



ELSEVIER

SCIENCE @ DIRECT®

Physics Letters B 554 (2003) 207–216

PHYSICS LETTERS B

www.elsevier.com/locate/npe

Abelian subset of second class constraints

F. Loran^{a,b}

^a *Department of Physics, Isfahan University of Technology (IUT), Isfahan, Iran*

^b *Institute for Studies in Theoretical Physics and Mathematics (IPM), PO Box 19395-5531, Tehran, Iran*

Received 6 November 2002; received in revised form 24 December 2002; accepted 27 December 2002

Editor: T. Yanagida

Abstract

We show that after mapping each element of a set of second class constraints to the surface of the other ones, half of them form a subset of Abelian first class constraints. The explicit form of the map is obtained considering the most general Poisson structure. We also introduce a proper redefinition of second class constraints that makes their algebra symplectic.

© 2003 Elsevier Science B.V. Open access under [CC BY license](http://creativecommons.org/licenses/by/3.0/).

1. Introduction

When Dirac introduced constrained systems [1], he classified constraints as first class and second class. First class constraints have been interesting since they turned out to be generators of gauge transformation. These constraints introduced a new class of symmetries, which for example, lead to Ward identities in the context of renormalization [2]. The main requirement in quantization of first class constraints is the covariance of observables under gauge transformations. In Dirac quantization, this requirement is satisfied by considering physical states as null eigen states of the generator of gauge transformation. The same idea is followed in BRST where a nilpotent BRST-charge generates BRST-transformation [3].

There are two major difficulties in both Dirac quantization and BRST. In general, first class constraints satisfy a closed algebra in which the structure coefficients are some functions of phase space coordinates. Consistency of these methods of quantization depends on the possibility of a definite operator ordering; the structure coefficients should stand on the left side of first class constraints. Another problem is obtaining the explicit form of the generator of gauge transformation or BRST-charge. Both difficulties can be overcome by making the first class constraints Abelian [4].

Second class constraints were thought to be redundant degrees of freedom that one should get rid of them before quantization, for example, by using Dirac bracket instead of Poisson bracket. But second class constraints are more important. For example, in Ref. [5], the gauge theory of second class systems is discussed. Or in closed string theory, it is claimed that boundary conditions lead to a set of second class constraints which give rise to

E-mail address: loran@cc.iut.ac.ir (F. Loran).

non-commutativity of spacetime [6]. On the other hand, covariant quantization, in general, is not consistent with classification of constraints as first and second class [7]. Consequently, we need a general method of quantization which treats both classes on the same footing.

- (1) One possibility is to convert second class constraints to first class [8]. Given a constraint system possessing second class constraints, in principle, one can consider an extended phase space and redefine second class constraints and the Hamiltonian to find an equivalent first class system. There are two difficulties in doing so. Firstly, it is not so easy to find out such redefinitions in general cases. Secondly, assuming that the conversion is done, one may still encounter the above mentioned difficulties in quantization of first class constraints. As is well known, all these problems can be remedied most easily provided one makes the algebra of second class constraints symplectic;
- (2) Another possibility is to consider half of second class constraints as first class constraints and the remaining ones as gauge fixing conditions [2]. This method, for example, is used to study gauge invariance in the Proca model [9].

In this Letter, we prove that after mapping each element of a set of second class constraints to the surface of the other ones, half of them form an *Abelian* subset. In addition we present a general method for redefining second class constraints to make their algebra symplectic. Although this method may not preserve covariance but it is still interesting since it works globally, and for a general Poisson structure. Therefore, it provides a simple conversion of second class constraints to first class ones.

In Ref. [4] it is shown that first class constraints become Abelian when they are mapped to the surface of each other. Thus, it seems that in this way, one can obtain Abelian subset of a given set of constraints in the most simple way.

The organization of Letter is as follows. In Section 2, we introduce necessary definitions and lemmas. The method is introduced in Section 3. We conclude our results in Section 4.

2. Definitions and lemmas

In this section we provide some general tools necessary for arguments of the next section. Consider a phase space defined by a set of coordinates z^μ satisfying the Poisson algebra,

$$\{z^\mu, z^\nu\} = J^{\mu\nu}(z), \quad (1)$$

in which $J^{\mu\nu}(z)$ is a full rank anti-symmetric tensor, e.g., the symplectic two form:

$$J = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}. \quad (2)$$

Assume a pair of conjugate functions $\phi(z)$ and $\omega(z)$ in \mathcal{F} , satisfying the relation,

$$\{\phi, \omega\} = 1, \quad (3)$$

where \mathcal{F} stands for the set of real analytic functions of the phase space coordinates. In fact for a given $\phi \in \mathcal{F}$, using the Cauchy–Kowalevski theorem [10], one can show that there exist at least one function $\omega \in \mathcal{F}$ that satisfies the relation,

$$\{\phi, \omega\} = a^\mu(z) \frac{\partial \omega}{\partial z^\mu} = 1, \quad a^\mu = \frac{\partial \phi}{\partial z^\nu} J^{\nu\mu}. \quad (4)$$

Corresponding to each $\xi \in \mathcal{F}$ an operator $\hat{\xi} : \mathcal{F} \rightarrow \mathcal{F}$ can be defined as follows,

$$\hat{\xi} \chi = \{\xi, \chi\}, \quad \chi \in \mathcal{F}. \quad (5)$$

It is easy to verify that,

$$\hat{\xi}(\chi_1 \chi_2) = \{\xi, (\chi_1 \chi_2)\} = \chi_1 \{\xi, \chi_2\} + \{\xi, \chi_1\} \chi_2 = (\hat{\xi} \chi_1) \chi_2 + \chi_1 (\hat{\xi} \chi_2). \quad (6)$$

Considering the operators $(\hat{\phi}, \hat{\omega})$ where $\{\phi, \omega\} = 1$, from Eq. (3) one can show that these operators satisfy the following relations:

$$[\hat{\phi}, \hat{\omega}] = 0, \quad (7)$$

$$[\hat{\phi}, \phi] = [\hat{\omega}, \omega] = 0, \quad (8)$$

$$[\hat{\phi}, \omega] = [\phi, \hat{\omega}] = 1. \quad (9)$$

These properties can be easily verified. For example, for an arbitrary function $\xi \in \mathcal{F}$, we have,

$$[\hat{\phi}, \hat{\omega}] \xi = \{\phi, \{\omega, \xi\}\} - \{\omega, \{\phi, \xi\}\} = -\{\xi, \{\phi, \omega\}\} = -\{\xi, 1\} = 0, \quad (10)$$

where in the second equality we have used the Jacobi identity. Considering the operators \hat{P}_ϕ and \hat{P}_ω [9],

$$\hat{P}_\phi \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \phi^n \hat{\omega}^n, \quad \hat{P}_\omega \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \omega^n \hat{\phi}^n, \quad (11)$$

one can use Eqs. (7), (8) to show that $[\hat{\phi}, \hat{P}_\phi] = [\hat{\omega}, \hat{P}_\omega] = 0$, and consequently,

$$[\hat{P}_\phi, \hat{P}_\omega] = 0. \quad (12)$$

Lemma 1. *The operators \hat{P}_ϕ and \hat{P}_ω satisfy the following properties:*

$$\hat{\omega} \hat{P}_\phi = 0, \quad (13)$$

$$\hat{\phi} \hat{P}_\omega = 0. \quad (14)$$

Proof. We prove the first equality. The second equality can be proved in the same way. Using Eq. (9), one can show that $[\hat{\omega}, \phi^n] = -n\phi^{n-1}$. Thus,

$$\hat{\omega} \hat{P}_\phi = [\hat{\omega}, \hat{P}_\phi] + \hat{P}_\phi \hat{\omega} = \sum_{n=0}^{\infty} \frac{1}{n!} [\hat{\omega}, \phi^n] \hat{\omega}^n + \hat{P}_\phi \hat{\omega} = - \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \phi^{n-1} \hat{\omega}^n + \hat{P}_\phi \hat{\omega} = 0. \quad (15)$$

Lemma 2. *Given conjugate functions $\omega, \phi \in \mathcal{F}$, the operator \hat{P}_ϕ is the projection map to the subspace of the phase space defined by $\phi = 0$.*

Proof. Using Eq. (13), it can be shown that $\hat{P}_\phi^2 = \hat{P}_\phi$. Assuming the (canonical) coordinate transformation,

$$z^\mu \rightarrow \phi, \omega, Z^{\mu'}, \quad (16)$$

one verifies that $\hat{\omega} = -\frac{\partial}{\partial \phi}$. Therefore, for an arbitrary $\xi \in \mathcal{F}$,

$$\hat{P}_\phi \xi(\phi, \omega, Z^{\mu'}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \phi^n \frac{\partial^n}{\partial \phi^n} \xi(\phi, \omega, Z^{\mu'}) = \xi(0, \omega, Z^{\mu'}) = \xi|_\phi. \quad (17)$$

Corollary 1. *The operator \hat{P}_ω is projection map to the subspace $\omega = 0$.*

This corollary can be proved noting that $\hat{\phi} = \partial/\partial \omega$ and consequently $\hat{P}_\omega \xi = \xi(\phi, 0, Z^{\mu'})$.

Lemma 3. *The operator $\widehat{P} \equiv \widehat{P}_\phi \widehat{P}_\omega$ is the projection map to the subspace, $\phi = \omega = 0$.*

Proof. From Eqs. (12)–(14), one obtains $\widehat{\phi} \widehat{P} = \widehat{\omega} \widehat{P} = \widehat{P}$, thus $\widehat{P}^2 = \widehat{P}$. Reviewing the proof of Lemma 2, one verifies that

$$\widehat{P}\xi(\phi, \omega, Z^{\mu'}) = \widehat{P}_\omega(\widehat{P}_\phi\xi(\phi, \omega, Z^{\mu'})) = \widehat{P}_\omega\xi(0, \omega, Z^{\mu'}) = \xi(0, 0, Z^{\mu'}). \tag{18}$$

Corollary 2. *If $\xi = \xi|_{\phi, \omega}$ then $\{\phi, \xi\} = \{\omega, \xi\} = 0$.*

The second equality, for example, can be proved noting that $\xi|_{\phi, \omega} = \widehat{P}\xi$ and $\widehat{\omega}\widehat{P} = 0$.

The above results become practically interesting if conjugate to a given $\phi \in \mathcal{F}$, one can obtain explicitly a function ω that satisfies Eq. (3). This can be easily done if there exist a function $H \in \mathcal{F}$, such that $\widehat{\phi}H \neq 0$ but $\widehat{\phi}^{M+1}H = 0$, for an integer $M \geq 1$. Since, in principle, ω exists and satisfies Eq. (3), using the coordinate transformation (16), H can be written as a polynomial in ω ,

$$H(z(\omega, \phi, Z^{\mu'})) = \sum_{m=0}^M \frac{A_m(0, \phi, Z^{\mu'})}{m!} \omega^m, \quad \widehat{\phi}A_m = \frac{\partial}{\partial \omega} A_m(0, \phi, Z^{\mu'}) = 0. \tag{19}$$

Thus,

$$\widehat{\phi}^M H = \frac{\partial^M}{\partial \omega^M} H = A_M, \tag{20}$$

$$\widehat{\phi}^{M-1} H = A_M \omega + A_{M-1}. \tag{21}$$

Comparing Eq. (20) with Eq. (21), one can verify ω as the coefficient of $\widehat{\phi}^M H$ in $\widehat{\phi}^{M-1} H$.

As an example suppose,

$$\phi = e^x - 1, \quad H = \frac{1}{2} p_x^2. \tag{22}$$

A simple calculation shows that $M = 2$ and

$$\widehat{\phi}H = e^x p_x, \tag{23}$$

$$\widehat{\phi}^2 H = e^{2x}. \tag{24}$$

Comparing Eq. (23) with Eq. (24) one reads $\omega = e^{-x} p_x$. This method can be used to obtain gauge fixing conditions conjugate to first class constraints [4].

Lemma 4. *Considering a function $\xi \in \mathcal{F}$ and a conjugate pair of functions ϕ and ω , we have $\xi = \xi|_\phi$ iff $\widehat{\omega}\xi = 0$.*

Proof.

(a) If $\xi = \xi|_\phi$ then from Lemma 2, $\xi = \widehat{P}_\phi \xi$. Therefore, using Eq. (13), $\widehat{\omega}\xi = \widehat{\omega}\widehat{P}_\phi \xi = 0$;

(b) if $\widehat{\omega}\xi = 0$ then $\xi = \widehat{P}_\phi \xi = \xi|_\phi$.

Corollary 3. *For arbitrary functions ξ and χ in \mathcal{F} ,*

$$\{\xi|_\phi, \zeta|_\phi\} = \{\xi|_\phi, \zeta|_\phi\}|_\phi, \tag{25}$$

$$\{\xi|_\phi, \phi\} = \{\xi|_\phi, \phi\}|_\phi. \tag{26}$$

Corollary 3 can be proved using the Jaccobi identity to show that the Poisson brackets of the LHS of Eqs. (25), (26) with ω is vanishing.

Lemma 5. *If $\phi = \phi|_\psi$ then $\psi = \psi|_\phi$.*

Proof. Since there exist a function ω conjugate to ϕ , one can write ψ as a polynomial in ϕ (similar to Eq. (19)),

$$\psi = \sum_{i=1} a_i \phi^i + \psi|_\phi, \tag{27}$$

where $\hat{\omega} a_i = 0, i \geq 1$. If a_i 's do not vanish, the assumption $\phi = \phi|_\psi$ implies that $\psi(\phi) = 0$. Thus if $\psi \neq 0$ then a_i 's should vanish and $\psi = \psi|_\phi$.

Lemma 6. *If ω_1 and ω_2 are conjugate to ϕ_1 and ϕ_2 , respectively, and*

$$\phi_2 = \phi_2|_{\phi_1}, \tag{28}$$

$$\omega_2 = \omega_2|_{\phi_1}, \tag{29}$$

then the operators \widehat{P}_{ϕ_1} and \widehat{P}_{ϕ_2} commute with each other.

Proof. It is sufficient to prove that $[(\phi_1^n \hat{\omega}_1^n), (\phi_2^m \hat{\omega}_2^m)] = 0$. Using the Jacobi identity and Lemma 4, one can show that,

$$[\hat{\omega}_1, \hat{\omega}_2] = \{\omega_1, \omega_2\} = 0, \quad [\hat{\omega}_1, \phi_2] = \{\omega_1, \phi_2\} = 0. \tag{30}$$

From Lemma 5 and Eq. (28) one verifies that $\phi_1 = \phi_1|_{\phi_2}$, thus $[\phi_1, \hat{\omega}_2] = \{\phi_1, \omega_2\} = 0$. This completes the proof.

Corollary 4. *The operators \widehat{P}_{ϕ_1} and \widehat{P}_{ω_2} commute, i.e., $[\widehat{P}_{\phi_1}, \widehat{P}_{\omega_2}] = 0$.*

Corollary 5. *The operators $\widehat{P}_i = \widehat{P}_{\omega_i} \widehat{P}_{\phi_i}, i = 1, 2$, commute if $\omega_2 = \widehat{P}_1 \omega_2$ and $\phi_2 = \widehat{P}_1 \phi_1$.*

This can be proved using Lemma 3 and Corollary 2.

Lemma 7. *If $\phi = \phi|_\psi, \{\phi, \psi\} = 0$ and $\{\phi, \omega\} = 1$, then $\{\phi, \omega|_\psi\} = 1$.*

Proof. Writing ω as a polynomial in ψ ,

$$\omega = \sum_{i=1} a_i \psi^i + \omega|_\psi, \tag{31}$$

one verifies that,

$$1 = \{\phi, \omega\} = \sum_{i=1} \{\phi, a_i\} \psi^i + \{\phi, \omega|_\psi\}. \tag{32}$$

Thus,

$$1 = \{\phi, \omega|_\psi\} \psi = \{\phi, \omega|_\psi\} \psi = \{\phi, \omega|_\psi\}, \tag{33}$$

where in the third equality we have used Eq. (25).

Corollary 6. *If $\{\phi, \omega\} = 1$, then $\{\phi, \omega|_\phi\} = 1$.*

Using Eq. (26), the proof is similar to the proof of Lemma 7.

Lemma 8. *If $\xi = \xi|_\phi, \psi = \psi|_\phi$ and $\{\phi, \psi\} = 0$, then $\tilde{\xi} = \tilde{\xi}|_\phi$ in which $\tilde{\xi} \equiv \xi|_\psi$.*

Proof. Lemma 7 implies that there exist a function ω_ψ conjugate to ψ such that $\omega_\psi = \omega_\psi|_\phi$. Consequently from Lemma 6, we know that $[\widehat{P}_\psi, \widehat{P}_\phi] = 0$. In addition, $\tilde{\xi} = \widehat{P}_\psi \xi$ and $\xi = \widehat{P}_\phi \tilde{\xi}$ (see Lemma 2). Thus,

$$\tilde{\xi} = \widehat{P}_\psi \widehat{P}_\phi \tilde{\xi} = \widehat{P}_\phi \widehat{P}_\psi \tilde{\xi} = \widehat{P}_\phi \tilde{\xi}. \quad (34)$$

This completes the proof.

3. Redefinition of second class constraints

In this section we show that the subspace \mathcal{M} of the phase space, defined by a set of irreducible second class constraints,

$$\phi_a = 0, \quad a = 1, \dots, 2k, \quad (35)$$

which satisfy the relation,

$$\det(\{\phi_a, \phi_b\})_{\mathcal{M}} \neq 0, \quad (36)$$

can be equivalently determined by a set of constraints $\tilde{\phi}_i, \tilde{\omega}_i, i = 1, \dots, k$, satisfying the symplectic algebra,

$$\{\tilde{\phi}_i, \tilde{\phi}_j\} = 0, \quad \{\tilde{\phi}_i, \tilde{\omega}_j\} = \delta_{ij}, \quad \{\tilde{\omega}_i, \tilde{\omega}_j\} = 0. \quad (37)$$

For this reason, we consider the following lemmas.

Lemma 9. *There exist at least one constraint, say ϕ_{k+1} , such that*

$$\{\phi_1, \phi_{k+1}\}_{\mathcal{M}} \neq 0. \quad (38)$$

Proof. If it was not the case, i.e., if $\{\phi_1, \phi_a\}_{\mathcal{M}} = 0, a = 1, \dots, 2k$, then,

$$\det(\{\chi_a, \chi_b\})_{\mathcal{M}} = 0, \quad (39)$$

contrary to the assumption Eq. (36).

Consider the constraints ϕ_1 and ϕ_{k+1} and the definition,

$$\omega'_1 \equiv \omega_1 - \omega_1|_{\phi_{k+1}}, \quad (40)$$

where $\omega_1 \in \mathcal{F}$ is some function conjugate to ϕ_1 .

Lemma 10. *If the equation $\phi_{k+1} = 0$ has a unique solution (the uniqueness condition) then the constraint $\omega'_1 \approx 0$ is equivalent to ϕ_{k+1} .*

Proof. Using the uniqueness condition, we show that $\phi_{k+1} = 0$ iff $\omega'_1 = 0$.

Consider the coordinate transformation $z^\mu \rightarrow (\omega_1, \phi_1, Z^{\mu'})$. The assumption,

$$\{\phi_1, \phi_{k+1}\} = \frac{\partial}{\partial \omega_1} \phi_{k+1} \neq 0, \quad (41)$$

reads,

$$\phi_{k+1} = \omega_1 \chi(\omega_1, \phi_1, z') + \xi(0, \phi_1, Z^{\mu'}), \quad (42)$$

for some functions χ and ξ . From Eq. (42) one can determine $\omega_1^0 \equiv \omega_1|_{\phi_{k+1}}$ as the solution of equation,

$$\omega_1^0 \chi(\omega_1^0, \phi_1, Z^{\mu'}) + \xi(0, \phi_1, z') = 0. \tag{43}$$

Inserting ξ from the above relation in Eq. (42), one verifies that,

$$\begin{aligned} \phi_{k+1} &= \omega_1 \chi(\omega_1, \phi_1, Z^{\mu'}) - \omega_1^0 \chi(\omega_1^0, \phi_1, Z^{\mu'}) \\ &= \omega_1 (\chi(\omega_1^0, \phi_1, Z^{\mu'}) + (\omega_1 - \omega_1^0) \chi'(\omega_1, \phi_1, Z^{\mu'})) - \omega_1^0 \chi(\omega_1^0, \phi_1, Z^{\mu'}) \\ &= (\omega_1 - \omega_1^0) (\chi(\omega_1^0, \phi_1, Z^{\mu'}) + \omega_1 \chi'(\omega_1, \phi_1, Z^{\mu'})) \\ &= \omega_1^0 \tilde{\chi}, \end{aligned} \tag{44}$$

where χ' , in the second equality, is some function that can be determined in terms of χ using Taylor expansion. In the last equality we have used definition (40). From Eq. (44) one finds two possible solutions for equation $\phi_{k+1} = 0$; $\omega_1^1 = 0$ and/or $\tilde{\chi} = 0$. Due to uniqueness condition these two solutions, if both possible, should coincide. Therefore, $\phi_{k+1} = 0$ if and only if $\omega_1^1 = 0$. Of course $\tilde{\chi}$ is non-vanishing because,

$$\begin{aligned} \det(\{\phi_a, \phi_b\})_{\mathcal{M}} &= \pm \det \begin{pmatrix} 0 & \{\phi_1, \phi_{k+1}\} & \cdots \\ \{\phi_{k+1}, \phi_1\} & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}_{\mathcal{M}} \\ &= \pm \tilde{\chi}^2 \det \begin{pmatrix} 0 & \{\phi_1, \omega_1^1\} & \cdots \\ \{\omega_1^1, \phi_1\} & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}_{\mathcal{M}} \neq 0. \end{aligned} \tag{45}$$

The above equation implies that not only the constraint ω_1^1 is equivalent to ϕ_{k+1} but also the set of constraints ϕ_a in which ϕ_{k+1} is replaced by ω_1^1 are second class.

Lemma 11. *The function ω_1^1 is conjugate to ϕ_1 , i.e., $\{\phi_1, \omega_1^1\} = 1$.*

Proof. If $\omega_1^0 = 0$, then proof is trivial. If $\omega_1^0 \neq 0$, one can prove Lemma 11 as follows. Consider the Taylor expansion of ϕ_{k+1} in terms of ω_1 ,

$$\phi_{k+1}(\omega_1, \phi_1, Z^{\mu'}) = \sum_{m=0} A_m(0, \phi_1, Z^{\mu'}) \omega_1^m. \tag{46}$$

Since $\omega_1^0 = \omega_1|_{\phi_{k+1}}$, we have,

$$\sum_{m=0} A_m(0, \phi_1, Z^{\mu'}) (\omega_1^0)^m = 0. \tag{47}$$

Consequently,

$$\{\phi_1, \omega_1^0\} \sum_{m=1} m A_m (\omega_1^0)^{m-1} = 0. \tag{48}$$

This has two solutions:

- (1) $A_{m>0} = 0$. In this case, the Poisson bracket of $\phi_{k+1} = A_0(0, \phi_1, Z^{\mu'})$ and ϕ_1 vanishes contrary to the assumption Eq. (38);
- (2) $\{\phi_1, \omega_1^0\} = 0$, which is the desired result.

Let us define $\tilde{\phi}_1 \equiv \phi_1$ and $\tilde{\omega}_1 \equiv \omega'_1$. Using Lemma 3 and Corollary 2, one can make the Poisson bracket of $\tilde{\phi}_1$ and $\tilde{\phi}_{k+1}$ with the other constraints vanishing by redefining the constraints ϕ_i and ϕ_{k+i} ($i > 1$) as follows,

$$\begin{aligned} \phi_i &\rightarrow \widehat{P}_1 \phi_i, \quad i = 2, \dots, k, \\ \phi_{i+k} &\rightarrow \widehat{P}_1 \phi_{i+k}, \end{aligned} \tag{49}$$

where $\widehat{P}_1 = \widehat{P}_{\tilde{\phi}_1} \widehat{P}_{\tilde{\omega}_1}$. Let us call these new constraints $\phi_{a_1}^1$, $a_1 = 1, \dots, 2k^1$, where $k^1 = k - 1$. The determinant of the matrix of Poisson brackets of the second class constraints $\tilde{\phi}_1$, $\tilde{\omega}_1$ and $\phi_{a_1}^1$'s is,

$$\det \begin{pmatrix} 0 & +1 & 0 \cdots 0 \\ -1 & 0 & 0 \cdots 0 \\ 0 & 0 & \\ \vdots & \vdots & (\{\phi_{a_1}^1, \phi_{b_1}^1\}) \\ 0 & 0 & \end{pmatrix} = \det(\{\phi_{a_1}^1, \phi_{b_1}^1\})_{\mathcal{M}} \neq 0. \tag{50}$$

Consequently there exist a constraint, say $\phi_{k^1+1}^1$, such that $\{\phi_1^1, \phi_{k^1+1}^1\}_{\mathcal{M}} \neq 0$. From Corollary 2 we know that $\{\phi_1^1, \tilde{\phi}_1\} = \{\phi_1^1, \tilde{\omega}_1\} = 0$. Thus, Lemma 7 guarantees the existence of a function ω_1^1 conjugate to ϕ_1^1 such that $\omega_1^1 = \omega_1^1|_{\tilde{\phi}_1, \tilde{\omega}_1}$. Lemma 10 says that, assuming the uniqueness condition, ω_1^1 ,

$$\omega_1^1 \equiv \omega_1^1 - \omega_1^1|_{\phi_{k^1+1}^1}, \tag{51}$$

is equivalent to $\phi_{k^1+1}^1$. Lemma 8 guarantees that $\omega_1^1 = \omega_1^1|_{\tilde{\phi}_1, \tilde{\omega}_1}$, because the Poisson brackets of $\phi_{k^1+1}^1$ with $\tilde{\phi}_1$ and $\tilde{\omega}_1$ vanish (see redefinition (49)). Therefore, from Lemma 3, $\omega_1^1 = \widehat{P}_1 \omega_1^1$. In addition Lemma 11 says that ω_1^1 is conjugate to ϕ_1^1 . We define $\tilde{\phi}_2 \equiv \phi_1^1$ and $\tilde{\omega}_2 \equiv \omega_1^1$. Similar to Eq. (50), one can show that the constraints,

$$\phi_{a_2}^2 \in \{\phi_i^1, \phi_{k^1+i}^1 | i = 1, \dots, k^1\}, \quad a_2 = 1, \dots, 2(k-2) \tag{52}$$

in which we have considered the redefinition,

$$\begin{aligned} \phi_i^1 &\rightarrow \widehat{P}_2 \phi_i^1, \quad i = 1, \dots, k^1 = k - 1, \\ \phi_{k^1+i}^1 &\rightarrow \widehat{P}_2 \phi_{k^1+i}^1, \end{aligned} \tag{53}$$

where $\widehat{P}_2 = \widehat{P}_{\tilde{\phi}_2} \widehat{P}_{\tilde{\omega}_2}$, form a set of secondary constraints, i.e.,

$$\det(\{\phi_{a_2}^2, \phi_{b_2}^2\})_{\mathcal{M}} \neq 0. \tag{54}$$

Since $\tilde{\phi}_2 = \widehat{P}_1 \tilde{\phi}_2$ and $\tilde{\omega}_2 = \widehat{P}_1 \tilde{\omega}_2$, from Corollary 5, it can be verified that $[\widehat{P}_1, \widehat{P}_2] = 0$. Therefore, using Corollary 2, one obtains $\{\phi_{a_2}^2, \tilde{\phi}_i\} = \{\phi_{a_2}^2, \tilde{\omega}_i\} = 0$, $i = 1, 2$. All the above process can be repeated until one ends up with a set of constraints satisfying Eq. (37).

Lemma 12. *The set of constraints $\tilde{\phi}_i$ and $\tilde{\omega}_i$, $i = 1, \dots, k$, satisfy Eq. (37).*

Proof. Since

$$\begin{aligned} \tilde{\phi}_i &= \widehat{P}_j \tilde{\phi}_i, \quad j < i, \\ \tilde{\omega}_i &= \widehat{P}_j \tilde{\omega}_i, \end{aligned} \tag{55}$$

where $\widehat{P}_i = \widehat{P}_{\tilde{\phi}_i} \widehat{P}_{\tilde{\omega}_i}$, Corollary 5 reads,

$$[\widehat{P}_i, \widehat{P}_j] = 0, \quad i, j = 1, \dots, k. \tag{56}$$

From Lemma 3, it can be verified that

$$\begin{aligned}\tilde{\phi}_i &= \tilde{\phi}_i|_{\tilde{\phi}_j, \tilde{\omega}_j}, \quad j < i, \\ \tilde{\omega}_i &= \tilde{\omega}_i|_{\tilde{\phi}_j, \tilde{\omega}_j}.\end{aligned}\tag{57}$$

Using Lemma 5 one obtains,

$$\begin{aligned}\tilde{\phi}_i &= \tilde{\phi}_i|_{\tilde{\phi}_j, \tilde{\omega}_j}, \quad i \neq j, \\ \tilde{\omega}_i &= \tilde{\omega}_i|_{\tilde{\phi}_j, \tilde{\omega}_j}.\end{aligned}\tag{58}$$

Finally, Corollary 2 guarantees the validity of Lemma 12.

When we have found second class constraints satisfying the symplectic algebra, we can convert them to first class constraints by extending the phase space to include new coordinates η_i 's and π_i 's, where

$$\{\eta_i, \eta_j\} = \{\pi_i, \pi_j\} = 0, \quad \{\pi_i, \eta_j\} = -\delta_{ij}, \quad \{\eta_i, z^\mu\} = \{\pi_i, z^\nu\} = 0,\tag{59}$$

and redefine constraints as follows:

$$\tilde{\phi}_i \rightarrow \Phi_i = \tilde{\phi}_i + \eta_i, \quad \tilde{\omega}_i \rightarrow \Phi_{k+i} = \tilde{\omega}_i - \pi_i.\tag{60}$$

It can be easily verified that the constraints Φ_a , $a = 1, \dots, 2k$ are Abelian,

$$\{\Phi_a, \Phi_b\} = 0.\tag{61}$$

Another interesting result is that, the operator \widehat{P} defined by the relation,

$$\widehat{P} \equiv \prod_{i=1}^k \widehat{P}_i,\tag{62}$$

is the projection map to the constraint surface \mathcal{M} and the projected coordinates $z_p^\mu \equiv \widehat{P}z^\mu = z^\mu|_{\mathcal{M}}$, are the coordinates of the constrained surface \mathcal{M} . In addition, from Corollary 2 it is clear that,

$$\{z_p^\mu, \dots\}_{\text{DB}} = \{z_p^\mu, \dots\},\tag{63}$$

where $\{, \}_{\text{DB}}$ stands for Dirac bracket respective to the constraints $\tilde{\phi}$'s.

Assume one maps each constraint $\tilde{\phi}_i$ to the surface of its conjugate $\tilde{\omega}_i$, i.e.,

$$\tilde{\phi}_i \rightarrow \tilde{\phi}_i|_{\tilde{\omega}_i}, \quad i = 1, \dots, k.\tag{64}$$

From Corollary 6, Lemmas 8 and 12, one verifies that, the algebra (37) is still satisfied. Recalling the constraints ϕ_a 's in Eq. (35) and the method we used to obtain $\tilde{\phi}_i$'s (see Eqs. (49), (53), (64) and Lemma 10), we verify that $\tilde{\phi}_i$'s are simply half of ϕ_a 's, mapped to the surface of ϕ_b 's, $b \neq a$.

Theorem. Given a set of second class constraints ϕ_a , $a = 1, \dots, 2k$, where

$$\phi_a = \phi_a|_{\phi_b}, \quad b \neq a,\tag{65}$$

there exist a permutation p such that the constraints $\tilde{\phi}_{p_i}$, $i = 1, \dots, k$, form a subset of Abelian (first class) constraints,

$$\{\phi_{p_i}, \phi_{p_j}\} = 0, \quad i, j = 1, \dots, k.\tag{66}$$

As an example see Ref. [9], where gauge invariance in the Proca model is studied considering the Abelian subset of second class constraints.

4. Conclusion

The main purpose of this Letter is to show that there exist an Abelian subset of second class constraints that can be obtained by mapping each constraint to the surface of other constraints. In addition we introduced a method that can be practically used to transform a given set of second class constraints to an equivalent set satisfying the symplectic algebra. In this way, second class constraints can be simply converted to Abelian first class ones.

In Ref. [4], it is proved that first class constraints become Abelian when they are mapped to the surface of each other. Therefore, one can conclude that, using the same technique, the Abelian subset of a given set of constraints, can be found independent of the details of their algebra.

Assuming the most general Poisson structure, we have found the projector operators that map functions of phase space to the constraint surface. It is shown that the Poisson brackets of these mapped functions with other functions are equivalent to the corresponding Dirac brackets.

Acknowledgement

The author would like to express his thanks to A. Shirzad for valuable discussions and useful comments.

References

- [1] P.A.M. Dirac, *Can. J. Math.* 2 (1950) 129;
P.A.M. Dirac, *Proc. R. Soc. London, Ser. A* 246 (1958) 326;
P.A.M. Dirac, *Lectures on Quantum Mechanics*, Yeshiva Univ. Press, New York, 1964.
- [2] M. Henneaux, C. Teitelboim, *Quantization of Gauge System*, Princeton Univ. Press, Princeton, NJ, 1992.
- [3] M. Henneaux, *Phys. Rep.* 126 (1985) 1.
- [4] F. Loran, *Phys. Lett. B* 547 (2002) 63, hep-th/0209180.
- [5] I.A. Batalin, E.S. Fradkin, *Phys. Lett. B* 180 (1986) 157;
I.A. Batalin, I.V. Tyutin, *Int. J. Mod. Phys. A* 6 (1991) 3225;
I.A. Batalin, R. Marnelius, *Mod. Phys. Lett. A* 16 (2001) 1505, hep-th/0106087.
- [6] C.S. Chu, P.M. Ho, *Nucl. Phys. B* 568 (2000) 447, hep-th/9906192;
I. Rudychev, *JHEP* 0104 (2001) 015, hep-th/0101039;
F. Loran, *Phys. Lett. B* 544 (2002) 199, hep-th/0207025.
- [7] I. Batalin, S. Lyakhovich, R. Marnelius, *Phys. Lett. B* 534 (2002) 201, hep-th/0112175;
N. Berkovits, *JHEP* 0109 (2001) 016.
- [8] L. Faddeev, *Phys. Lett. B* 145 (1984) 81;
L. Faddeev, S. Shatashvili, *Phys. Lett. B* 167 (1986) 225;
E.Sh. Egorian, R.P. Manvelian, *Theor. Math. Phys.* 94 (1993) 173;
I.A. Batalin, E.S. Fradkin, *Nucl. Phys. B* 279 (1987) 514;
I.A. Batalin, E.S. Fradkin, T.A. Fradkina, *Nucl. Phys. B* 314 (1989) 158.
- [9] A.S. Vytheswaran, *Int. J. Mod. Phys. A* 13 (1998) 765, hep-th/9701050.
- [10] F. John, in: *Partial Differential Equations*, Vol. 1, 4th Edition, Springer-Verlag, New York, 1981.