Dismantlings and iterated clique graphs

M.E. Frías-Armenta\textsuperscript{a,1}, V. Neumann-Lara\textsuperscript{b,2}, M.A. Pizaña\textsuperscript{c}

\textsuperscript{a}Universidad de Sonora, División de Ciencias Exactas y Naturales, Depto. de Matemáticas. Blvd. Rosales y Blvd. Luis Ensenas s/n. Hermosillo Sonora, Mexico

\textsuperscript{b}Instituto de Matemáticas, U.N.A.M. Circuito Exterior, C.U. México 04510 D.F. Mexico

\textsuperscript{c}Universidad Autónoma Metropolitana, Depto. de Ingeniería Eléctrica. Av. San Rafael Atlixco 186, Col. Vicentina, Del Iztapalapa. México 09340 D.F. Mexico

Received 18 June 2002; received in revised form 29 November 2003; accepted 11 December 2003

Dedicated to the memory of our beloved friend Victor Neumann-Lara who passed away on 26 February 2004.

Abstract

Given a graph $G$ and two vertices $x, y \in V(G)$, we say that $x$ is dominated by $y$ if the closed neighbourhood of $x$ is contained in that of $y$. Here we prove that if $x$ is a dominated vertex, then $G$ and $G - \{x\}$ have the same dynamical behaviour under the iteration of the clique operator.

\textcopyright 2004 Elsevier B.V. All rights reserved.

Keywords: Clique graphs; Iterated clique graphs; Dismantlings; Clique behaviour

1. Introduction and terminology

All our graphs are finite, simple and loopless. We shall identify induced subgraphs with their vertex sets, in particular, we shall write $x \in G$ instead of $x \in V(G)$. Given $x \in G$, the closed neighbourhood $N_G[x]$ of $x$ is the set consisting of $x$ and all its neighbours. Given $x, y \in G$ we say that $x$ is dominated by $y$ (in $G$) if $N_G[x] \subseteq N_G[y]$. Note that every vertex is dominated by itself, however we say that $x$ is dominated (without specifying who is $y$) only when $x$ is dominated by a different vertex. Given two graphs $G$ and $H$ we say that $G$ is dismantleable to $H$ if there is a sequence of graphs $G_0, G_1, \ldots, G_n$ satisfying $G = G_0$, $H \cong G_0$, and $G_{i+1} = G_i - \{x_i\}$, where $x_i$ is a dominated vertex of $G_i$.

A clique of $G$ is a maximal complete subgraph. The clique graph $k(G)$ of $G$ is the intersection graph of all cliques of $G$: every clique is a vertex, two of them being adjacent if they share at least one vertex. Similarly, $c(G)$ is the intersection graph of all complete subgraphs of $G$. Clearly, $k(G)$ is an induced subgraph of $c(G)$. We define inductively the \textit{iterated clique graphs} by the formulas $k^0(G) = G$ and $k^{n+1}(G) = k(k^n(G))$. Iterated clique graphs have been studied in several papers, for a large bibliography see [10,11]. It is known (and easy to prove) that a graph $G$ is either $k$-divergent (i.e. $\lim_{n \to \infty} |k^n(G)| = \infty$) or $k$-stationary (i.e. $k^n(G) \cong k^m(G)$ for some $n < m$). A special case of a $k$-stationary graph is a $k$-null graph: for some $n$, $k^n(G)$ is isomorphic to the one vertex graph $K_1$. We say that two graphs $G$ and $H$ have the same $k$-behaviour if both are $k$-divergent or both are $k$-stationary and both are $k$-null or both are not $k$-null.

Given two graphs $G$ and $H$, we say that $H$ is a \textit{retract} of $G$ if there are two weak morphism of graphs (images of adjacent vertices are adjacent or equal) $x: H \to G$ and $\beta: G \to H$ such that $\beta \circ x$ is the identity in $H$. 

E-mail addresses: eduardo@gauss.mat.uso.mx (M.E. Frías-Armenta), neumann@matem.unam.mx (V. Neumann-Lara), map@xanum.uam.mx (M.A. Pizaña).

URL: http://xamanek.izt.uam.mx/map.

\textsuperscript{1}Partially supported by CONACyT, Grants 1-36596-E, 489100-1-010260.

\textsuperscript{2}Partially supported by CONACyT, Grant 400333-5-27968E.
Since whenever $G$ is dismantleable to $H$, we have that $H$ is a retract of $G$, Neumann–Lara’s retraction theorem \cite{7,8} tells us that if $H$ is $k$-divergent then so is $G$. Also, Prisner proved \cite{9} that if $G$ is dismantleable to $K_1$ then $G$ is $k$-null.

Our main Theorem (Theorem 5) states a stronger result: If $G$ is dismantleable to $H$ then $G$ and $H$ have the same $k$-behaviour. A special kind of dismantlings will play a key rôle in what follows:

**Definition 1.** Let $G$ and $H$ be graphs, we say that $G \xrightarrow{k} H$ if $H$ is isomorphic to an induced subgraph $H_0$ of $G$ such that every vertex $x$ in $G$ is dominated by some (not necessarily different) vertex $y$ in $H_0$.

It is straightforward to verify that $G \xrightarrow{k} H$ implies that $G$ is dismantleable to $H$. Also $G$ is dismantleable to $H$ iff there is a sequence of graphs satisfying $G \xrightarrow{k} G_0 \xrightarrow{k} G_1 \rightarrow \cdots \rightarrow G_r = H$. Note that $c(G) \xrightarrow{k} k(G)$ for every graph $G$.

2. Dismantlings and $k$-behaviour

**Lemma 2.** Assume $H_0$ is an induced subgraph of $G$ satisfying that every vertex in $G$ is dominated by some vertex in $H_0$. Let $Q_1, Q_2 \in k(G)$ (not necessarily different), then $Q_1 \cap Q_2 \neq \emptyset$ iff $Q_1 \cap Q_2 \cap H_0 \neq \emptyset$.

**Proof.** Take $Q_1, Q_2 \in k(G)$ and $x \in Q_1 \cap Q_2$, as $x$ is dominated by some $y \in H_0$ (possibly $y = x$) it follows that $Q_1 \cup Q_2 \subseteq N_0[x] \subseteq N_0[y]$, therefore $Q_1 \cap Q_2 \cap H_0 \supseteq \{y\} \neq \emptyset$. \(\square\)

**Theorem 3.** If $G \xrightarrow{k} H$, then $k(G) \xrightarrow{k} k(H)$.

**Proof.** Let $H_0 \cong H$ be a induced subgraph of $G$ such that every vertex in $G$ is dominated by some vertex in $H_0$. For each clique $Q \in k(H_0)$ select a fixed clique $f(Q) \in k(G)$ satisfying $Q \subseteq f(Q)$. Obviously $Q = f(Q) \cap H_0$, so we know $f$ to be injective. Now Lemma 2 tells us that $Q_1, Q_2 \in k(H_0)$ are adjacent iff $f(Q_1)$ and $f(Q_2)$ are adjacent (in $k(G)$). It follows that $k(H) \cong k(H_0) \cong f(k(H_0))$, where $f(k(H_0))$ is the subgraph of $k(G)$ induced by $\{f(Q) : Q \in k(H_0)\}$.

Finally, if $Q \in k(G)$ let $Q_0 \in k(H_0)$ satisfying $Q \cap H_0 \subseteq Q_0$. We claim that $Q$ is dominated by $f(Q_0)$. By Lemma 2 for every $Q_i \in k(G)$ we have $Q_i \cap Q \neq \emptyset$ iff $Q_i \cap Q \cap H_0 \neq \emptyset$, but $Q_i \cap Q \cap H_0 \subseteq Q_i \cap Q_0 \subseteq Q_i \cap f(Q_0)$. \(\square\)

**Theorem 4.** If $G \xrightarrow{k} H$ then $k(G) \rightarrow k^2(G)$.

**Proof.** Let $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_r\} \subseteq k^2(G)$. We know by Lemma 2 that $\{Q_1 \cap H_0, \ldots, Q_r \cap H_0\}$ is a set of pairwise intersecting completes of $H_0$. Then for every clique $\mathcal{Q} \subseteq k^2(G)$ select a fixed clique $f(\mathcal{Q}) \in k(G)$ satisfying $f(\mathcal{Q}) \supseteq \{Q_1 \cap H_0, \ldots, Q_r \cap H_0\}$. We claim that $f$ is an isomorphism onto its image and that every vertex in $k(G)$ is dominated by a vertex in $k^2(G)$.

Let $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_r\}, \mathcal{P} = \{P_1, P_2, \ldots, P_s\} \subseteq k^2(G)$. If $f(\mathcal{Q}) \neq f(\mathcal{P})$ we have $Q_i \cap H_0 \neq f(\mathcal{P})$ for all $i = 1, \ldots, r$, since $f(\mathcal{P})$ is a clique, we have $Q_i \cap H_0 \cap P_j \neq \emptyset$ for all $i$ and $j$. Then $Q_i \cap P_j \neq \emptyset$ for all $i$ and $j$. It follows that $\mathcal{Q} \neq \mathcal{P}$ and therefore $f$ is injective. Obviously $f$ preserves adjacencies. If $f(\mathcal{Q})$ is adjacent to $f(\mathcal{P})$ for some $\mathcal{Q}, \mathcal{P} \subseteq k^2(G)$, let $C_0 \in f(\mathcal{Q}) \cap f(\mathcal{P})$ and let $Q_0$ be any clique in $k(G)$ containing $C_0$. Then $Q_0 \in \mathcal{Q} \cap \mathcal{P}$ and therefore $\mathcal{Q}$ and $\mathcal{P}$ are adjacent in $k^2(G)$. Thus $f$ is an isomorphism onto its image.

Now take $\mathcal{Q} = \{C_1, \ldots, C_l\} \subseteq k^2(G)$. Let $\mathcal{Q} \subseteq k^2(G)$ such that $\{Q_1, \ldots, Q_r\} \subseteq \mathcal{Q}$. We claim that $f(\mathcal{Q})$ dominates $\mathcal{Q}$: If $\mathcal{P} \subseteq k(G)$ is adjacent (or equal!) to $\mathcal{Q}$, without loss, assume $C_1 \in \mathcal{Q} \cap \mathcal{P}$. Now $Q_i \cap H_0 \in \mathcal{Q} \cap \mathcal{P}$ since every complete of $H_0$ intersecting $C_1$ also intersects $Q_i \cap H_0 \supseteq C_1$. It follows that $\mathcal{P}$ is also adjacent to $f(\mathcal{Q})$. \(\square\)

**Theorem 5.** If $G$ is dismantleable to $H$, $G$ and $H$ have the same $k$-behaviour. In particular, if $x$ is a dominated vertex of $G$, $G \xrightarrow{k} k(H)$ have the same $k$-behaviour.

**Proof.** Obviously, we only have to prove this in the case $G \xrightarrow{k} H$.

If $H$ is $k$-null we have $k^n(G) \xrightarrow{k} k^n(H) \cong K_1$ for some $n$, but then $k^n(G)$ must be a cone (must have a universal vertex), then $k^{n+2}(G) \cong K_1$. On the other hand, if $G$ is $k$-null we have $K_1 \cong k^n(G) \xrightarrow{k} k^n(H)$ which implies $k^n(H) \cong K_1$. 

If $H$ is $k$-divergent, then $k^*(G) \overset{\leq}{\to} k^*(H)$ implies $|k^*(G)| \geq |k^*(H)|$ and therefore $G$ is also $k$-divergent. Now, let us assume $H$ to be $k$-stationary, hence $k^*(H) \cong k^{n,m}(H)$ for some $n \geq 0, m \geq 1$. Using Theorem 3 we know that $k^{n+m}(G) \overset{\leq}{\to} k^{n+m}(H) \cong k^n(H)$ for all $j$. Then Theorem 4 gives us $kck^n(H) \overset{\leq}{\to} k^{n+m}(G)$ for all $j$. Since any finite graph may only be dismantled to a finite number of (non-isomorphic) graphs, it follows that $k^{n+m}(G) \cong k^{n+m+2}(G)$ for some $i < j$. Thus, $G$ is also $k$-stationary.

If $k^t(G) \cong k^{t+\psi}(G)$ for some minimum $p \geq 1$ and some $t \geq 0$, we say that $p$ is the period of $G$ (we set $p = \infty$ for $k$-divergent graphs). The previous theorem tells us that the finiteness of $p$ is invariant under dismantlings, we shall show now that $p$ itself is not. Consider the graph $R$ obtained from Fig. 1 identifying the following pairs of vertices: \{a, a'\}, \{b, b'\} and \{c, c'\}.

It has three dominated vertices: $u$, $v$ and $w$. The period of $R$ is 3, but $R - \{u\}$ and $R - \{v\}$ have periods 6 and 1 respectively. You may check this either by computer (we used GAP [2]) or by applying the theory of clockwork graphs developed in [4]. Clockwork graphs have been successfully used to construct examples in [6] (see also [5]) and others.

Precursors of clockwork graphs were also used to construct examples in [1,3].

References


