( D \leq M \). Now, \( M \) is \( \mathcal{F} \)-abnormal in \( G \). Since \( G \in \mathcal{F} \), \( D \) is an \( \mathcal{F} \)-covering subgroup of \( G \). Thus, \( D \) is a normal \( \mathcal{F} \)-projector of \( M \). From minimality of \( G \), we have \( M \in \mathcal{F} \) and \( D = M \trianglelefteq G \). This implies that \( G/M \in \mathcal{N} \subseteq \mathcal{F} \), a contradiction. \( \square \)
3.2. Lemma. Let $G$ be a group. The $\mathcal{F}$-residual, $T$, of $L_\mathcal{F}(G, p)$ is contained in $\Phi_p(G)$. Thus, $T$ is a soluble group.

**Proof.** Suppose that $T$ is not contained in $\Phi_p(G)$. Then there exists a maximal subgroup $M$ of $G$, such that $|G: M|_p = 1$ and $G = TM$. Since $\mathcal{F}$ is subgroup-closed, $T \leq G^\mathcal{F}$ and $G = G^\mathcal{F}M$. Thus, $M$ is an $\mathcal{F}$-abnormal maximal subgroup of $G$. This implies that $L_\mathcal{F}(G, p) \leq M$, a contradiction. □

3.3. Theorem. Let $G$ be a group and $p$ a prime. Then, $L_\mathcal{F}(G, p)$ is an $\mathcal{F}_p\mathcal{F}$-group.

**Proof.** Denote by $S(X) = L_\mathcal{F}(X, p)$ for every group $X$. We split the proof into three steps:

**Step 1:** Every Sylow $p$-subgroup of $G^\mathcal{F} \cap S(G)$ is a normal subgroup of $G$. Let $P$ be a Sylow $p$-subgroup of $G^\mathcal{F} \cap S(G)$; we have that $G = (G^\mathcal{F} \cap S(G))N_G(P)$. Suppose that $P$ is not normal in $G$. Then there exists a maximal subgroup $M$ of $G$ containing $N_G(P)$. We have that $G = G^\mathcal{F}M$ and $M$ is $\mathcal{F}$-abnormal in $G$. Since $|G:M|_p = 1$, $S(G) \leq M$, a contradiction. Thus $P$ is a normal subgroup of $G$.

**Step 2.** If the $\mathcal{F}$-residual of $S(G)$ is a $p'$-group, then $S(G)$ is an $\mathcal{F}$-group. Denote by $T$ the $\mathcal{F}$-residual of $S(G)$ and suppose that $T$ is a $p'$-group. Let $M$ be an $\mathcal{F}$-abnormal maximal subgroup of $G$. If $T$ is not contained in $M$, then $G = TM$. This implies $|G:M|_p = 1$. But then $S(G) \leq M$, a contradiction. Thus, $T$ is contained in $L_\mathcal{F}(G)$. Now, let $D$ be an $\mathcal{F}$-projector of $S(G)$. Since $T$ is a soluble group, Proj$_\mathcal{F}(S(G))$ is a conjugacy class of subgroups of $S(G)$. Then, $G = S(G)N_G(D)$. If $N_G(D)$ is a proper subgroup of $G$, there exists a maximal subgroup $M$ of $G$ such that $N_G(D) < M$. On the other hand, $S(G) = TD$. Then, $G = TM$ and $M$ is an $\mathcal{F}$-abnormal maximal subgroup of $G$. Therefore, $T \leq L_\mathcal{F}(G) \leq M$ and $G = M$, a contradiction. Consequently, $D$ is a normal subgroup of $G$. Applying Lemma 3.1, we have that $S(G)$ is an $\mathcal{F}$-group.

**Step 3.** Conclusion. If $T$ is a $p'$-group, then $S(G) \in \mathcal{F} \subset \mathcal{F}_p\mathcal{F}$. Therefore, we can suppose that $p$ divides the order of $T$. By Step 1, there exists a Sylow $p$-subgroup $P$ of $T$ such that $P$ is a normal subgroup of $G$. Now, $S(G/P) = S(G)/P$ and the $\mathcal{F}$-residual of $S(G)/P$ is $T/P$. Since $T/P$ is a $p'$-group, we have that $T = P$ by step 2. Consequently, $S(G)$ is an $\mathcal{F}_p\mathcal{F}$-group. □

It is not true that $L_\mathcal{F}(G, p)$ is an $\mathcal{F}$-group as the following example shows:

**Example 1.** Consider $\mathcal{F} = \mathcal{U}$, the formation of supersoluble groups, and $G = G_1 \times G_2$ the direct product of a non-abelian group $G_1$ of order $qr$ ($5 \leq q < r$, $q, r$ prime numbers) and $G_2$, the alternating group of degree 4. The $\mathcal{U}$-abnormal maximal subgroups of $G$ are precisely the Hall 2'-subgroups of $G$. Then, $L_\mathcal{U}(G, 2) = G$ which is not supersoluble.

However, if $p$ is a prime not dividing the order of $G$ we obtain:
3.4. Corollary. For every group $G$, $L_{\mathcal{F}}(G)$ is an $\mathcal{F}$-group. \qed

Taking $\mathcal{F}=\mathcal{N}$ in the above corollary, we obtain the well-known result of Gaschütz mentioned in the introduction: $L(G)$ is nilpotent for every group $G$.

If $\mathcal{F}$ does not contain $\mathcal{N}$ the above theorems fail. Take, for instance, $\mathcal{F}=\mathcal{P}_3$, the class of 3-groups and $G=\text{SL}(2,3)$. The $\mathcal{P}_3$-abnormal maximal subgroups of $G$ are $\{N_G(S) \mid S \in \text{Syl}_3(G)\}$ and $L_{\mathcal{F}}(G,3)=L_{\mathcal{F}}(G)=Z(G)$ which is not a 3-group ($Z(G)\cong Z_2$).

3.5. Corollary. If $\mathcal{F}$ is composed by soluble groups, for each group $G$ and for each prime $p$, $L_{\mathcal{F}}(G,p)$ is a soluble group. In particular, if $\mathcal{F}=\mathcal{N}$ the formation of nilpotent groups, $L_{\mathcal{N}}(G,p)$ is a metanilpotent group. \qed

3.6. Corollary. Let $G$ be a group such that $G^{\mathcal{F}}$ is soluble. If each $\mathcal{F}$-abnormal maximal subgroup of $G$ is an $\mathcal{F}$-group, then $G^{\mathcal{F}}$ is a $p$-group for some prime $p$.

Proof. It is clear that $\text{Proj}_{\mathcal{F}}(G) \supseteq \{M \mid M$ is an $\mathcal{F}$-abnormal maximal subgroup of $G\}$. Since $G \in \mathcal{F}G$, $\text{Proj}_{\mathcal{F}}(G)$ is a conjugacy class of subgroups of $G$. Let $M$ be an $\mathcal{F}$-abnormal maximal subgroup of $G$ and $p$ the prime such that $|G:M|=p^r$. Then, $L_{\mathcal{F}}(G,p)=G \in \mathcal{F}_pG$ by Theorem 3.3. \qed

3.7. Proposition. Let $G$ be a group such that either $G^{\mathcal{F}}$ is a soluble group or for some prime $p$, $L_{\mathcal{F}}(G,p)$ is a soluble group. Then: $L_{\mathcal{F}}(G)=L_{\mathcal{F}}(G,p) \cap L_{\mathcal{F}}(G,q)$ for every prime $q \neq p$.

Proof. Denote by $L_p=L_{\mathcal{F}}(G,p)$ and $L_q=L_{\mathcal{F}}(G,q)$. Suppose that there exists an $\mathcal{F}$-abnormal maximal subgroup $M$ of $G$, such that $L_p \cap L_q$ is not contained in $M$. Then $G=ML_p=ML_q=G^{\mathcal{F}}M$. Now, under the hypothesis of the proposition $M$ is a maximal subgroup of type 1. Then $|G:M|=r^t$ for some prime $r$. Thus, $|G:M|_p=1$ or $|G:M|_q=1$. Consequently, $L_p \leq M$ or $L_q \leq M$, a contradiction. Thus, $L_{\mathcal{F}}(G)=L_p \cap L_q$. \qed

Next, we describe the subgroups $L_{\mathcal{F}}(G)$ and $L_{\mathcal{F}}(G,p)$, where $\mathcal{U}$ denotes the class of supersoluble groups. For this, we need a preliminary lemma.

3.8. Lemma. Let $G$ be a group and $M$ a maximal subgroup of $G$ of type 2 such that $|G:M|$ is prime. Then there exists a maximal subgroup $T$ of $G$ of type 2 such that $|G:T|$ is composite and $M_G=T_G$.

Proof. Arguing by induction on $|G|$, we can suppose $M_G=1$. Then, $G \in \mathcal{P}_2$. Suppose $|G:M|=p$, then $|M|$ divides $(p-1)!$. Thus, $p$ is the largest prime dividing the order of $G$. Let $P$ be a Sylow $p$-subgroup of $\text{Soc}(G)$. Then, $N_G(P)<G$ and
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\[ G = \text{Soc}(G)N_G(P). \]

Let \( T \) be a maximal subgroup of \( G \) such that \( N_G(P) \leq T \). Then, \( G = \text{Soc}(G)T \). Consequently, \( T_G = 1 \) and \( |G : T| \) is composite. \( \square \)

Applying Lemma 3.8, it is easy to see that \( L_\mathcal{U}(G) = \bigcap \{ M \mid M \) is a maximal subgroup of \( G \) such that \( |G : M| \) is composite \} for every group \( G \). Thus, taking \( \mathcal{F} = \mathcal{U} \) in Corollary 3.4, we obtain a well-known result of Bathia (see [2, Theorem 3]): \( L_\mathcal{U}(G) \) is supersoluble.

**3.9. Proposition.** Let \( G \) be a group and \( p \) a prime. If either \( p \) does not divide the order of \( L_\mathcal{U}(G, p) \) or \( p \) is the largest prime dividing the order of \( L_\mathcal{U}(G, p) \), then \( L_\mathcal{U}(G, p) \) has a Sylow tower of supersoluble type.

**Proof.** If \( p \) does not divide the order of \( L_\mathcal{U}(G, p) \), then \( L_\mathcal{U}(G, p) = L_\mathcal{U}(G, p) \) is supersoluble by Corollary 3.4 and the result is true. Suppose that \( p \) is the largest prime dividing the order of \( L_\mathcal{U}(G, p) \). Then \( L_\mathcal{U}(G, p) \) has a normal Sylow \( p \)-subgroup \( P \). By Theorem 3.3, \( L_\mathcal{U}(G, p)/P \) is supersoluble. Then \( L_\mathcal{U}(G, p) \) has a Sylow tower of supersoluble type. \( \square \)

**Example 2.** The \( \mathcal{U} \)-abnormal maximal subgroups of \( G = \text{Alt}(4) \), the alternating group of degree 4, are its Sylow 3-subgroups. Then \( L_\mathcal{U}(G, 2) = G \) which does not possess a Sylow tower of supersoluble type. Here, 2 is not the largest prime dividing \( |L_\mathcal{U}(G, 2)| \).

Finally, we study the subgroup \( \text{CL}_\mathcal{F}(G) \).

**3.10. Proposition.** Let \( G \) be a group. Then the \( \mathcal{F} \)-residual of \( \text{CL}_\mathcal{F}(G) \) is nilpotent, i.e. \( \text{CL}_\mathcal{F}(G) \in \mathcal{N}_\mathcal{F} \).

**Proof.** We use induction on \( |G| \). Denote by \( T \) the \( \mathcal{F} \)-residual of \( \text{CL}_\mathcal{F}(G) \). We can suppose \( T \neq 1 \). Let \( p \) be the largest prime dividing the order of \( T \). Then, \( T \) has a normal Sylow \( p \)-subgroup \( P \). Let \( N \) be a minimal normal subgroup of \( G \) such that \( N \leq P \). By induction \( T/N \) is nilpotent. If \( B \) is another minimal normal subgroup of \( G \), then \( TB/B \) is nilpotent. Therefore, \( T \) is nilpotent and the proposition is proved. Thus, we can assume that \( G \) possesses a unique minimal normal subgroup \( N \), which is a \( p \)-group. Suppose that \( N \leq \Phi(G) \). Then, \( T/T \cap \Phi(G) \) is nilpotent. Applying [1, Corollary 2.3.1], \( T \) is nilpotent. Consequently, we can suppose that \( G \) is a primitive group of type 1 and there exists a maximal subgroup \( M \) of \( G \) such that \( G = MN \). Then \( |G : M| \) is prime and \( N \) is cyclic. Therefore, \( G \) is supersoluble and the commutator subgroup, \( G' \), of \( G \) is nilpotent. Since \( \mathcal{F} \) contains the formation of nilpotent groups, \( T \leq G' \). Thus, \( T \) is nilpotent and the proposition is proved. \( \square \)

It is clear that we cannot improve the above theorem by replacing nilpotent by abelian; if \( G \) is a supersoluble group whose \( \mathcal{N} \)-residual is nilpotent of class 2, (for
instance, take \( P = \langle x, y \mid x^3 = y^3 = z^3 = 1, z = [x, y] \rangle \) the extraspecial group of order 27 and exponent 3 and \( \alpha \) an automorphism of \( P \) such that \( x^\alpha = x^{-1}, y^\alpha = y^{-1} \) and consider the semidirect product \( G = P(\alpha) \), then \( G \) obviously verifies that the \( \mathcal{N} \)-residual of \( \text{CL}_{\mathcal{F}}(G) = G \) is a non-abelian nilpotent group.

In general, \( \text{CL}_{\mathcal{F}}(G) \) is not an \( \mathcal{F} \)-group. Take for instance the group \( G \) in Example 1 and \( \mathcal{F} = \mathcal{N} \). The \( \mathcal{N} \)-abnormal maximal subgroups of \( G \) of composite index are the Hall 2'-subgroups again and \( \text{CL}_{\mathcal{F}}(G) = G_1 \) which is not nilpotent.

Obviously, \( \text{CL}_{\mathcal{F}}(G) \) is not always a supersoluble group: take, for example, a non-supersoluble formation \( \mathcal{F} \) and a non-supersoluble \( \mathcal{F} \)-group. In fact, we next prove that the formations \( \mathcal{F} \) such that \( \text{CL}_{\mathcal{F}}(G) \) is supersoluble for all groups \( G \) are exactly those composed by supersoluble groups.

### 3.11. Theorem

**Theorem.** \( \mathcal{F} \subseteq \mathcal{U} \), the formation of supersoluble groups, if and only if \( \text{CL}_{\mathcal{F}}(G) \) is supersoluble for every group \( G \).

**Proof.** Suppose, first, that \( \mathcal{F} \) is contained in \( \mathcal{U} \). We show that \( \text{CL}_{\mathcal{F}}(X) \) is supersoluble for every group \( X \). Assume that the result is false and take \( G \) a minimal counterexample. Then \( T = \text{CL}_{\mathcal{F}}(G) \neq 1 \). Let \( N \) be a minimal normal subgroup of \( G \) such that \( N \leq T \). From minimality of \( G \), we have that \( \text{CL}_{\mathcal{F}}(G)/N = \text{CL}_{\mathcal{F}}(G/N) \) is supersoluble. Consequently, the \( \mathcal{U} \)-residual of \( \text{CL}_{\mathcal{F}}(G) \) is contained in \( N \). Thus, \( N = \text{CL}_{\mathcal{F}}(G) \). Let \( R \) be an \( \mathcal{U} \)-projector of \( \text{CL}_{\mathcal{F}}(G) \). Since \( \text{Proj}_{\mathcal{U}}(\text{CL}_{\mathcal{F}}(G)) \) is a conjugacy class of subgroups of \( \text{CL}_{\mathcal{F}}(G) \), we have that \( G = \text{CL}_{\mathcal{F}}(G)N_G(R) = NN_G(R) \). If \( N_G(R) \) is a proper subgroup of \( G \), \( N_G(R) \) is an \( \mathcal{F} \)-abnormal maximal subgroup of \( G \) of prime index. Thus, \( N \) is cyclic. Consequently, \( \text{CL}_{\mathcal{F}}(G) \) is supersoluble, a contradiction. Therefore, \( R \) is a normal subgroup of \( G \). Applying Lemma 3.1, we conclude that \( \text{CL}_{\mathcal{F}}(G) \) is supersoluble, final contradiction.

Conversely, suppose \( \text{CL}_{\mathcal{F}}(X) \) is supersoluble for every group \( X \). If \( G \) is an \( \mathcal{F} \)-group, all maximal subgroups of \( G \) are \( \mathcal{F} \)-normal. Thus, \( \text{CL}_{\mathcal{F}}(G) = G \) which is supersoluble. Therefore, \( \mathcal{F} \) is contained in \( \mathcal{U} \). □

### 4. The subgroup \( \text{CL}_{\mathcal{F}}(G, \pi) \)

In this section, \( \mathcal{F} \) will denote a subgroup-closed saturated formation.

#### 4.1. Theorem

**Theorem.** Let \( G \) be a group and \( p \) a prime. If the \( \mathcal{F} \)-residual of \( \text{CL}_{\mathcal{F}}(G, p) \) is \( p \)-soluble, then it is soluble.

**Proof.** We use induction on the order of \( G \). Denote by \( T \) the \( \mathcal{F} \)-residual of \( \text{CL}_{\mathcal{F}}(G, p) \). The result is clear if \( T = 1 \). Thus, we can suppose \( T \neq 1 \). Let \( N \) be a minimal normal subgroup of \( G \) such that \( N \leq T \). Since \( T \) is \( p \)-soluble, \( N \) is a \( p \)-group or a \( p' \)-group. If \( N \) is a \( p \)-group, then \( N \) is a soluble group. By induction \( T/N \) is soluble. Thus, \( T \) is soluble and the theorem is proved.
Consequently, we can assume that $N$ is a $p'$-group. Let $q$ be the largest prime dividing the order of $N$. Let $Q$ be a Sylow $q$-subgroup of $N$. Then $G = NN_G(Q)$. If $N_G(Q)$ is a proper subgroup of $G$, there exists a maximal subgroup $M$ of $G$ such that $N_G(Q) \leq M$. Then, $G = NM$ and $M$ is an $\mathcal{F}$-abnormal maximal subgroup of $G$. Since $N$ is a $p'$-group, $|G:M|_p = 1$. If $|G:M|$ is composite, $T \leq CL_{\mathcal{F}}(G, p) \leq M$, a contradiction. Thus, $|G:M|$ is a prime number. So, $|G:M| = 1 + kq$ is a prime dividing the order of $N$ which is a contradiction to the fact that $q$ is the largest prime dividing $|N|$. Hence $Q$ is a normal subgroup of $G$ and $N = Q$. By induction, $T/N$ is a solvable group. Since $N$ is solvable, we have that $T$ is a solvable group and the theorem is proved.

4.2. Corollary. Suppose that $\mathcal{F}$ is composed by solvable groups. Let $G$ be a group. Then: $CL_{\mathcal{F}}(G, p)$ is a solvable group if and only if the $\mathcal{F}$-residual of $CL_{\mathcal{F}}(G, p)$ is a $p$-soluble group.

Notice that if $\mathcal{F}$ is the trivial formation, $CL_{\mathcal{F}}(G, p)$ is the subgroup $S(G)$ defined in [9]. In this case the $\mathcal{F}$-residual of $S(G)$ is $S(G)$ itself. Thus, [9, Theorem 8(ii)] can be improved in the following way: $S(G)$ is solvable if and only if $S(G)$ is $p$-soluble.

4.3. Theorem. Let $\pi$ be a set of primes and $\mathcal{H}$ be the class of all groups $T$ such that $CL_{\mathcal{F}}(T, \pi) = T$. Then $\mathcal{H}$ is a Schunck class. Moreover, if $G$ is a soluble group with a normal $\mathcal{H}$-projector, then $G$ lies in $\mathcal{H}$.

Proof. Denote by $S(X) = CL_{\mathcal{F}}(X, \pi)$ for every group $X$. Clearly, $\mathcal{H}$ is a homomorph. Let $G$ be a group such that $G/\Phi(G) \in \mathcal{H}$. $S(G)/\Phi(G) = S(G/\Phi(G)) = G/\Phi(G)$. Then, $S(G) = G$ and $G \in \mathcal{H}$. Thus, $\mathcal{H}$ is a saturated homomorph. Now, let $G$ be a group in the boundary of $\mathcal{H}$. Then, $S(G) = 1$; otherwise, let $N$ be the minimal normal subgroup of $G$ such that $N \leq S(G)$. Since $S(G/N) = S(G)/N$ and $G/N \in \mathcal{H}$, we have that $S(G) = G$, a contradiction; thus, $S(G) = 1$. Now, let $A$ be a minimal normal subgroup of $G$. There exists an $\mathcal{F}$-abnormal maximal subgroup of $G$ such that $|G:M|_\pi = 1$, $|G:M|$ is composite and $G = AM$. If $B$ is another minimal normal subgroup of $G$, then $G = BM$; otherwise, $M/B$ will be an $\mathcal{F}$-abnormal maximal subgroup of $G$ such that $|G:B:M/B|_\pi = 1$ and $|G:B:M/B|$ composite. Thus, $S(G/B)$ is contained in $M/B$, a contradiction. Consequently, $G = BM$ for every minimal normal subgroup $B$ of $G$. Thus, $G = 1$ and $G$ is a primitive group.

On the other hand, if $q$ is a prime, then the cyclic group of order $q$ lies in $\mathcal{H}$. Then, if $G$ is a solvable group with a normal $\mathcal{H}$-projector, arguing as in the proof of Lemma 3.1, we conclude that $G \in \mathcal{H}$.

4.4. Theorem. Let $G$ be a group and $\pi$ a set of primes such that $CL_{\mathcal{F}}(G, \pi)$ is a soluble group. Then $CL_{\mathcal{F}}(G, \pi)$ is an $\mathcal{H}$-group.
**Proof.** Denote again $S(X) = \text{CL}_\mathcal{F}(X, \pi)$ for every group $X$. Arguing by induction on the order of $G$, we can assume that $S(G) \neq 1$. Let $N$ be a minimal normal subgroup of $G$ such that $N \leq S(G)$. Then, $N$ is a $q$-group of a prime $q$. We distinguish two cases:

(a) $q \in \pi$. If $M$ is a maximal subgroup of $S(G)$ such that $|S(G) : M|_q = 1$, then $N \leq M$. Thus, $N \leq \Phi_q(S(G)) \leq S(S(G))$. By induction, $S(S(G/N)) = S(G/N)$. Now, $S(S(G/N)) = S(S(G))/N$ and $S(G/N) = S(G)/N$. Thus, $S(S(G)) = S(G)$ and $S(G)$ lies in $\mathcal{H}$.

(b) $q \notin \pi$. Let $M$ be a maximal subgroup of $G$. If $N$ is not contained in $M$, then $G = NM$. So $|G : M|_N = 1$. If $|G : M|$ is composite, then $S(G) \leq M$ and so $G = M$, a contradiction. Thus $|G : M|$ is a prime number. Thus, $N$ is a cyclic group. Therefore, $N$ is contained in every maximal subgroup $B$ of $S(G)$ such that $|S(G) : B|$ is composite. Thus, $N \leq S(S(G))$. Arguing as in (a), we conclude that $S(G) \in \mathcal{H}$. Consequently, we can assume that $N \leq \Phi(G)$. Consider $P$ an $\mathcal{H}$-projector of $S(G)$. Since $S(G)/N \in \mathcal{H}$, we have that $S(G) = NP$. Now, $\text{Proj}_\mathcal{H}(S(G))$ is a conjugacy class of subgroups of $S(G)$. Then, $G = S(G)N_G(P)$. Then $G = N_G(P)$. Applying the above theorem, $S(G)$ is an $\mathcal{H}$-group. 

$\text{CL}_\mathcal{F}(G, \pi)$ in the case $\mathcal{F} = \{1\}$ is the subgroup $S_\pi(G)$ defined in [2]. Therefore, in [2, Theorem 10] the hypothesis of $\pi$-solubility of $G$ is unnecessary.

**Acknowledgment**

This paper is part of the author’s doctoral thesis at the University of Valencia (Spain) under the supervision of Pr. Dr. L.M. Ezquerro. The author is grateful him for his numerous suggestions about the preprint.

We also thank the referee his kind suggestions which helped to improve our exposition.

**References**


