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IRREDUCIBLE RECURRENCES AND REPRESENTATION THEOREMS FOR ${}_3F_2(1)$

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Abstract—By examining the irreducibility of a certain recurrence, we show that the hypergeometric function of the title cannot be represented by gamma functions.

1. INTRODUCTION

The function $F = {}_3F_2(1)$ is one of the basic functions of mathematical physics and combinatorics. F is a five parameter family of functions, $F: C^5 \to C$. For certain values of these parameters F has a convenient series representation,

$$F = {}_3F_2\begin{pmatrix} a, b, c \\ e, f \end{pmatrix} = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k}{(e)_k(f)_k k!}, \tag{1}$$

$$e, f \neq 0, -1, -2, \ldots, \operatorname{Re}(a + b + c - e - f) < 0,$$

these restrictions being necessary to ensure the meaning and convergence of the series. However, the function has an analytic continuation in all its parameters, and F is analytic everywhere in C^5 except where

$$e, f = -m, e + f - a - b - c = -m, m = 0, 1, 2, \ldots, \tag{2}$$

see [9]. Further, the singularities of F are poles.

For discussion of F and a survey of many of its interesting properties, see any of the references [2, 5, 8].

For special values of its parameters, F may be represented in closed form, i.e. as a simple ratio of gamma functions; (I will assign a precise meaning to the expression "closed form" later). For instance if one of (a, b, c) equals e or f, two of the parameters cancel and F may be expressed as a ratio of gamma functions via the famous formula of Gauss ([2], Vol 1, p. 104 (46)], which sums a ${}_2F_1$ of unit argument. Another such case is when $e = 2a, f = (b + c + 1)/2$. Then F may be summed by Watson's formula ([2], Vol. 1, p. 189 (6))

$${}_3F_2\begin{pmatrix} a, & b, & c \\ 2a, & \dfrac{b+c+1}{2} & \end{pmatrix} = \frac{\sqrt{\pi}\,\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(\frac{b+c+1}{2}\right)\Gamma\left(a+\frac{1}{2}-\frac{b}{2}-\frac{c}{2}\right)}{\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\frac{c+1}{2}\right)\Gamma\left(a+\frac{1}{2}-\frac{b}{2}\right)\Gamma\left(a+\frac{1}{2}-\frac{c}{2}\right)}. \tag{3}$$

Note, however, that the above is essentially a three parameter family of functions, since two of the parameters are linear combinations of the others. Many such formulas are known for F, but in all cases the functions represented are not general; rather, some of the parameters depend linearly on others. The references [5, 6] contain a fairly complete compendium of all those cases where F is known to be expressible in terms of simpler functions.

In this paper I shall call the situation where the parameters a, b, c, e, f are not interrelated the case of the *unrestricted* F. It has long been wondered whether the unrestricted F could be written in terms of simpler functions, in particular, gamma functions, as in the case of Watson's formula. (All the special cases in which F can be summed are of this type.) However, attempts to extend the results in [5, 6] in any dramatic way have led to failure. There seemed to be some barrier keeping us from writing the unrestricted F in simple terms. But it was not clear exactly what constituted the barrier. As we shall see the "barrier" is, indeed, real. It is a consequence of the irreducibility of a certain linear difference equation. Recall, the expression

$$\sum_{k=0}^{\sigma} A_k(n)y(n+k)=0, \quad A_0(n)A_\sigma(n)\not\equiv 0, \quad n=0,1,2,\ldots, \tag{4}$$

is called a (homogeneous) linear difference equation (or recurrence relation) of order σ.

Definition 1

The equation (4) for $y(n)$ with rational coefficients $A_k(n)$ is called *optimal* (or *minimal*) if $y(n)$ satisfies no such recurrence of lower order.□

A closely related concept is embodied in

Definition 2

The equation (4) with rational coefficients $A_k(n)$ is called *irreducible* if it has no solutions in common with an equation with rational coefficients of lower order.□

(Obviously, the class of equations considered under these two definitions needs to be restricted for the definitions to be productive, since any function $y(n)$ satisfies a difference equation of arbitrary order σ. Requiring the equation to have rational coefficients seems to cover most cases of interest. Here we shall discuss exclusively difference equations with rational coefficients.)

Clearly, a difference equation for a function $y(n)$ is minimal if it is irreducible but not the other way around. Theoretical information on reducibility is almost non-existent. Demonstrating whether a given equation is reducible is usually exceedingly difficult, and often has to depend on techniques special to the particular equation. Our cases are not exceptions. It is clear, however, that a second order equation is reducible if and only if it has a solution which can be written in terms of gamma functions since a solution must satisfy the first order equation

$$y(n)+r(n)y(n+1)=0,$$

with $r(n)$ rational and consequently

$$y(n)=c\lambda^n\prod_{j=0}^{p}\Gamma(\lambda_j+n)\Big/\prod_{j=0}^{q}\Gamma(\mu_j+n), \quad c\neq 0, \quad \lambda\neq 0. \tag{5}$$

Conversely, any function of the form (5) satisfies a first order equation.

My own work ([5], Vol. 2, p. 159 ff) shows that the function

$$F(n)\stackrel{\text{def}}{=}{}_3F_2\binom{n+a,n+b,n+c}{2n+e,n+f}$$

satisfies a recurrence of order three with coefficients $A_k(n)$ which are rational in n.

If a recurrence of order two could be found for $F(n)$ and could be shown to be irreducible it would be established that $F(n)$ could not be represented in the form (5). This, conceivably, could be used to establish that $F(0)=F$ could not be written as a simple ratio of gamma functions. Recent work of Lewanowicz [4] on minimal recurrences for the coefficients for the expansion of a function satisfying a differential equation with polynomial coefficients, as well as results of Askey and Gasper [1], strongly indicated that $F(n)$ satisfies a second order recurrence. In fact, the recurrence can be thought of as "interpreting" for noninteger n the recurrence given by Askey and Gasper for $V(k)$ in the expansion

$$P_n^{(a,b)}(x)=\sum_{k=0}^{n} V(k)P_k^{(\alpha,\beta)}(x),$$

and is shown in exactly the same way.

I give this recurrence and prove that it is irreducible; the representation theorem for $F(0)$ then follows in a straight-forward manner.

Throughout I will tacitly assume that the parameters of the F under consideration are such that the function makes sense. Even when this is not true, however, simple redefinitions and limit procedures always result in expressions which do make sense. Thus, in the interests of simplicity, I will not clutter the paper with rather synthetic restrictions on a, b, c, e, f.

2. AN EXPANSION IN JACOBI POLYNOMIALS AND THE BASIC RECURRENCE RELATION

The following expansion, due to Fields and myself, can be found in ([5], Vol. 2, p. 29 (1)):

$$G(x) = \sum_{n=0}^{\infty} C(n)p_n(x),$$

where

$$G(x) \stackrel{\text{def}}{=} {}_2F_1\left(\begin{matrix} a, b \\ e \end{matrix} \middle| \frac{1+x}{2}\right),$$

$$p_n(x) \stackrel{\text{def}}{=} P_n^{(\alpha,\beta)}(x),$$

$$C(n) \stackrel{\text{def}}{=} \frac{(a)_n(b)_n}{(e)_n(n+\gamma)_n} {}_3F_2\left(\begin{matrix} n+\beta+1, n+a, n+b \\ 2n+\gamma+1, n+e \end{matrix}\right) \tag{6}$$

$$\gamma \stackrel{\text{def}}{=} \alpha + \beta + 1.$$

The expansion converges for $-1 < x < 1$ when G is defined, α, β are real, $\alpha > -1$, $\beta > -$ and

$$\min\left[\operatorname{Re}(e-a-b)+\alpha+1, \operatorname{Re}(e-a-b)+\frac{\alpha}{2}+\frac{3}{4}\right] > 0,$$

as can be deduced from the formulas ([2], Vol. 1, p. 108 (1), Vol. 2, p. 212 (3)). However, the convergence of (6) is not really relevant since, once a recurrence is developed for $C(n)$, the permanence principle of functional equations can be used to extend the validity of the recurrence to complex α, β, a, b, e. For a discussion of this principle, see [9].

The formulas ([2], Vol. 1, p. 56 (1), Vol. 2, p. 169 (14)) show G satisfies

$$(1-x^2)y'' + [(2e-a-b-1)-(a+b+1)x]y' - aby = 0 \tag{7}$$

and p_n satisfies

$$(1-x^2)z'' + [(\beta-\alpha)-(\gamma+1)x]z' + n(n+\gamma)z = 0. \tag{8}$$

Putting the expansion (6) in (7) and using (8) and a bit of algebra shows we must have

$$\sum_{n=0}^{\infty} C(n)\{[(2e-a-b-1+\alpha-\beta)+(\gamma-a-b)x]p_n' - [n(n+\gamma)+ab]p_n\} = 0. \tag{9}$$

Now

$$xp_n' = (xp_n)' - p_n,$$

so (9) can be written

$$\sum_{n=0}^{\infty} C(n)[d_1 p_n' + d_2(xp_n)' + d_3 p_n] = 0,$$

$$d_1 \stackrel{\text{def}}{=} 2e - a - b - 1 + \alpha - \beta, \tag{10}$$

$$d_2 \stackrel{\text{def}}{=} \gamma - a - b,$$

$$d_3 \stackrel{\text{def}}{=} a + b - \gamma - ab - n(n + \gamma).$$

I now require a

LEMMA

For all except isolated values of α, β,

$$xp_n = \mu_1 p_{n+1} + \mu_2 p_n + \mu_3 p_{n-1}, \tag{11}$$

$$p_n = \nu_1 p'_{n+1} + \nu_2 p'_n + \nu_3 p'_{n-1} \tag{12}$$

where

$$\mu_1 \stackrel{\text{def}}{=} 2(n+1)(n+\gamma)/(2n+\gamma)(2n+\gamma+1),$$

$$\mu_2 \stackrel{\text{def}}{=} (\beta^2 - \alpha^2)/(2n+\gamma-1)(2n+\gamma+1),$$

$$\mu_3 \stackrel{\text{def}}{=} 2(n+\alpha)(n+\beta)/(2n+\gamma-1)(2n+\gamma),$$

$$\nu_1 \stackrel{\text{def}}{=} 2(n+\gamma)/(2n+\gamma)(2n+\gamma+1),$$

$$\nu_2 \stackrel{\text{def}}{=} 2(\alpha-\beta)/(2n+\gamma-1)(2n+\gamma+1),$$

$$\nu_3 \stackrel{\text{def}}{=} -2(n+\alpha)(n+\beta)/(n+\gamma-1)(2n+\gamma)(2n+\gamma-1).$$

These relations hold for $n \geq 0$ provided one interprets $p_{-1} = 0$.

Proof. These results are well known. (11) is merely a restatement of the recurrence for p_n, and (12) is due to Askey and Gasper [1].□

Differentiating (11) and substituting the result along with (12) into the sum (9) and setting the coefficient of p_n' to zero, which must be true since $\{p_n'\}$ is an orthogonal system, gives a recurrence of *second order* for $C(n)$:

THEOREM 1

For all except isolated values of α, β, a, b, e, $C(n)$ satisfies the recurrence

$$\sigma_1(n)u(n-1) + \sigma_2(n)u(n) + \sigma_3(n)u(n+1) = 0, \quad n \geq 1, \tag{13}$$

$$\sigma_1 \stackrel{\text{def}}{=} -2(n+\gamma-1)(n+a-1)(n+b-1)/(2n+\gamma-1)(2n+\gamma-2),$$

$$\sigma_2 \stackrel{\text{def}}{=} e_1 + e_2/(2n+\gamma-1)(2n+\gamma+1), \tag{14}$$

$$e_1 \stackrel{\text{def}}{=} 2e - a - b - 1 - \frac{(\beta-\alpha)}{2}, \quad e_2 = \frac{(\beta-\alpha)}{2}(\gamma+1-2a)(\gamma+1-2b),$$

$$\sigma_3 \stackrel{\text{def}}{=} 2(n+\alpha+1)(n+\beta+1)(n+\gamma+1-a)(n+\gamma+1-b)/(n+\gamma)$$
$$\times (2n+\gamma+1)(2n+\gamma+2). \square$$

Letting $n \to n+1$ produces a recurrence of the form (3), but it is more convenient for our purposes to allow (13) to stand as written. It is also convenient to allow the parameters of $C(n)$ to stand as they are, rather than making the identifications $\beta + 1 \to a$, etc., to get $F(n)$.

3. OTHER SOLUTIONS AND THEIR ASYMPTOTIC BEHAVIOR

Before proceeding it is necessary to analyze in detail the recurrence (13), including the nature of its solutions and their asymptotic behaviors. Since the computations are very tedious, I will spare the reader the messy particulars, sketching only briefly the arguments.

Let

$$b_1 \stackrel{\text{def}}{=} \alpha + 1, \quad b_2 \stackrel{\text{def}}{=} \gamma + 1 - a, \quad b_3 \stackrel{\text{def}}{=} \gamma + 1 - b, \quad a_1 \stackrel{\text{def}}{=} \gamma + 1 - e,$$

$$C_h(n) \stackrel{\text{def}}{=} \frac{(2n+\gamma)\Gamma(n+\gamma)\Gamma(n+\gamma+1-b_h)(-1)^n}{\Gamma(n+\beta+1)\Gamma(n+b_h)}$$
$$\times {}_3F_2{}^*\left(\begin{matrix} 1-b_h-n, 1-b_h+n+\gamma, 1-b_h+a_1 \\ 1-b_h+b_1, 1-b_h+b_2, 1-b_h+b_3 \end{matrix}\right), \quad h = 1, 2, 3, \tag{15}$$

where the (*) indicates the denominator parameter corresponding to 1 is to be deleted.

That $C_2(n)$ satisfies the recurrence (13) may be shown as follows. Let $x = a + 1 - e$ and write

$$C_2(n) \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} A_k(n)(x)_k.$$

Substituting this in the recurrence, using the fact that $(x)_k$ satisfies $x(x)_k = (x)_{k+1} - k(x)_k$ and equating like coefficients of $(x)_k$ yields, essentially, a polynomial of degree 3 in k. It is easily seen that this polynomial is 0 when $k = n + \alpha, n + \alpha + 1, -n - \beta$, and, further, that the coefficient of k^3 is zero. Thus the polynomial vanishes identically so $C_2(n)$ is, indeed, a solution of the recurrence.

Making the substitution $u^*(n) = (\beta+1)_n u(n)/(\gamma)_n$ yields a recurrence invariant under the change of variable $n \to -n - \gamma$. Thus $u^*(-n-\gamma)$ must also be a solution, and this can be used to show that

$$C_4(n) \stackrel{\text{def}}{=} \frac{\Gamma(n+\gamma)\Gamma(n+\gamma+1-e)\Gamma(2n+\gamma+1)}{\Gamma(n+\gamma+1-a)\Gamma(n+\gamma+1-b)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}$$
$$\times {}_3F_2\left(\begin{matrix} -\alpha-n, \alpha-\gamma-n, b-\gamma-n \\ 1-\gamma-2n, e-\gamma-n \end{matrix}\right) \tag{16}$$

is also a solution of (13). But the functions C, C_h, C_4 with argument x instead of unity satisfy a recurrence of third order ([5], Vol. 2, Section 12.2, p. 135) and so any four of them are linearly dependent (in n). Thus C_2, C_3 satisfy the recurrence. By the same reasoning, another solution is

$$C_5(n) \stackrel{\text{def}}{=} \frac{(-1)^n(2n+\gamma)\Gamma(n+\gamma)\Gamma(n+e-1)}{\Gamma(n+\beta+1)\Gamma(n+\gamma+2-e)}\,{}_3F_2\left(\begin{matrix}\beta+2-e, a+1-e, b+1-e\\ n+\gamma+2-e, 2-e-n\end{matrix}\right). \quad (17)$$

The standard techniques of asymptotic analysis[10] show the recurrence for $C(n)$, see (13), has a basis of solutions u_1, u_2, with the properties

$$u_1 \sim n^{\theta_1}\left\{1+\frac{\rho_1}{n}+\frac{\rho_2}{n^2}+\cdots\right\}, \quad n\to\infty,$$

$$u_2 \sim (-1)^n n^{\theta_2}\left\{1+\frac{\rho_1^*}{n}+\frac{\rho_2^*}{n^2}+\cdots\right\}, \quad n\to\infty, \quad (18)$$

$$\theta_1 \stackrel{\text{def}}{=} 2a+2b-2e-\alpha-1, \quad \theta_2 \stackrel{\text{def}}{=} 2e-\beta-3.$$

The asymptotic behavior of $C(n)$ can easily be determined from the formula ([5], Vol. 2, p. 104 (10)). (The work of Fields[3] also provides the required result.) We conclude that $C(n)$ is a constant multiple of u_1 and, in fact,

$$C(n) \sim \frac{2\Gamma(e)\Gamma(\alpha+1-a-b+e)}{\Gamma(a)\Gamma(b)} n^{\theta_1}\left\{1+\frac{\rho_1}{n}+\frac{\rho_2}{n^2}+\cdots\right\}, n\to\infty, \quad (19)$$

except for isolated values of the parameters. Since clearly,

$$C_5(n) = 2(-1)^n n^{\theta_2}\{1+0(n^{-2})\}, \quad n\to\infty,$$

C_5 must be a constant multiple of $u_2(n)$. Further, C_1, C_2, C_3, C_4 are linear combinations of u_1 and u_2.

4. THE REDUCIBILITY OF A SPECIAL RECURRENCE

I will need only a particular case of (13), that is, where $\alpha = \beta$. Making this specialization and letting

$$v(n) \stackrel{\text{def}}{=} \frac{(\alpha+1)_n u(n)}{(2\alpha+1)_n(n+\alpha+\frac{1}{2})},$$

gives the recurrence

$$-(n+a-1)(n+b-1)v(n-1)+2(2e-a-b-1)(n+\alpha+\tfrac{1}{2})v(n)$$
$$+(n+2\alpha+2-a)(n+2\alpha+2-b)v(n+1)=0. \quad (20)$$

A solution proportional to C is

$$D(n) \stackrel{\text{def}}{=} \frac{(a)_n(b)_n}{2^{2n}(\alpha+\frac{3}{2})_n(e)_n}\,{}_3F_2\left(\begin{matrix}n+\alpha+1, n+a, n+b\\ 2n+2\alpha+2, n+e\end{matrix}\right). \quad (21)$$

From (18) it follows that this recurrence has a basis of solutions v_1, v_2 with the behavior

$$v_1(n) \sim n^{\phi_1}\left\{1+\frac{\sigma_1}{n}+\frac{\sigma_2}{n^2}+\cdots\right\}, \quad n\to\infty,$$

$$v_2(n) \sim (-1)^n n^{\phi_2}\left\{+\frac{\sigma_1^*}{n}+\frac{\sigma_2^*}{n^2}+\cdots\right\}, \quad n\to\infty, \tag{22}$$

$$\phi_1 \overset{\text{def}}{=} 2(a+b-e-\alpha-1), \quad \phi_2 \overset{\text{def}}{=} 2(e-\alpha-2).$$

THEOREM 2

The recurrence (20) is reducible only if either

(i) $$p_1(\alpha+1-a)+p_2(\alpha+1-b)=p_3+\alpha+1+e-a-b \tag{23}$$

or

(ii) $$p_1(\alpha+1-a)+p_2(\alpha+1-b)=p_3+\alpha+2-e, \tag{24}$$

for integer $p_1, p_2, p_3 \geq 0$.

Conversely, when p_1 or $p_2 = 0$ or 1, the equation is reducible. For case (i) the solution is a multiple or limiting case of D and of the form

$$M\prod_{j=1}^{q}\frac{\Gamma(n+\lambda_j)}{\Gamma(n+2\alpha+2-\lambda_j)}, \tag{25}$$

where M, λ_j are appropriate constants. For case (ii) the solution is of form above times $(-1)^n$, and corresponds to one or more of the functions C_j, (see (15), (17)).

Proof. The difference equation is reducible iff a solution can be written in the form (5). Since there is a basis of solutions of the equation having the behavior $v_1 \sim n^{\phi_1}$, $v_2 \sim (-1)^n n^{\phi_2}$ we must have $p = q$ and either $\lambda = 1$ or $\lambda = -1$. Since the equation is invariant under the substitution $n \to -n - 2\alpha - 1$ and since this substitution preserves the asymptotic character of either kind of solution, any solution must be of the form (25), or (25) times $(-1)^n$. To prevent the trivial cancellation of factors, assume

$$\lambda_i \neq 2\alpha+2-\lambda_j, \quad 1 \leq i,j \leq p. \tag{26}$$

Take the first case. Substituting (25) in the recurrence produces the requirement that a polynomial in n vanish identically. This will happen if and only if it vanishes identically in the complex variable

$$z \overset{\text{def}}{=} n+\alpha+1/2.$$

Let

$$\bar{\mu} \overset{\text{def}}{=} \alpha+3/2-\mu,$$

for any parameter μ. That polynomial equation in n becomes the polynomial equation in z,

$$\begin{aligned}&-(z-\bar{a})(z-\bar{b})\prod(z+\bar{\lambda}_j)(z+\bar{\lambda}_j-1)\\&+2(2\bar{e}-\bar{a}-\bar{b}-1)z\prod(z-\bar{\lambda}_j)(z+\bar{\lambda}_j)\\&+(z+\bar{a})(z+\bar{b})\prod(z-\bar{\lambda}_j+1)(z-\bar{\lambda}_j)=0,\end{aligned} \tag{27}$$

with the requirement

$$1-\bar{\lambda}_i \neq \bar{\lambda}_j,$$

which comes from (26). Putting $z = \bar{\lambda}_m$ shows that either $\bar{\lambda}_m = \bar{a}$, $\bar{\lambda}_m = \bar{b}$, or $\bar{\lambda}_m = -\bar{\lambda}_j$ for some j. Thus we may group the $\bar{\lambda}_j$. Assume the last p_1 are equal to $\bar{a}$, the p_2 before that equal to $\bar{b}$ and only these $\bar{\lambda}_j$ take those values. We have

$$\begin{aligned}
&-(z-\bar{a})(z-\bar{b})(z+\bar{a})^{p_1}(z+\bar{a}-1)^{p_1}(z+\bar{b})^{p_2}(z+\bar{b}-1)^{p_2}\prod(z+\bar{\lambda}_j)(z+\bar{\lambda}_j-1)\\
&+2(2\bar{e}-\bar{a}-\bar{b}-1)z(z-\bar{a})^{p_1}(z+\bar{a})^{p_1}(z-\bar{b})^{p_2}(z+\bar{b})^{p_2}\prod(z-\bar{\lambda}_j)(z+\bar{\lambda}_j)\\
&+(z+\bar{a})(z+\bar{b})(z-\bar{a})^{p_1}(z-\bar{a}+1)^{p_1}(z-\bar{b})^{p_2}(z-\bar{b}+1)^{p_2}\prod(-z+\bar{\lambda}_j)(-z+\bar{\lambda}_j-1)=0.
\end{aligned}$$

Now let $z = \bar{\lambda}_1$. This shows that $\bar{\lambda}_1 = 1-\bar{a}$ or $1-\bar{b}$ or $\bar{a}$ or $\bar{b}$ or $\bar{\lambda}_1 = -\bar{\lambda}_j$ for some $\bar{\lambda}_j$ in the product. The first four possibilities are ruled out. Thus assume the last. Without loss of generality, let $\bar{\lambda}_1 = -\bar{\lambda}_2$. The term $(z+\bar{\lambda}_1)(z-\bar{\lambda}_2)$ can now be factored out of every product, leaving the terms, respectively,

$$(z+\bar{\lambda}_1-1)(z+\bar{\lambda}_2-1)\prod_{j=3},\quad (z+\bar{\lambda}_1)(z+\bar{\lambda}_2)\prod_{j=3},\quad (z+\bar{\lambda}_1+1)(z+\bar{\lambda}_2+1)\prod_{j=3}.$$

Now let $z = \bar{\lambda}_3$. This means that $\bar{\lambda}_3 = 1-\bar{a}$, $1-\bar{b}$, $\bar{a}$, $\bar{b}$, $1-\bar{\lambda}_1$, $1-\bar{\lambda}_2$ or $-\bar{\lambda}_j$ for some $j \geq 3$. Of these, only the last can occur. Thus, let $\bar{\lambda}_3 = -\bar{\lambda}_4$ and factor out from each term $(z+\bar{\lambda}_3)(z-\bar{\lambda}_3)$.

Continuing to factor out factors in this fashion gives an equation where the products above are, respectively, replaced by the products

$$\prod_{j=1}^{2p_3}(z+\bar{\lambda}_j-1),\quad \prod_{j=1}^{2p_3}(z+\bar{\lambda}_j),\quad \prod_{j=1}^{2p_3}(z+\bar{\lambda}_j+1),$$

and $\bar{\lambda}_{j+1} = -\bar{\lambda}_j$. Thus $\Sigma_{j=1}^{2p_3}\bar{\lambda}_j = 0$.

Now equate the coefficient of $z^{2p_1+2p_2+2p_3+1}$ to zero. This gives the necessary condition (23).

In the cases mentioned in the converse statement of i), the equation is reducible. For instance let $p_1 = p_2 = 0$; then $e = a + b - \alpha - 1 - p_3$. Assume p_3 non-integral at first and use formula ([5], Vol. 1, p. 104 (10)), dividing the result by $\Gamma(-p_3)$. The result is a ${}_3F_2$ with one numerator parameter $= -p_3$. Thus it terminates and is a rational function of n. But any rational function of n may be written in the form (5).

The rest of the proof is similar.□

5. REPRESENTATION THEOREMS

I now proceed to a description of what I will call a closed-form expression for a hypergeometric function.

Let g: $C^m \to C$ and

$$g(x+n(1,1,\ldots,1)) = g(x) + \delta n,\quad n = 0, 1, 2, \ldots,$$

where $\delta \equiv \delta(g)$ is a rational number. Such a function will be called a *unicial*.

Note any linear function with rational coefficients is a unicial.

Of course, any definition of such a concept as "closed-form" reflects a personal preference, but the definition incorporated in the following theorem covers all those cases where F is known to be summable.

THEOREM 3

The unrestricted ${}_{k+1}F_k(1)$, $k > 0$, can be written in the form

$$K\lambda^L \prod_{j=1}^{p}\Gamma(\xi_j)\Big/\prod_{j=1}^{q}\Gamma(\omega_j), \tag{28}$$

where K, L, ξ_j, ω_j are unicial functions of some or all of the parameters, $\delta(K) = 0$, if and only if $k = 1$.

Proof. Note that if such a representation is possible for some $p > 1$, it is possible for F. This follows by allowing a sufficient number of numerator parameters to cancel denominator parameters.

Thus it is sufficient to show such a representation cannot hold when $k = 2$. I shall actually show more, namely, the alternate

THEOREM 4

Watson's formula (3) cannot be generalized.

Proof. By "generalized" I mean the following formula cannot hold for unrestricted a, b, c, e,

$$ {}_3F_2\binom{a, b, c}{2a, e} = K\lambda^L \prod_{j=1}^{p} \Gamma(\xi_j) \Big/ \prod_{j=1}^{q} \Gamma(\omega_j), \tag{29}$$

where K, L, ξ_j, ω_j are unicials in a, b, c, e, $\delta(K) = 0$. Assume otherwise, then

$$W(n) \stackrel{\text{def}}{=} {}_3F_2\binom{a+n, b+n, c+n}{2a+2n, e+n} = K\lambda^{L+\mu n} \prod_{j=1}^{p} \Gamma(\xi_j + r_j n) \Big/ \prod_{j=1}^{q} \Gamma(\omega_j + s_j n),$$

where μ, r_j, s_j are rationals. Let

$$\rho \stackrel{\text{def}}{=} \sup \{\text{denominator } r_j, s_j\},$$

when r_j, s_j are expressed in lowest terms. Clearly, $W(n + \rho - 1)/W(n-1)$ is rational in n. Making an obvious identification of parameters shows that $D(n)$ (see (21)) satisfies the equation

$$t(n-1) - r(n)t(n + \rho - 1) = 0, \tag{30}$$

where $r(n)$ is rational. This, obviously, is not possible when $\rho = 1$ for then (30) would violate the fact that (20) is irreducible. Assume $\rho > 1$. The functions

$$t_h(n) = \pi_h^n D(n), \quad 1 \le h \le \rho, \tag{31}$$

are all solutions of the equation, π_h being a ρth-root of unity. Furthermore, they are linearly independent, since their Casorati determinant is proportional to the van der Monde determinant of the π_h, which cannot vanish. (One can easily construct from t_h real linearly independent solutions.)

The solutions (31) have complete asymptotic expansions

$$t_h(n) \sim \pi_h^n n^{\phi_1}\left\{1 + \frac{\sigma_1}{n} + \frac{\sigma_2}{n^2} + \cdots,\right\}, \quad n \to \infty,$$

$$\phi_1 = 2(a + b - e - \alpha - 1).$$

Since (30) has a solution in common with the irreducible equation (20), it must admit every solution of (20) ([7], p. 336). Thus it must have a solution

$$v_2(n) \sim (-1)^n n^{\phi_2}\left\{1 + \frac{\sigma_1^*}{n} + \frac{\sigma_2^*}{n^2} + \cdots\right\}, \quad n \to \infty,$$

$$\phi_2 = 2(e - \alpha - 2).$$

But no linear combination of the t_h can furnish, *in general*, a solution with this behavior, since taking linear combinations will change the power of n by integer values. For instance, let e be irrational, a, b, α rational. (Note in this case (23), (24) reveal that the equation is, indeed, irreducible). On the other hand, when ϕ_1 and ϕ_2 differ by integers, such a possibility may occur, for instance, $e = (a + b + 1)/2$. This gives an F summed by Watson's formula. □

There is a slick alternate proof for $\rho = 2$ that does not depend on asymptotic analysis. Then the equation (30) becomes

$$t(n-1) - r(n)t(n+1) = 0.$$

But irreducible equations are unique, hence the middle term of (20) must vanish, which means $e = (a + b + 1)/2$.

Of course, the theorem is applicable only in general, and does not tell us when any particular F has a closed-form expression. The following result, a weak form of a converse for the previous theorem, is of interest.

THEOREM 5

Let the equation (13) be reducible. Then F can be written in terms of gamma functions.

Proof. If the equation is reducible, then it must have a solution of the form (5). But this can happen only when $p = q$, and either $\lambda = 1$ or -1. For $\lambda = 1$, the solution represented must be a multiple of $C(n)$. This implies

$$C(n) = C(0) \prod_{j=1}^{p} \frac{(\lambda_j)_n}{(\mu_j)_n}.$$

Using the asymptotic formula for $C(n)$, (19), gives

$$F = \frac{2\Gamma(f)\Gamma(e + f - a - b - c)}{\Gamma(b)\Gamma(c)} \prod_{j=1}^{p} \frac{\Gamma(\lambda_j)}{\Gamma(\mu_j)}.$$

When $\lambda = -1$, the solution must be a multiple of $C_5(n) \sim 2(-1)^n n^{\phi_2}$, and this case is argued similarly. □

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