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Residually Finite Extensions of Periodic Groups

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1. INTRODUCTION

The object of this paper is to establish the existence of a residually finite group A and a homomorphism θ defined on A with some useful properties. First of all, a copy of any countable group is embedded in $A\theta$. Second, when θ is restricted to certain subgroups of A to be defined in the sequel, the kernel of θ is periodic. The group A arises in a natural way as a group of automorphisms. Our methods enable us to establish:

THEOREM A. *Given any finitely generated group T generated by elements of finite order, we may find a finitely generated residually finite group $B = B_T$ generated by elements of finite order and a periodic normal subgroup $K = K_T$ of B with $B/K \approx T$. If T is finitely presented, K can be taken to be the normal closure of a finite number of elements of B.*

We note that if T is periodic the above implies that B is periodic so that Theorem A provides a new tool in the study of periodic groups. (See [1] for a beautiful discussion and a list of references.)

For example, an unresolved problem posed by Hanna Neumann [4, p. 113] is the following: Can one find a prime p such that the verbal group

$$\langle a, b; X^p = 1 \rangle \tag{1}$$

is not hopfian? Our methods together with the existence of a prime p with (1) not residually finite (see [4, p. 113]) easily lead to:

THEOREM B. *Either the groups of (1) are not all hopfian for all p or we can find a prime p and a residually finite periodic group R which has a presentation of the form*

$$R = \langle a, b; a^p = 1, b^p = 1, W_i = 1, i = 1, 2, \ldots \rangle,$$

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where the W_i are certain words expressible as a product of pth powers and such that R/R^p is not residually finite.

Since any countable group can be embedded in a group generated by two elements of finite order (see [2, proof of Theorem 10.4, p. 283], Theorem A has the

COROLLARY. *Any countable group H is embeddable in a group B/K where B is a residually finite group generated by two elements of finite order and where K is periodic.*

The above corollary leads to the construction of a wide class of residually finite groups M such that the elements of finite order of M form a normal subgroup. For example, if H is torsion free and M is a subgroup of B with $H \approx M/K$, then the elements of finite order in M consist precisely of K. If H is not residualy finite, K is clearly not a direct factor of M. If for every countable torsion free group H we associate a fixed residually finite countable group $\bar{H} = M$ as above this association is one-to-one on isomorphism classes. That is, for countable torsion free groups H_1, H_2 $\bar{H}_1 \approx \bar{H}_2$ implies $H_1 \approx H_2$.

Subgroups of residually finite groups are residually finite. If $\bar{B}$ is a group generated by r generators b_i and $\bar{B}$ has a homomorphism α onto the free group F on r free generators x_i with $b_i\alpha = x_i$ then $\bar{B}$ is free and the b_i are free generators of $\bar{B}$. These remarks together with the above corollary yield a quick proof of the well-known fact that free groups are residually finite. (See [2 p. 195]). A quick proof of the known result that free products of cyclic groups of finite order are residually finite also follows. (See Section 5).

2. THE CONSTRUCTION OF A

We begin with a finitely generated group $\tilde{G}$ which may be decomposed in a sequence of direct decompositions of the form

$$\tilde{G} = G_1 \times G_2 \times \cdots \times G_n \times F_n, \qquad F_n = G_{n+1} \times F_{n+1}, \qquad n \geqslant 1.$$

We assume further that each G_{i+1} is isomorphic to a proper direct factor of G_i and α_i is an isomorphism of G_{i+1} onto a proper direct factor of G_i. For example, we may begin with a finitely generated group $\bar{G}$ which has a direct decomposition (see [3]),

$$\bar{G} = A \times B, \qquad A \approx \bar{G} \approx B, \qquad \bar{G} \neq 1.$$

If $\varepsilon_1, \varepsilon_2$ are isomorphisms of $\bar{G}$ onto A and B, respectively, then for $n \geqslant 0$,

$$\bar{G} = B \times B\varepsilon_1 \times B\varepsilon_1^2 \times \cdots \times B\varepsilon_1^n \times A\varepsilon_1^n, \qquad A\varepsilon_1^n = B\varepsilon_1^{n+1} \times A\varepsilon_1^{n+1}$$

so that we may take $G_n = B\varepsilon_1^{n-1}$, $F_n = A\varepsilon_1^{n-1}$, $n \geqslant 1$. The α_i may be defined in terms of $\varepsilon_1, \varepsilon_2$.

If we set $G_n^n = G_n$ for all n, then for all k, $1 \leqslant k < n$, we define G_k^n by

$$G_k^n = (G_{k+1}^n)\alpha_k.$$

We pictorially represent the first "five levels" below:

$$\begin{array}{l}
G_1 \xleftarrow{\alpha_1} G_2 \xleftarrow{\alpha_2} G_3 \xleftarrow{\alpha_3} G_4 \xleftarrow{\alpha_4} G_5 \\
G_1^2 \xleftarrow{\alpha_1} G_2^2 \\
G_1^3 \xleftarrow{\alpha_1} G_2^3 \xleftarrow{\alpha_2} G_3^3 \\
G_1^4 \xleftarrow{\alpha_1} G_2^4 \xleftarrow{\alpha_2} G_3^4 \xleftarrow{\alpha_3} G_4^4 \\
G_1^5 \xleftarrow{\alpha_1} G_2^5 \xleftarrow{\alpha_2} G_3^5 \xleftarrow{\alpha_3} G_4^5 \xleftarrow{\alpha_4} G_5^5
\end{array}$$

Let L_n^n be a direct factor of G_n such that $L_n^n \neq 1$ and

$$G_n = L_n^n \times G_n^{n+1}, \qquad G_{n+1}\alpha_n = G_n^{n+1}. \tag{2}$$

Each L_n^n determines L_j^n, $1 \leqslant j \leqslant n$, by using

$$L_k^n = L_{k+1}^n \alpha_k.$$

We list below pictorially the relation between these subgroups and the α_i

$$\begin{array}{ccccccc}
L_1^1, & L_1^2, & L_1^3 & \cdots & L_1^{n-1}, & L_1^n, & G_1^{n+1} \\
 & \uparrow & \uparrow & & \uparrow & \uparrow & \uparrow \\
 & L_2^2, & L_2^3 & \cdots & L_2^{n-1}, & L_2^n, & G_2^{n+1} \\
 & & \uparrow & & \uparrow & \uparrow & \uparrow \\
 & & L_3^3 & \cdots & L_3^{n-1}, & L_3^n, & G_3^{n+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & \uparrow & \uparrow & \uparrow \\
 & & & & L_{n-1}^{n-1}, & L_{n-1}^n, & G_{n-1}^{n+1} \\
 & & & & & \uparrow & \uparrow \\
 & & & & & L_n^n, & G_n^{n+1} \\
 & & & & & & \uparrow \\
 & & & & & & G_{n+1}^{n+1}
\end{array}$$

The arrows on the ith level above indicate the action of α_{i-1}. We have

$$
\begin{array}{lll}
G_1 & = L_1^1 \times L_1^2 \times L_1^3 \times \cdots \times L_1^{n-1} \times L_1^n \times G_1^{n+1} \\
G_2 & = \quad L_2^2 \times L_2^3 \times \cdots \times L_2^{n-1} \times L_2^n \times G_2^{n+1} \\
G_3 & = \quad\quad L_3^3 \times \cdots \times L_3^{n-1} \times L_3^n \times G_3^{n+1} \\
\cdots & \cdots \\
G_{n-1} & = \quad\quad\quad L_{n-1}^{n-1} \times L_{n-1}^n \times G_{n-1}^{n+1} \\
G_n & = \quad\quad\quad\quad L_n^n \times G_n^{n+1} \\
G_{n+1} & = \quad\quad\quad\quad\quad G_{n+1}^{n+1}
\end{array}
$$

On the kth level above

$$G_k^{n+1} = L_k^{n+1} \times G_k^{n+2}. \tag{3}$$

By using (2) we define θ_n as the projection of G_n onto G_n^{n+1}. The kernel of θ_n is L_n^n. Consider the map $\theta_n \alpha_n^{-1}$. This maps G_n onto G_{n+1}. If L_n is the subgroup generated by the L_n^j, $j \geqslant n$, then L_n is the direct product of the L_n^j

$$L_n = L_n^n \times L_n^{n+1} \times \cdots.$$

The complete preimage of L_{n+1} under $\theta_n \alpha_n^{-1}$ is L_n. Hence, $G_n/L_n \approx G_{n+1}/L_{n+1}$. If L is the subgroup generated by the L_n and if $T_n = G_n L/L$, then $T_n \approx G_n/(G_n \cap L) = G_n/L_n$ so that $T_n \approx T_{n+1}$ for all n. Since G_n is finitely generated but L_n is not finitely generated, $T_n \neq 1$.

2.1. *Permutation Automorphisms*

Let G represent the subgroup of $\tilde{G}$ generated by the G_i so that G is the internal restricted direct product

$$G = G_1 \times G_2 \times G_3 \times \cdots.$$

We now define a group A of automorphisms of G. We may quickly grasp the idea which we will describe in further detail below by thinking of the subgroup E_n generated by the G_i^n, $i \leqslant i \leqslant n$ (which is an internal direct product)

$$E_n = G_1^n \times G_2^n \times \cdots \times G_n^n$$

as an external direct product of n copies of G_1^n so that E_n has a group of automorphisms S_n which is obtained by permutation of coordinates. Here S_n is the symmetric group on n elements. S_n can clearly be extended to a group of automorphisms of G by writing $G = E_n \times \bar{E}_n$ and by defining the action of S_n on $\bar{E}_n$ to be the identity. The idea is then to generate A from the S_n in a suitable manner.

2.2. *Permutation Automorphisms in More Detail*

If $i < n$ we define an automorphism θ_i^n of G which we call a transposition of G_i^n and G_{i+1}^n. The definition is as follows:

$$\begin{array}{lll} x\theta_i^n = x\alpha_i^{-1}, & x \in G_i^n & \\ x\theta_i^n = x\alpha_i, & x \in G_{i+1}^n & \\ x\theta_i^n = x, & x \in L_i^j, & i \leqslant j \leqslant n-1 \\ x\theta_i^n = x, & x \in L_{i+1}^j, & i+1 \leqslant j \leqslant n-1 \\ x\theta_i^n = x, & x \in G_j, & j \neq i, \quad j \neq i+1. \end{array}$$

We write $\theta_i^n = (G_i^n, G_{i+1}^n) = (G_{i+1}^n, G_i^n)$. Clearly the θ_i^n, $i < n$, generate a group of automorphisms of G which are isomorphic to S_n the symmetric group on n elements. If $i < j \leqslant n$, we define

$$(G_i^n, G_j^n) = \theta_i^n \theta_{i+1}^n \cdots \theta_{j-1}^n = (G_j^n, G_i^n).$$

With these definitions, we may freely use notation borrowed from the symmetric group. Thus

$$\alpha = (G_1^8, G_3^8, G_4^8)(G_5^8, G_6^8)(G_7^8, G_2^8) \tag{4}$$

is a well-defined automorphism of G. We call an automorphism generated by the θ_i^n, $1 \leqslant i \leqslant n$, an nth level permutation automorphism. Thus α above is an 8th level permutation automorphism. The subscripts of an nth level permutation automorphism are those i such that G_i is moved. In (4), the subscripts of α are 1, 2, 3, 4, 5, 6, 7. The subscripts of the identity automorphism are the null set. Finally let γ_i, $i \geqslant 2$, be a sequence of automorphisms of G such that $\gamma_i = 1$ or γ_i is an ith level permutation automorphism and such that if $i \neq j$, the subscripts of γ_i and γ_j are disjoint. Then the product

$$\gamma = \gamma_2 \gamma_3 \gamma_4 \cdots \tag{5}$$

represents a well-defined automorphism of G in an obvious way. We call γ a permutation automorphism of G. The group A consists of all automorphisms generated by permutation automorphisms. Permutation automorphisms of G on any level of the G_i^n induce corresponding automorphisms on lower levels. For example if α is as in (4) then α restricted to the subgroup generated by the G_i^9, $1 \leqslant i \leqslant 9$, is precisely

$$(G_1^9, G_3^9, G_4^9)(G_5^9, G_6^9)(G_7^9, G_2^9).$$

We may note if $k \neq j$,

$$(G_k^n, G_j^n)(G_k^{n+1}, G_j^{n+1}) = (L_k^n, L_j^n),$$

where (L_k^n, L_j^n) has the obvious meaning.

3. Some Properties of A

We point out that if γ is an element of A then

$$x \in G_i \quad \text{and} \quad x\gamma \in G_i \quad \text{imply} \quad x\gamma = x. \tag{6}$$

Now we claim if $\gamma \neq 1$ then we can find integers n, j, and s with $j \neq s$ and $L_j^n \gamma = L_s^n$. To see this, by (6) we may first choose i such that $G_i \gamma \neq G_i$. Suppose γ is a product of r permutation automorphisms γ_i, $1 \leqslant i \leqslant r$. Let $i_1 = i$ and let n_k and i_k be such that

$$G_{i_k}^{n_k} \gamma_k = G_{i_{k+1}}^{n_k}, \qquad k = 1, 2, ..., r$$

Let e be the maximum of the n_i. Then by the preservation of lower levels property,

$$G_{i_k}^e \gamma_k = G_{i_{k+1}}^e \qquad \text{and} \qquad L_{i_k}^e \gamma_k = L_{i_{k+1}}^e$$

for all k, $1 \leqslant k \leqslant r$. Hence

$$G_i^e \gamma = G_{i_{r+1}}^e \qquad \text{and} \qquad L_{i_1}^e \gamma = L_{i_{r+1}}^e.$$

If $i \neq i_{r+1}$ we may take $n = e$, $j = i$, $s = i_{r+1}$. If $i = i_{r+1}$ then by (6), γ fixes each element of G_i^e. However,

$$G_i = L_i^i \times L_i^{i+1} \times \cdots \times L_i^{e-1} \times G_i^e.$$

Since $G_i \gamma \neq G_i$, there must be a u, $i \leqslant u \leqslant e-1$, and x, $x \in L_i^u$, with $x\gamma \notin L_i^u$. Hence γ moves L_i^u. However, the image of L_i^u under any element of A is L_p^u for some p. Hence we must have $L_i^u \gamma = L_p^u$, $i \neq p$. This proves the assertion.

We may now state

Theorem 1. *A is a residually finite group.*

Proof. If $\gamma \in A$, then γ induces a permutation γ_* on the groups L_i^n. If we set

$$U_n = \{L_1^n, L_2^n, ..., L_n^n\}$$

γ permutes elements of U_n for each n. Consequently γ_* may be viewed as an element in the unrestricted direct product

$$S_2 \times S_3 \times S_4 \times \cdots,$$

where S_n above is the symmetric group on elements of U_n. Consequently

$$\gamma \to \gamma_* \tag{7}$$

may be viewed as a homomorphism from A into a subgroup of the unrestricted direct product of symmetric groups. By our remarks preceding Theorem 1 if $\gamma \neq 1$, then $\gamma_* \neq 1$ so that the map in (7) is actually an isomorphism so that we see that A is embedded in a residually finite group.

Remark. A is of cardinality of the continuum.

4. The Homomorphism θ

Any permutation automorphism γ either fixes the L_i^n or permutes the L_i^n. Hence $L\gamma = L$ for all $\gamma \in A$. Hence if $\gamma \in A$, γ induces an automorphism $\bar{\gamma}$ of G/L. Namely if $g \in G$, $(gL)\bar{\gamma} = (g\gamma)L$. Moreover the map θ defined by

$$\gamma \to \bar{\gamma}$$

is a homomorphism on A. Since

$$G/L = (G_1 L/L) \times (G_2 L/L) \times \cdots$$

and if $T_i = G_i L/L$ is as in Section 2, we may view θ as a homomorphism of A into the automorphism group of the group $\tilde{T}$ generated by the T_i. $\tilde{T}$ is the direct product of the T_i. If γ is a permutation automorphism and $G_i^n \gamma = G_j^n$ then

$$T_i = G_i L/L = G_i^n L/L \xrightarrow{\bar{\gamma}} G_j^n L/L = G_j L/L = T_j.$$

Hence $T_i \bar{\gamma} = T_j$ and we see $\bar{\gamma}$ may be viewed as a permutation of the symbols T_i. $\bar{\gamma}$ is obtained from γ by following subscripts. For example if in (5), $\gamma_{2i} = (G_{2i}^{2i}, G_{2i-1}^{2i})$, $\gamma_{2i+1} = 1$, $i \geqslant 1$,

$$\bar{\gamma} = (T_1, T_2)(T_3, T_4)(T_5, T_6) \cdots$$

To summarize, in the sequel θ will be interpreted as a homomorphism defined on A such that if $\gamma \in A$ and $\gamma\theta = \bar{\gamma}$ then $\bar{\gamma}$ is a permutation on a infinite countable set of symbols, T_i, $i \geqslant 1$.

5. Groups Generated by Elements of Finite Order

In this section we study infinite groups T which can be generated by a finite number of elements of finite order. By using the right representation of T we may assume without loss of generality that T is an infinite permutation group on a countable number of symbols. To conform with notation of the previous section, it is convenient to assume that T is an infinite permutation group on symbols

$$T_1, T_2, T_3 \cdots. \tag{8}$$

If $t = (T_{i_1}, T_{i_2}, ..., T_{i_q})$ is a permutation of the symbols (8) which is a cycle of order q, let n be the maximum of the integers $i_1, i_2, ..., i_q$. Define $\bar{t} \in A$ by

$$\bar{t} = (G^n_{i_1}, G^n_{i_2}, ..., G^n_{i_q}). \tag{9}$$

Suppose T has r generators t_i, $1 \leqslant i \leqslant r$, where t_i is of order d_i. Hence each t_i is expressible in terms of a product of disjoint cycles U_{ij} on the symbols (8). The order of any cycle in this product is a divisor of d_i. We express this decomposition of t_i into disjoint cycles by writing

$$t_i = U_{i1} U_{i2} U_{i3} \cdots.$$

Define $\gamma_i \in A$ by

$$\gamma_i = \overline{U_{i1}}\, \overline{U_{i2}}\, \overline{U_{i3}} \cdots.$$

Clearly then $\gamma_i\theta = t_i$, $1 \leqslant i \leqslant r$, and γ_i has the same finite order as t_i. If $\bar{A} = \bar{A}_T$ designates the subgroup generated by the γ_i, $\bar{A}\theta = T$. We note that if T is a free product of cyclic groups $\langle t_i \rangle$ of finite order then θ is an isomorphism on $\bar{A}$. For in this case there is a homomorphism θ_* from T onto $\bar{A}$ with $t_i\theta_* = \gamma_i$ so that $\theta\theta_* = 1$.

5.1. *Kernel* $\theta \cap \bar{A}$

We claim that kernel $\theta \cap \bar{A}$ is a periodic group. Let $\gamma \in \bar{A}$ and write

$$\gamma = \gamma_{i_1}^{\varepsilon_1} \gamma_{i_2}^{\varepsilon_2} \cdots \gamma_{i_q}^{\varepsilon_q}, \qquad \gamma\theta = t_{i_1}^{\varepsilon_1} t_{i_2}^{\varepsilon_2} \cdots t_{i_q}^{\varepsilon_q}$$

for suitable ε_i, $\varepsilon_i = 1$ or $\varepsilon_i = -1$. Suppose $\gamma\theta = 1$. We examine the effect of γ on an arbitrary G_{u_1}. Define u_k, $1 < k \leqslant q+1$, by

$$T_{u_k} \xrightarrow{t_{i_k}^{\varepsilon_k}} T_{u_{k+1}}, \qquad 1 \leqslant k \leqslant q.$$

Hence $T_{u_1}(\gamma\theta) = T_{u_{q+1}}$ so $u_{q+1} = u_1$. Define automorphisms δ_k of G as follows: If $u_k = u_{k+1}$, let δ_k be the identity automorphism of G. If $u_k \neq u_{k+1}$ let δ_k be that cycle of $\gamma_{i_k}^{\varepsilon_k}$ that contains a u_k subscript, say

$$\delta_k = (G_{u_k}^{n_k}, G_{u_{k+1}}^{n_k} \cdots). \tag{10}$$

Let δ be the product of the δ_i, $1 \leqslant i \leqslant q$. Let n be any positive integer. Then one may verify that the effect of δ^n on G_{u_1} is precisely the effect of γ^n on G_{u_1}. This follows from the fact that a permutation automorphism either leaves a particular G_i fixed or acts on that G_i via its unique cycle that has an i subscript and from the fact that $u_1 = u_{q+1}$. Now we examine δ_* using $*$ of Section 3. Note $(\delta_k)_* = 1$ or else $(\delta_k)_*$ has a representation in the unrestricted direct product of the S_n such that for $r \geqslant n_k$, the component of $(\delta_k)_*$ in S_r is obtained from (10) as

$$(L_{u_k}^r, L_{u_{k+1}}^r \cdots).$$

Consequently the component of δ_* in any S_r may be viewed as a permutation on d or fewer symbols where d is the sum of the d_{i_k}, $1 \leqslant k \leqslant q$. Hence if $e = d!$, $\delta_*^e = 1$ so that $\delta^e = 1$ and so $\gamma^e = 1$ on G_{u_1}. Since e is independent of u_1, $\gamma^e = 1$. This proves the assertion. We remark that the same proof shows that if $\tilde{A}$ is any subgroup of A generated by a finite number of permutation automorphisms of finite orders, then $\tilde{A} \cap (\text{kernel}\,\theta)$ is periodic. Also, we note that kernel θ, in itself, is not a periodic group. For example, if

$$\alpha_1 = (G_1^2, G_2^2)(G_3^5, G_4^5, G_5^5)(G_6^9, G_7^9, G_8^9, G_9^9)(G_{10}^{14}, G_{11}^{14}, G_{12}^{14}, G_{13}^{14}, G_{14}^{14}) \cdots$$

$$\alpha_2 = (G_1^3, G_2^3)(G_3^6, G_4^6, G_5^6)(G_6^{10}, G_7^{10}, G_8^{10}, G_9^{10})(G_{10}^{15}, G_{11}^{15}, G_{12}^{15}, G_{13}^{15}, G_{14}^{15}) \cdots$$

then clearly $\alpha_1\theta = \alpha_2\theta$ so that $(\alpha_1\alpha_2^{-1})\theta = 1$. However,

$$\alpha_1\alpha_2^{-1} = (L_1^2, L_2^2)(L_3^5, L_4^5, L_5^5)(L_6^9, L_7^9, L_8^9, L_9^9)(L_{10}^{14}, L_{11}^{14}, L_{12}^{14}, L_{13}^{14}, L_{14}^{14}) \cdots$$

so that $\alpha_1\alpha_2^{-1}$ has infinite order.

5.2. *Finitely Presented T*

If T is finitely presented, we claim that

THEOREM. (Kernel θ) $\cap \bar{A}$ *is the normal closure of a finite number of elements of* $\bar{A}$.

Proof. Suppose T has a finite number of defining relations $w_i = 1$, $1 \leqslant i \leqslant n$, where $w_i = w_i(t_1, t_2, ..., t_r)$ is a word in the t_j. Let N be the normal closure in $\bar{A}$ of the elements $\bar{w}_i = w_i(\gamma_1, \gamma_2, ..., \gamma_r)$, $1 \leqslant i \leqslant n$. Then $\bar{w}_i \theta = w_i$ so $\bar{w}_i$ is in the kernel of θ. Also the map θ_1

$$t_i \to \gamma_i N$$

induces a homomorphism from T onto $\bar{A}/N$. But the map θ_2 defined by

$$\gamma N \to \gamma\theta$$

induces a homomorphism from $\bar{A}/N$ onto T and $\theta_2\theta_1 = 1$ so that θ_2 is an isomorphism. Consequently if $\gamma \in \bar{A}$ and $\gamma\theta = 1$, then $\gamma \in N$. Hence

$$(\text{kernel } \theta) \cap \bar{A} = N.$$

6. Some Concluding Remarks

The work of the previous section concludes the proof of Theorem A of the Introduction. In addition we have shown that if T has r generators t_i, $1 \leqslant i \leqslant r$, with respective finite orders d_i, $1 \leqslant i \leqslant r$, then $\bar{A}$ has r generators γ_i, $1 \leqslant i \leqslant r$, with respective orders d_i.

Finally we would like to make some remarks on the unresolved problem of [4, p. 113] posed by Hanna Neumann. If R is any group let R^n be the subgroup of R generated by nth powers. The problem of Hanna Neumann may be stated as follows: If F is the free group on two generators can one find a prime p such that F/F^p is not hopfian? By [4, p. 113], we may choose a prime p such that the verbal group

$$R_2 = \langle a_2, b_2; X^p = 1 \rangle$$

is not residually finite. From our present work, let R be a residually finite group generated by two elements a and b of order p and let θ be a homomorphism of R onto R_2 with a periodic kernel and with $a\theta = a_2$ and $b\theta = b_2$. We examine the structure of R assuming that R_2 is hopfian. For this purpose, let R_1 be the group

$$R_1 = \langle a_1, b_1; a_1^p = b_1^p = 1 \rangle$$

and let θ_1 be a homomorphism of R_1 onto R with $a_1\theta_1 = a$, $b_1\theta_1 = b_1$. First let us examine the kernels of θ_1 and θ. Clearly R_1^p is in the kernel of $\theta_1\theta$ and $R_1/R_1^p \approx R_2$. If R_2 is hopfian this implies that if K is the kernel of $\theta_1\theta$ then $K = R_1^p$. On the other hand if $r \in R$ and $r\theta = 1$, let $r_1 \in R_1$ with $r_1\theta_1 = r$, so that $r_1\theta_1\theta = 1$. This implies that $r_1 \in R_1^p$ so $r \in R^p$. Hence the kernel of

θ is precisely R^p. Also since R is residually finite but R/R^p is not, there exists an element $r \in R$ with $r^p \neq 1$. By using the fact that the kenel of θ_1 is contained in K we see that the kernel of θ_1 consists of elements which are expressible in R_1 as products of pth powers. Translated to R, this means that R has a presentation of the form

$$R = \langle a, b; a^p = 1, b^p = 1, w_i = 1 \rangle \tag{11}$$

for a certain set of words w_i where each w_i is expressible as a product of pth powers. Since θ has a periodic kernel and R_2 is periodic, R is periodic. This concludes the proof of Theorem B of the Introduction.

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