Multiplicative representation of disjointness preserving operators

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SUMMARY

Let X and Y be vector lattices and let T : X → Y be a disjointness preserving linear operator, i.e., |

\[ |Tx_1| \wedge |Tx_2| = 0 \text{ if } |x_1| \wedge |x_2| = 0; \ x_1, x_2 \in X. \]

Necessary and sufficient conditions are obtained under which T can be written as a multiplication, i.e.,

\[ Tx(\cdot) = e(\cdot)x(\tau(\cdot)), \]

where \( \tau \) is a continuous mapping from a topological space on which Y can be represented as a vector lattice of extended real valued continuous functions into a topological space on which X can be similarly represented. If X and Y are normed lattices and T is continuous, these conditions are satisfied and hence T can be represented as such a multiplication. With the author's permission, several remarks made by C.B. Huijsmans and B. de Pagter are incorporated in the text.

INTRODUCTION

For any extremally disconnected compact space Q, we denote by \( C_\infty(Q) \) the vector lattice of all extended real valued continuous functions on Q which may assume the values ±\( \infty \) on nowhere dense subsets of Q (see e.g. [11], section 47). Let \( Q_1 \) and \( Q_2 \) be extremally disconnected compact spaces, E a closed subset of \( Q_2 \), \( \tau \) a continuous mapping from E into \( Q_1 \) and e a function in \( C_\infty(Q_2) \) having its support contained in E. The set \([Q_1, Q_2, E, \tau, e]\) determines an operator \( \psi_{\tau, e} \) from \( C_\infty(Q_1) \) into \( C_\infty(Q_2) \), defined by

\[
\psi_{\tau, e}(x)(q) = \begin{cases} 
e(x(q)) & \text{if } q \in E, \\ 0 & \text{if } q \notin E, \end{cases}
\]
for all \(x \in C_\infty(Q_1)\) and all \(q \in Q_2\). Some precautions are necessary when applying this formula because infinite values may be multiplied by zero. We shall come back to this point. It is obvious that \(\psi_{\tau,e}\) is disjointness preserving. In other words, \(\psi_{\tau,e}\) is a \(d\)-homomorphism.

The purpose of the present paper is to indicate necessary and sufficient conditions for a disjointness preserving operator \(T\) which will guarantee the validity of a multiplicative representation for \(T\) as mentioned above. The most interesting case in which this is possible is the case of a continuous \(d\)-homomorphism between normed lattices. The possibility to obtain a convenient representation for a rather large class of operators is of interest in itself and for \(d\)-homomorphisms this is so in particular because these operators have recently attracted a great deal of attention (see [1], [2], [12], [15], [16], [17]). Most of the work on this subject has been devoted to a special type of \(d\)-homomorphisms, namely to orthomorphisms and stabilizers in the terminology of [12] and [16] or to unextending operators in the terminology of [1], [2]. We recall that if \(X\) is a vector sublattice of the vector lattice \(W\), then the linear operator \(T : X \to W\) is called unextending if \(|x_1| \wedge |x_2| = 0\) implies \(|Tx_1| \wedge |x_2| = 0\).

The structure of these operators is less complex than the structure of \(d\)-homomorphisms in general. The multiplicative representation of order bounded unextending operators was obtained by Wickstead in [15]. Later, in [1] and [2], it was shown that in the most important case of Banach lattices the additional assumption of order boundedness is superfluous. Besides, a general definition of the multiplicative representation of operators acting between vector lattices was given in [1], [2] and a theorem on the multiplicative representation of \(d\)-isomorphisms was established. But only now we are able to present such a representation for \(d\)-homomorphisms. Though, while writing [1] and [2], the authors did not know how to obtain such a representation, it is so nevertheless that a great deal of the work done at that time comes in handy in section 3. In particular, the proof of Proposition 3.3 below is similar in some details to that of Theorem 3.1 in [1] and it depends heavily on a formula defining a mapping \(\tau\) which is constructed in Proposition 3.1 and which is essentially due to A. Koldunov. Proposition 3.2 is a generalization of Theorem 4.1 in [1] to the case of \(d\)-homomorphisms. It should be noted also that for the very special kind of order bounded \(d\)-homomorphisms either on \(L_p\)-spaces or \(C(K)\)-spaces the multiplicative representation was obtained by Ivanik [4] and Arendt [3] respectively. Some interesting applications of results from [1], [2] are contained in [6], where the spectral properties of \(d\)-isomorphisms are investigated. We may expect (and the results in [3] point in that direction) that multiplicative representations can be useful in the study of spectral properties.

All vector lattices that we consider are over the field of real numbers, but all results can be extended to the complex case. The paper is organized as follows. Section 1 contains definitions and some preliminary results. Section 2 contains the main Theorem A and its proof under the hypothesis that Propositions B and C hold. These propositions are proved separately in sections 3 and 4 respectively. The concluding section 5 contains an example of a Dedekind complete
Banach lattice $Z$ and a $d$-homomorphism $T : Z \to Z$ which is neither regular nor order bounded, nor continuous, nor possessing a multiplicative representation. This is in contrast to the fact that every unextending operator in a Banach lattice is regular, continuous and admits a multiplicative representation ([11], [2]).

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1. NOTATION AND PRELIMINARY RESULTS

All non-defined terms from the theory of partially ordered vector spaces can be found in [5], [11], [14]. All vector lattices considered are assumed to be Archimedean. For elements $x_1, x_2$ of a vector lattice $X$ we write $x_1 \perp x_2$ if the elements are disjoint, i.e., if $|x_1| \wedge |x_2| = 0$. The Dedekind completion of a vector lattice $X$ is denoted by $kX$. For $x_0 \in X$ we denote by $X(x_0)$ the ideal generated by $x_0$, i.e., $X(x_0)$ is the set of all $x \in X$ satisfying $|x| \leq \lambda |x_0|$ for an appropriate $\lambda \in \mathbb{R}^+_\infty$.

If $B$ is a compact (always Hausdorff) space, then $C(B)$ denotes the vector lattice of all real continuous functions on $B$. For $x \in C(B)$ its support is the set

$$\text{supp } (x) = \text{cl } \{t \in B : x(t) \neq 0\}.$$  

The function identically equal to 1 on $B$ is denoted by $1_B$. The collection of all extended (real) continuous functions on $B$ which may assume the values $\pm \infty$ on nowhere dense sets is denoted by $D_\infty(B)$. If $D_\infty(B)$ turns out to be a vector space we write $C_\infty(B)$. This is certainly the case if $B$ is extremally disconnected (see e.g. [11], section 47). For each $x \in D_\infty(B)$ we define $R(x)$ to be the set of all $t \in B$ for which $|x(t)| < \infty$.

For any vector lattice $X$ we denote by $Q(X)$ the Stone space of $X$. This is, therefore, an extremally disconnected compact space such that $kX$ can be represented as an order dense ideal (foundation) in the Dedekind complete vector lattice $C_\infty(Q)$.

If $F$ is a subset of the compact space $B$ and if $x_1$ and $x_2$ belongs to $D_\infty(B)$, we write $(x_1 - x_2)[F] = 0$ if there exists a neighbourhood $U$ of $F$ such that $x_1(t) = x_2(t)$ for all $t \in U$. If $x_2 = \lambda 1_B$, we simply write $x_1[F] = 0$ instead of $(x_1 - \lambda 1_B)[F] = 0$.  

If $X$ and $Y$ are vector lattices and $T : X \to Y$ is linear, then $T$ is called a $d$-homomorphism if $x_1 \perp x_2$ implies $Tx_1 \perp Tx_2$. Observe that in this case $|Tx| = |T|x|$ for all $x \in X$. Each positive $d$-homomorphism is an order homomorphism (Riesz homomorphism; vector lattice homomorphism), and then $|Tx| = T(|x|)$ for all $x \in X$. Meyer proved in [12] that each order bounded $d$-homomorphism $T : X \to Y$ admits a decomposition $T = T_+ - T_-$, where $T_+ = T \vee 0$ and $T_- = (-T) \vee 0$. The operators $T_+$ and $T_-$ are Riesz homomorphisms satisfying $T_+ x = (Tx)_+$ and $T_- x = (Tx)_-$ for all $0 \leq x \in X$ (see also [19]).

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We now recall the definition of the multiplicative representation of an operator as given in [1]. Let $B_1$ and $B_2$ be compact spaces, $E$ a closed subset of $B_2$, $\tau$ a continuous mapping from $E$ into $B_1$ and $e$ a function belonging to $D_\infty(B_2)$ with $\text{supp}(e) \subset E$. The collection $[B_1, B_2, E, \tau, e]$ generates an operator $\psi_{\tau, e} : D_\infty(B_1) \to D_\infty(B_2)$ by the formula

$$
(\psi_{\tau, e}x)(q) = \begin{cases} 
  e(q)x(\tau(q)) & \text{if } q \in E, \\
  0 & \text{if } q \in B_2 \setminus E.
\end{cases}
$$

This definition needs some comment since the multiplication on the right is not always possible and even if it is possible, the function so obtained need not be in $D_\infty(B_2)$. First of all we restrict ourselves to those $x$ in $D_\infty(B_1)$ for which $\tau^{-1}(R(x))$ is dense in $E$. For $q \in B_2 \setminus E$ we define $(\psi_{\tau, e}x)(q) = 0$ as already observed, and for $q \in E \cap R(e) \cap \tau^{-1}(R(x))$ we define $(\psi_{\tau, e}x)(q) = e(q)x(\tau(q))$. If this function has an extension to an extended realvalued continuous function on the whole of $B_2$, we define this (necessarily unique) extension to be $\psi_{\tau, e}x$. For these $x$ we clearly have $\psi_{\tau, e}x \in D_\infty(B_2)$. We consider $\psi_{\tau, e}x$ only for those functions $x$ for which the above definition makes sense.

**DEFINITION 1.1.** Let $T$ be an operator from the vector lattice $X$ into the vector lattice $Y$. We shall say that $T$ admits a multiplicative representation if there exists a collection $[B_1, B_2, E, \tau, e]$ as above such that $X$ and $Y$ can be represented as vector sublattices of $D_\infty(B_1)$ and $D_\infty(B_2)$ respectively in such a way that $T$ is exactly the restriction of $\psi_{\tau, e}$ to $X$, i.e., for any $x \in X$ the function $\psi_{\tau, e}x$ is well defined and $(Tx)(q) = (\psi_{\tau, e}x)(q)$ for all $q \in B_2$.

Each $T$ admitting a multiplicative representation is obviously an order bounded $d$-homomorphism. In [1] it was proved that if $T$ admits a multiplicative representation on appropriate representing compact spaces for $X$ and $Y$, then $T$ admits as well a multiplicative representation on the Stone spaces $Q(X)$ and $Q(Y)$. Since it is our purpose here to prove the existence of a multiplicative representation for a $d$-homomorphism (subject to certain conditions), we shall at once consider only such representations on Stone spaces, leaving aside the problem of finding a multiplicative representation on other compact spaces.

2. FORMULATION OF THE RESULTS

Let $T$ be a (linear) operator from the vector lattice $X$ into the vector lattice $Y$.

**DEFINITION 2.1.** The operator $T : X \to Y$ is said to satisfy condition (R) if

$$
\inf \{|Tx'_n| + |Tx''_n|\} = 0
$$

for all sequences $\{x'_n\}$ and $\{x''_n\}$ in $X$ that converge relatively uniformly to zero.

We recall that a sequence $\{x_n\}$ in $X$ converges relatively uniformly to zero (notation $x_n \xrightarrow{\tau} 0$) if there exists an element $r \in X_+$ and a sequence $\varepsilon_n \downarrow 0$ of
positive numbers such that $|x_n| \leq \varepsilon_n r$ for $n = 1, 2, \ldots$. The sequence converges to zero in order if there exists a sequence $p_n \downarrow 0$ in $X_+$ such that $|x_n| \leq p_n$ for $n = 1, 2, \ldots$ (notation $x_n \downarrow 0$). The operator $T$ is $r$-continuous (sequentially) if $x_n \downarrow 0$ implies $Tx_n \downarrow 0$ and $T$ is $r$-o-continuous (sequentially) if $x_n \rightarrow 0$ implies $Tx_n \rightarrow 0$. It is easy to see that almost any kind of continuity of an operator $T$ (for example, $r$-continuity and $r$-o-continuity) implies condition (R). In particular, if $X$ and $Y$ are normed lattices and $T$ is norm continuous, then $T$ satisfies (R). Therefore, condition (R) is not very restrictive. Compare condition (R) with the condition that appears in Theorem 3.9 of [19], where the absolute values are replaced by the positive parts. We shall now formulate the main result of this paper.

THEOREM A. Let $X$ and $Y$ be arbitrary (Archimedean) vector lattices and let $T : X \rightarrow Y$ be a $d$-homomorphism. Then the following conditions are equivalent.

1. For any $x \in X$ the restriction of $T$ to the ideal $X(x)$ generated by $x$ admits a multiplicative representation (on Stonian spaces $Q(X)$ and $Q(Y)$);
2. $T$ is regular;
3. $T$ is sequentially $r$-continuous;
4. $T$ is sequentially $r$-o-continuous;
5. $T$ satisfies (R).

If $T$ is order continuous, then $T$ admits a multiplicative representation on the whole of $X$.

It is evident that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). Thus, the essence of the theorem lies in (5) $\Rightarrow$ (1) if $T$ is an arbitrary $d$-homomorphism and in obtaining a "universal" representation if in addition $T$ is order continuous. Without order continuity such a universal representation is not possible in general. An example is given in section 5.

COROLLARY 1. Let $X$ and $Y$ be normed vector lattices and let $T$ be a norm continuous $d$-homomorphism from $X$ into $Y$. Then $T$ satisfies the conditions mentioned in Theorem A.

If $X$ is a normed vector lattice, then its Dedekind completion $kX$ can be made into a normed vector lattice by defining the norm of any $x \in kX$ as

$$|\hat{x}| = \inf \{ ||x|| : x \in X, |\hat{x}| \leq |x| \}.$$  

COROLLARY 2. If $X$ and $Y$ are normed vector lattices and $T$ is a norm continuous $d$-homomorphism from $X$ into $Y$, then there exists a norm continuous $d$-homomorphism $T^* : kX \rightarrow kY$ such that $T^*$ extends $T$ and $||T^*|| = ||T||$.

The proof of Theorem A (i.e., (5) $\Rightarrow$ (1)) depends on two auxiliary results, Propositions B and C below. The proofs of these will be presented in sections.
3 and 4 respectively. Except for these two propositions the rest of the proof of Theorem A is rather simple and will be presented now.

PROPOSITION B. If $X$ and $Y$ are vector lattices and $T$ is a $d$-homomorphism from $X$ into $Y$ satisfying condition (R), then $T$ is order bounded.

We proceed as if Proposition B has been proved. This allows us to use Meyer's result mentioned in section 1 according to which $T_+$ and $T_-$ are Riesz homomorphisms. Hence, instead of dealing with an order bounded $d$-homomorphism $T$ we may assume now that $T$ is a Riesz homomorphism from $X$ into $Y$. We may just as well consider $T$ as a Riesz homomorphism from $X$ into the Dedekind completion $kY$. This does not affect the validity of Theorem A, since we try to find at least one multiplicative representation for $T$ and not a multiplicative representation on every compact space on which $Y$ can be represented. In the latter case we could not replace $Y$ by $kY$. The next simplifying step is to use a theorem by Lipecki [9] and Luxemburg-Schep [10] stating that each Riesz homomorphism $T$ from a vector lattice $X$ into a Dedekind complete vector lattice $Y$ can be extended to a Riesz homomorphism $T^\wedge$ from $kX$ into $Y$. Therefore, we may (and shall) regard $X$ and $Y$ in Theorem A as Dedekind complete vector lattices and $T : X \to Y$ as a Riesz homomorphism. The last step, which concludes the proof, is as follows.

PROPOSITION C. If $X$ and $Y$ are Dedekind complete vector lattices and $T$ is a Riesz homomorphism from $X$ into $Y$, then, for any $x \in X$, the restriction of $T$ to the ideal $X(x)$ admits a multiplicative representation. If $T$ is order continuous, then $T$ admits a multiplicative representation on the whole of $X$.

REMARK. Having read the manuscript of the present paper, A.V. Koldunov informed me that some results close to Proposition C are contained in his dissertation [7], but these have not been published elsewhere.

3. PROOF OF PROPOSITION B

Let $B$ be a compact space, let $X$ be a vector sublattice of $C(B)$ which contains the constant functions and separates the points of $B$ and let $\phi$ be a linear operator from $X$ to a space $C_\omega(Q)$, where $Q$ is an extremally disconnected compact space. We fix the following notations:

\[ E_\phi = \{ q \in Q : (\phi(x))(q) \neq 0 \text{ for at least one } x \in X \}, \]
\[ e_\phi = \phi(1) \text{ and } E^- = \text{cl } E. \]

Note that $E$ is empty if and only if $\phi$ is the null operator. In the following we shall sometimes write $(\phi x)(q)$ or $\phi(x)(q)$ instead of $(\phi(x))(q)$. Recall that $x[s] = 0$ means that $x(s') = 0$ for all $s' \in$ a neighbourhood of the point $s \in B$.

PROPOSITION 3.1. If $\phi$ is a $d$-homomorphism (not the null operator), then there exists a continuous mapping $\tau = \tau_\phi$ from $E$ into $B$ such that the following
condition \((N1)\) is satisfied:

\[(N1) \quad \text{If } q \in E, \; x \in X \text{ and } x[\tau(q)] = 0, \text{ then } (\varphi x)(q) = 0.\]

**PROOF.** For each \(q \in E\) we put \(X_q = \{x \in X : (\varphi x)(q) \neq 0\}\). Since \(|\varphi x| = |\varphi|_x|\) for all \(x \in X\), it is clear that \(x \in X_q\) implies \(|x| \in X_q\). We now define the subset \(\tau(q)\) of \(B\) by

\[\tau(q) = \cap \{\text{supp } (x) : x \in X_q\}.\]

We shall prove that \(\tau(q)\) consists of one point only and that the thus defined mapping \(\tau\) has the desired properties. The proof is divided into several steps.

(a) We prove that the set \(\tau(q)\) is not empty. The set is an intersection of closed subsets of the compact set \(B\). Hence, to show that \(\tau(q)\) is not empty, it is sufficient to show that any finite intersection of the sets \(\text{supp } (x)\); \(x \in X_q\), is not empty. For this it is sufficient to show that \(x_1, x_2 \in X_q\) implies \(|x_1| \cap |x_2| \in X_q\). We may assume that \(x_1\) and \(x_2\) are positive. Writing \(x_i = i - i x_2 (i = 1, 2)\), we have \(x_1 \perp x_2\), so \(\varphi(x_1) \perp \varphi(x_2)\), which implies that one at least of these, say \(\varphi(x_1)\) vanishes at \(q\). Then

\[0 = \varphi(x_1)(q) - \varphi(x_1 - x_1 \wedge x_2)(q) - \{\varphi(x_1) - \varphi(x_1 \wedge x_2)\}(q),\]

whence \(\varphi(x_1 \wedge x_2)(q) = \varphi(x_1)(q) \neq 0\).

(b) We prove that \(\tau(q)\) consists of one point only. Let \(s \in \tau(q)\) and \(s' \neq s\). There exists a function \(x \in X\) such that \((\varphi x)(q) \neq 0\). There also exists a function \(x' \in X\) such that \((x' - x)s] = 0\) and \(x'[s'] = 0\). Indeed, since \(X\) separates the points of \(B\), there exists \(0 \leq y \in X\) satisfying \(y(s) = 1\) and \(y(s') = 0\). Put \(z = (2y - 1)_{B_+}\) and define \(x'\) by

\[x' = \{(x(s) + 1)z\} \wedge x.\]

Obviously, we have \(x' \in X\), \((x' - x)s] = 0\) and \(x'[s'] = 0\). Observe already for later purposes that it follows from a compactness argument that for each open neighbourhood \(U\) of \(q\) there exists \(x'' \in X\) such that \((x - x'')s = 0\) and \(\text{supp } (x'') \subset U\). We return to \(x'\). It follows from \((x - x')s] = 0\) that \((\varphi x')(q) \neq 0\). Indeed, assuming \((\varphi x')(q) = 0\), we should have \(\varphi(x' - x)(q) \neq 0\), which implies \(x' - x \in X_q\). But then \(s \in \tau(q) \subset \text{supp } (x' - x)\), which contradicts \((x - x')s] = 0\). Hence \((\varphi x')(q) \neq 0\), i.e., \(x' \in X_q\). Its follows, by the definition of \(\tau\), that \(\tau(q) \subset \text{supp } (x')\). Since \(x'(s') = 0\), this shows that \(s'\) is not contained in \(\tau(q)\). It follows that \(\tau(q)\) consists of \(s\) only.

(c) We show that the mapping \(\tau\) satisfies \((N1)\). For this purpose, let \(q \in E\) satisfy \(x[\tau(q)] = 0\). If we should have \((\varphi x)(q) \neq 0\), then \(x \in X_q\) and hence \(\tau(q) \subset \text{supp } (x)\), contrary to our assumption that \(x[\tau(q)] = 0\). Hence \((\varphi x)(q) = 0\).

(d) We prove that \(\tau\) is continuous, i.e., we prove that \(\tau^{-1}(U)\) is open for every open subset \(U\) of \(B\). If \(\tau^{-1}(U)\) is empty, there is nothing more to prove. Assume, therefore, that \(\tau^{-1}(U)\) contains at least one point \(q\). Then \(q \in E\), so there exists \(x \in X\) such that \((\varphi x)(q) \neq 0\). We may assume that \(x \geq 0\). Since \(\tau(q) \in U\), it follows from the remarks made in part (b) that there exists \(x'' \in X\)
such that \( \text{supp}(x') \subseteq U \) and \( (x - x')[\tau(q)] = 0 \). By (N1) this implies \( \varphi(x') (q) = -\varphi(x)(q) \neq 0 \). Replacing \( x \) by \( x'' \), we may assume immediately that there exists \( x \in X \) such that \( \text{supp}(x) \subseteq U \) and \( \varphi(x)(q) \neq 0 \). In virtue of (N1) once more this implies that \( \varphi(x)(q') = 0 \) for each \( q' \) not contained in \( \tau^{-1}(U) \). Thus, the open set \( \varphi(x)^{-1}(0, \infty) \) is a subset of \( \tau^{-1}(U) \). Since \( q \in \varphi(x)^{-1}(0, \infty) \), we have shown thus that \( \tau^{-1}(U) \) is a neighbourhood of \( q \). This proves that every \( q \in \tau^{-1}(U) \) is an interior point of \( \tau^{-1}(U) \), and so \( \tau^{-1}(U) \) is open. Hence, \( \tau \) is continuous.

**REMARKS.**

1. The continuity of \( \tau \) and property (N1) imply immediately that the following stronger version of (N1) holds:

   If \( q \in E \), \( x \in X \) and \( x[\tau(q)] = 0 \), then \( \varphi(x)(q) = 0 \).

2. In the converse direction, it is not true in general that if \( \varphi(x)(q) = 0 \), then \( x[\tau(q)] = 0 \).

3. The assumption that \( Q \) is extremally disconnected is not used in the proof of Proposition 3.1. This assumption becomes important if we want to extend in a continuous manner the mapping \( \tau \) from the open set \( E \) onto its closure \( E^- \). For \( Q \) extremally disconnected this is always possible and whenever desired we shall assume that \( \tau \) has been extended continuously to \( E^- \). In view of the continuity of \( \tau \) the property (N1) continues to hold after the extension.

Having found the mapping \( \tau = \tau_\varphi : E^- \rightarrow B \), we can write \( e = \varphi(1_B) \) and now make the corresponding operator \( \psi_{\tau,e} : D_{\omega}(B) \rightarrow C_{\omega}(Q) \) as described in section 1. It is a natural question whether this operator (or rather its restriction to \( X \)) is exactly the original operator \( \varphi \). In general this is not so (see section 5). Some additional assumptions on \( \varphi \) must be made to obtain the desired equality \( \varphi = \psi_{\tau,e}|X \).

**PROPOSITION 3.2.** With the same hypotheses and notations as before, the following conditions for the \( d \)-homomorphism are equivalent.

1. The set \( E_0 \) is dense in \( E \), where \( E_0 \) is the set of all \( q \in E \) such that if \( x \in X \) and \( x(\tau(q)) = 0 \), then \( \varphi(x)(q) = 0 \).

2. \( \varphi \) is a quasi-lattice mapping [8], i.e., for arbitrary \( x_1, x_2 \in X \) we have

   \[
   |\varphi(x_1 \lor x_2)| - |\varphi(x_1)| \lor |\varphi(x_2)|.
   \]

3. \( \varphi(x) = \psi_{\tau,e} x \) for all \( x \in X \), where \( e = \varphi(1_B) \) and \( \tau : E^- \rightarrow B \) is the continuous mapping from Proposition 3.1.

4. \( \varphi \) is order bounded.

5. \( \varphi \) is regular.

**PROOF.** We shall prove (1) \( \Rightarrow \) (3) \( \Rightarrow \) (5) \( \Rightarrow \) (4) \( \Rightarrow \) (2) \( \Rightarrow \) (1). The implications (3) \( \Rightarrow \) (5) \( \Rightarrow \) (4) are obvious.

(1) \( \Rightarrow \) (3). Since \( E_0 \) is dense in \( E \) it is sufficient to show that \( \varphi(x)(q) = -e(q)x(\tau(q)) \) holds for every \( x \in X \) and every \( q \in E_0 \). For this purpose, define \( x' \in X \) by \( x' = x - x(\tau(q))1_B \). Then \( x'(\tau(q)) = 0 \) and hence, by (1), \( \varphi(x')(q) = 0 \). This is equivalent to \( \varphi(x)(q) = e(q)x(\tau(q)) \).

(4) \( \Rightarrow \) (2). If \( \varphi \) is order bounded, then by Meyer's result [12] as mentioned
earlier, we have $\varphi = \varphi_+ - \varphi_-$ with $\varphi_+$ and $\varphi_-$ Riesz homomorphisms satisfying $\varphi_+ x = (\varphi x)_+$ and $\varphi_- x = (\varphi x)_-$ for all $0 \leq x \in X$. Hence $\varphi_+ x \perp \varphi_- x$ for $x \geq 0$. For an arbitrary $x \in X$ it follows from $x_+ \perp x_-$ that $\varphi x_+ \perp \varphi x_-$ and hence $|\varphi(x)| = = \varphi(|x|)$. For the proof of the formula in (2) we may assume, therefore, that $x_1, x_2 \geq 0$. Writing $u = x_1 \vee x_2$ for brevity, the left hand side of (2) is

$$
|\varphi(u)| = |\varphi_+ u - \varphi_- u| = \varphi_+ u + \varphi_- u = \varphi_+ u \vee \varphi_- u =
\{\varphi_+ (x_1 \vee x_2)\} \vee \{\varphi_- (x_1 \vee x_2)\} = \varphi_+ x_1 \varphi_+ x_2 \vee \varphi_- x_1 \varphi_- x_2.
$$

Similarly $|\varphi(x_1)| = \varphi_+ x_1 \varphi_- x_1$ and $|\varphi(x_2)| = \varphi_+ x_2 \varphi_- x_2$. The formula in (2) follows now.

(2) $\Rightarrow$ (1). Assume that $\mathcal{E}_0$ is not dense in $E$. Since $E$ is open and \( F = \{ q \in Q : e(q) = \pm \infty \} \) is nowhere dense, there exists a point $q \in E \setminus (E_0 \cup F)$. Since $q$ is not in $\mathcal{E}_0$, there exists $x \in X^*$ such that $x(\tau(q)) = 0$, but $(\varphi x)(q) = \pm a \neq 0$ (it is possible that $a = +\infty$ or $a = -\infty$). Choose a positive real number $\lambda$ such that $|\lambda| e(q) | < |a|$ and apply the formula in (2) to the functions $x$ and $\lambda 1_B$ at the point $q$. This gives

$$
|\varphi(x \lambda \lambda 1_B)|(q) = \{ |\varphi(x)|(q) \} \vee \{ |\varphi(\lambda 1_B)|(q) \}.
$$

The right hand side is $|a| \vee |\lambda| e(q) | = |a|$. We estimate the value on the left. Since $x(\tau(q)) = 0$, we have $(x \lambda \lambda 1_B)(\tau(q)) = \lambda$. Hence, by (N1),

$$
|\varphi(x \lambda \lambda 1_B)|(q) \leq |\varphi(\lambda 1_B)|(q) = |\lambda e(q) | < |a|.
$$

We thus obtain a contradiction. It follows that $\mathcal{E}_0$ is dense in $E$. This concludes the proof of Proposition 3.2.

For the special case that $X = C(B)$ and $\varphi(C(B)) = C(Q)$ the implication (4) $\Rightarrow$ (3) has also been proved by Arendt in [3]. Condition (3) shows that $\text{supp}(e) = E^-$ and that for any $x \in X$ the inequality $|\varphi(x)| \leq \| x \|_{\infty} \cdot |e|$ holds. The following corollary is now obvious.

**Corollary.** Let $\varphi$ satisfy the conditions in Proposition 3.2 and let $\varphi(X) \subset Y$, where $Y$ is a vector sublattice of $C_{\infty}(Q)$. Then $\varphi : X \to Y$ is order bounded (even regular).

As already observed, an arbitrary d-homomorphism $\varphi$ need not satisfy the conditions in Theorem 3.2. The condition (R) introduced in section 2 changes the situation.

**Proposition 3.3.** If the d-homomorphism $\varphi : X \to C_{\infty}(Q)$ satisfies the condition (R), then $\varphi$ satisfies all conditions mentioned in Proposition 3.2.

**Proof.** We shall prove that $\varphi$ satisfies condition (1) of Proposition 3.2. Precisely, we shall prove that for any $q \in E$ where $e(q) \neq \pm \infty$ and for any $x \in X$ it follows from $x(\tau(q)) = 0$ that $(\varphi x)(q) = 0$. We may assume that $|e(q)| = 1$ for every $q \in E_1 = \text{supp}(e)$, because otherwise we replace $\varphi$ by $(e^{-1} x_{E_1} + x_{Q \setminus E_1}) \varphi$. Note here that $E_1$ is both open and closed, so that $x_{E_1}$ and $x_{Q \setminus E_1}$ are con-
tinuous. Assume now that (1) fails to hold, i.e., there exist \( q \in E \) and \( x \in X_+ \) such that \( x(\tau(q)) = 0 \) but \( (\varphi x)(q) \neq 0 \). Then there exists an open neighbourhood \( V \) of \( q \) such that \( V \subset E \) and \( |(\varphi x)(q')| \geq \lambda > 0 \) for all \( q' \in V \). For convenience we may assume that \( 0 \leq x < 1_B \). The remaining part of the proof is divided into several steps.

(a) We prove that \( q \in \text{cl } \tau^{-1}(C) \), where \( C \) is the cozero set of \( x \), i.e., \( C = x^{-1}(0, 1) \). For this purpose, note first that \( \tau(q) \) is not contained in \( \text{int } x^{-1}(0) \), because otherwise we should have \( (\varphi x)(q) = 0 \) by (N1). Hence \( \tau(q) \in \text{cl } C \). Assume now that the statement in (a) is not true, i.e., \( q \) not contained in \( \text{cl } \tau^{-1}(C) \). Fix a neighbourhood \( U \) of \( q \) such that \( U \subset \tau^{-1}(x^{-1}(0)) \).

|\((\varphi x)(q')| > \frac{1}{2} \lambda \) for every \( q' \in U \).

Since \( U \) is disjoint from \( \tau^{-1}(C) \), it is obvious that

\( U \subset \tau^{-1}(x^{-1}(0)) \).

For each \( n = 1, 2, \ldots \) there exists a function \( x_n \in X \) such that

\( x_n = 0 \) on the set \( x^{-1}(n, 1) \),

\( x \) on the set \( x^{-1}([0, (n + 1)^{-1}) \),

and moreover \( 0 \leq x_n \leq x \). Indeed, take \( x_n = x \wedge (e - nx)^+ \) for \( n = 1, 2, \ldots \). We have \( x_n \to 0 \) and hence, by (R),

\( \inf |\varphi(x_n)| = 0 \).

From (3) and (2) we infer by (N1) that \( (\varphi x_n)(q') = (\varphi x)(q') \) for every \( q' \in U \), so that according to (1) we have \( |(\varphi x_n)(q')| > \frac{1}{2} \lambda \) for all \( q' \in U \). This contradicts (4) and hence the statement in (a) holds. A slightly stronger statement is needed, as follows:

(a') If \( \varepsilon > 0 \) and \( C_\varepsilon = x^{-1}(0, \varepsilon) \), then \( q \in \text{cl } \tau^{-1}(C_\varepsilon) \). If this does not hold, it would follow from (a) that \( q \in \text{cl } \tau^{-1}(x^{-1}[\varepsilon, 1]) \).

and by the continuity of \( \tau \) this would imply that \( \tau(q) \) is a point of \( \text{cl } x^{-1}[\varepsilon, 1] \). But this is impossible since \( x(\tau(q)) = 0 \).

(b) Fix a natural number \( n \) and for \( p = 0, 1, \ldots, 2n - 1 \) define the sets

\( F_p^{(n)} = x^{-1}\left(\left[ \frac{p}{2n}, \frac{p + 1}{2n} \right]\right) \).

We shall prove that there exist functions \( x'_n, x''_n \in X \) such that

\( |x - x'_n| \leq n^{-1} \cdot 1_B, \ |x - x''_n| \leq n^{-1} \cdot 1_B \)

and

\( x'_n[F_p^{(n)}] = p/2n \) for \( p \) even.
Indeed, as observed in the proof of Proposition 3.1, for \( s \in B \) and \( U \) an open neighbourhood of \( s \), there exists \( y \in X \) such that \( y[s] = 1 \) and \( \text{supp}(y) \subseteq U \). Using a compactness argument it follows that if \( G \subseteq U \subseteq B \) with \( G \) closed and \( U \) open, then there exists \( y \in X \) such that \( y[G] = 1 \) and \( \text{supp}(y) \subseteq U \). Using this remark it is straightforward to construct functions \( x_n, x_n^* \in X \) satisfying (5), (6), (7). Condition (5) implies that \( (x - x_n^*) \xrightarrow{I} 0 \) and \( (x - x_n) \xrightarrow{I} 0 \). Hence, in view of (R), we have

\[
(8) \quad \inf g_n = 0 \text{ for } g_n = |\phi(x - x_n^*)| + |\phi(x - x_n)|.
\]

(c) Fix a natural number \( n_0 \) such that \( n_0^{-1} \leq \frac{1}{2} \lambda \). In virtue of (a') the set

\[
W = V \cap \tau^{-1}(x^{-1}(0, n_0^{-1}))
\]

is a non-empty open subset of \( V \). Choose any \( n \geq n_0 \) and a point \( q' \in W \) satisfying \( |(\phi x)(q')| < \infty \). For the point \( \tau(q') \) there exists an integer \( p \) such that \( 0 \leq p \leq 2n - 1 \) and \( x(\tau(q')) \in F_p^{(n)} \), i.e.,

\[
p/2n \leq x(\tau(q')) \leq (p + 1)/2n.
\]

If \( p \) is even, we derive from (6) and (N1) that

\[
(\phi x_n^*)(q') = (p/2n) \cdot |c(q')| - p/2n \leq x(\tau(q')) < n_0^{-1} \leq \frac{1}{2} \lambda,
\]

whence

\[
(9) \quad |\phi(x - x_n^*)(q')| = |(\phi x)(q') - (\phi x_n^*)(q')| \geq |(\phi x)(q')| - |(\phi x_n^*)(q')| \geq \lambda - \frac{1}{2} \lambda = \frac{1}{2} \lambda.
\]

If \( p \) is odd, we obtain similarly from (7) that

\[
(10) \quad |\phi(x - x_n)(q')| \geq \frac{1}{2} \lambda.
\]

Hence, it follows from (9) and (10) that \( g_n(q') \geq \frac{1}{2} \lambda \) for each \( q' \in W \) with \( |(\phi x)(q')| < \infty \), and so \( g_n(q') \geq \frac{1}{2} \lambda \) for all \( q' \in W \). This contradicts (8). Thus, our assumption that condition of Proposition 3.2 fails to hold is false. It follows that the conditions in Proposition 3.2 hold.

**PROPOSITION B.** If \( X \) and \( Y \) are vector lattices and \( T \) is a \( d \)-homomorphism from \( X \) into \( Y \) satisfying condition \((R)\), then \( T \) is order bounded.

**PROOF.** Fixing \( x_0 \in X_+ \), we have to prove that \( \{Tx : 0 \leq x \leq x_0\} \) is order bounded in \( Y \). We restrict \( T \) to the ideal \( X(x_0) \) generated by \( x_0 \). By the Krein-Kakutani theorem \( X(x_0) \) can be represented as a vector sublattice \( X' \) of a vector lattice \( C(B) \) on a compact space \( B \) such that \( X' \) contains the constant functions and separates the points of \( B \). It is obvious that the restriction \( T|X(x_0) \) is a \( d \)-homomorphism from \( X(x_0) \) into \( Y \) satisfying condition \((R)\). Hence, in view of Proposition 3.3, \( T|X(x_0) \) satisfies all conditions of Proposition 3.2. By the corollary to Proposition 3.2, \( T|X(x_0) \) is an order
bounded operator from $X(x_0)$ into $Y$. Thus, the set $\{T x : 0 \leq x \leq x_0\}$ is order bounded.

4. PROOF OF PROPOSITION C

Let $[B_1, B_2, E, \tau, e]$ and $\psi_{\tau, e} : D_\infty(B_1) \to D_\infty(B_2)$ be as in section 1.

**Lemma 4.1.** The operator $\psi_{\tau, e}$ is order continuous if and only if the following conditions is satisfied:

if $E_0$ is a closed subset of $\text{supp } (e)$ and $\text{int } E_0$ is not empty, then $\text{int } \tau(E_0)$ is not empty.

We omit the simple proof of this essentially known result. Note incidentally that in topology the mappings satisfying the mentioned condition are called skeleton mappings.

**Proposition C.** If $X$ and $Y$ are Dedekind complete vector lattices and $T$ is a Riesz homomorphism from $X$ into $Y$, then, for any $x \in X$, the restriction of $T$ to the ideal $X(x)$ admits a multiplicative representation. If $T$ is order continuous, then $T$ admits a multiplicative representation on the whole of $X$.

**Proof.** The first statement follows immediately from Proposition 3.2. Indeed, if we fix some $x \in X_+$ and we represent $X$ on $\mathcal{Q}(X)$ in such a manner that the image of $x$ is the characteristic function of some open and closed subset $Q_x$ of $\mathcal{Q}(X)$, then the image of $X(x)$ is $C(Q_x)$ and we can apply Proposition 3.2 (3) to $T|C(Q_x)$.

A little more work is needed to prove the proposition for order continuous $T$. First we assume that there exists a weak unit $x$ in $X$. Fix the representation of $X$ on $\mathcal{Q}(X)$ such that the image of $x$ is $1_{\mathcal{Q}(X)}$. As above, the restriction of $T$ to $C(\mathcal{Q}(X))$ admits a multiplicative representation $\psi_{\tau, e}$. Since $T$ is order continuous the mapping $\tau$ is skeleton by Lemma 4.1. We shall prove now that the same representation $\psi_{\tau, e}$ for $T$ is valid on the whole of $X$. Given any $x' \in X_+$, there exists an upwards directed system $(x_\beta)$ in $C(\mathcal{Q}(X))$ such that $0 \leq x_\beta \leq x'$. Then $Tx_\beta \uparrow Tx'$ since $T$ is order continuous. Each $x_\beta$ is an element of $C(\mathcal{Q}(X))$ and thus

$$Tx_\beta = \psi_{\tau, e}(x_\beta) = e(\cdot)x_\beta(\tau(\cdot)).$$

Since $\tau$ is skeleton we have $x_\beta(\tau(\cdot)) \uparrow x'(\tau(\cdot))$, which gives $Tx_\beta \uparrow e(\cdot)x'(\tau(\cdot))$. Hence $Tx' = \psi_{\tau, e}x'$.

The general case (i.e., the case that $X$ has no weak unit) can be reduced to the above case by choosing in $X$ a maximal family of pairwise disjoint elements.

5. EXAMPLES

We begin by recalling that in [1], [2] it was indicated for an arbitrary extremally disconnected compact space $Q$ without isolated points how to construct an unextending operator $T : C_\infty(Q) \to C_\infty(Q)$ which admits no
multiplicative representation. From this example it follows that some additional assumptions are necessary if we want to obtain a multiplicative representation for a \(d\)-homomorphism. On the other hand, if \(X\) is a Banach lattice and \(T\) is an unextending operator from \(X\) into \(X\) (or even into any normed vector lattice \(Y\)), then (by [1], [2]) \(T\) is continuous, regular and admits a multiplicative representation without any further assumption on \(T\). In this case the norm completeness of \(X\) amends the situation. It were naturally to expect that for a \(d\)-homomorphism on a Banach lattice the situation would be the same. However, this is not the case. We present an example of a Dedekind complete Banach lattice \(Z\) and a \(d\)-homomorphism \(T : Z \rightarrow Z\) which is neither regular nor order bounded, nor continuous and which admits no multiplicative representation. This example was not included in [1] due to lack of space.

**EXAMPLE 1.** Let \(Q\) be an extremally disconnected space and let \(q_0 \in Q\) be a non-isolated point in \(Q\). Let \(X = (C(Q), \|\cdot\|_\infty)\) and let \(X_0\) be the set of all \(x \in X\) such that \(x[q_0]\) is constant. Choose a function \(x_0 \in X\) such that \(x_0(q_0) = 0\) but \(x_0[q_0] \neq 0\). Let \(\pi : X \rightarrow X/X_0\) be the natural quotient mapping and let \(\varphi\) be a linear functional on \(X/X_0\) such that \(\varphi(\pi(x_0)) = 1\). Let

\[Y = X \oplus \mathbb{R} \text{ with } \|(x, \lambda)\| = \|x\| \vee |\lambda|,\]

and define the operator \(T : X \rightarrow Y\) by

\[Tx = (x, \varphi(\pi x)).\]

It is easy to verify that \(T\) is a \(d\)-homomorphism and \(T\) is neither regular nor order bounded. Also, \(T\) is not continuous, does not satisfy condition \((R)\) and does not have a multiplicative representation. We outline the proof why \(T\) is not continuous. Put \(x_n = x_0 \wedge n^{-1}1_Q\) for \(n = 1, 2, \ldots\). Then \(\|x_n\| \rightarrow 0\). On the other hand \((x_0 - x_n)[q_0] = 0\), i.e., \(x_0 - x_n \in X_0\). Hence \(\pi(x_0 - x_n) = 0\), so \(\varphi(\pi x_n) = 1\) for all \(n\). It follows that \(Tx_n = (x_n, 1)\) does not tend to zero in norm.

It is of interest to construct for this example the mapping \(\tau\) of Proposition 3.1. Since \(Q(Y) = Q \oplus \{p\}\), where \(p\) is an isolated point, it can be easily verified that \(\tau(q) = q\) for every \(q \in Q(Y) \setminus \{p\}\) and \(\tau(p) = q_0\). Thus, \(q_0 = \tau(p)\) is the only point in \(Q(X)\) where the mapping \(\tau\) does not satisfy the conditions of Proposition 3.2. Indeed, \(x_0(\tau p) = x_0(q_0) = 0\) but \((Tx_0)(p) = 1\).

A small modification gives us the desired example of a "bad" \(d\)-homomorphism from a Banach lattice \(Z\) into itself. We simply put

\[Z = X \oplus Y, \|(x, y)\| = \|x\| \vee \|y\|,\]

and we define \(T_1 : Z \rightarrow Z\) by \(T_1(x, y) = (0, Tx)\). It is easy to see that \(T_1\) has the desired properties.

**EXAMPLE 2.** In conclusion we construct an example showing that neither in Theorem A nor in Proposition C we can in general obtain a multiplicative representation that is valid on the entire lattice \(X\). The construction of this example is based on ideas of I. Kaplansky, A. Koldunov and A. Veksler.
Let $Q$ be an extremally disconnected compact space without isolated points. Then we can find an order dense ideal $X$ in $C_\infty(Q)$, a family $(x_i : i \in I)$ in $X_+$ and a subset $(q_i : i \in I)$ of $Q$ such that the following properties hold:

(a) $(q_i)$ is dense in $Q$;
(b) $x_i(q) \geq 1$ for each $q \in Q$ and $i \in I$;
(c) $x_i(q_i) = \infty$ for each $i \in I$;
(d) for each $x \in X_+$ there is a constant $K(x)$ such that $(x/x_i)(q_i) \leq K(x)$ for each $i \in I$. We recall here that in $C_\infty(Q)$ the quotient $x/x_i$ exists (see [11], [14]).

We define a linear operator $T$ from $X$ into $Y = c_0(I)$ by putting $(Tx)(i) = (x/x_i)(q_i)$. Obviously $T$ is a Riesz homomorphism and $(Tx_i)(i) = 1$ for each $i \in I$. We shall prove that $T$ does not admit a universal multiplicative representation on $X$. The lattice $X$, as an ideal in $C_\infty(Q)$, is thus trivially represented on $Q$, but there may be other representations of $X$ on $Q$. Any representation of this kind can be described by an operator $j : X \rightarrow C_\infty(Q)$. For each such $j$ there exists a function $x_0 \in C_\infty(Q)$ such that $j(x) = x/x_0$ for all $x \in X$.

Let us assume, contrary to what we wish to prove, that there exists a representation $j$ of $X$ and a multiplicative representation $\psi_{\tau,e}$ such that $T|X = \psi_{\tau,e}|X$. Then, for any $x \in X$, we have

$$Tx = \psi_{\tau,e}(jx) = e(\cdot)jx(\tau(\cdot)) = e(\cdot)((x/x_0)(\tau(\cdot))).$$

Hence

$$1 = (Tx_i)(i) = e(i)((x_i/x_0)(\tau(i))).$$

It is easy to see (from the definition of $\tau$ in the proof of Proposition 3.1) that $\tau(i) = q_i$ for each $i \in I$. In view of (1) and (c) this implies that $x_0(q_i) = \infty$ for each $i \in I$. But this is impossible by (a). Hence, $T$ does not admit a multiplicative representation on $X$.

REFERENCES