A Determinant Representation for the Distribution of Quadratic Forms in Complex Normal Vectors

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Let the column vectors of $X: M \times N$, $M < N$, be distributed as independent complex normal vectors with the same covariance matrix $\Sigma$. Then the usual quadratic form in the complex normal vectors is denoted by $Z = XLX^H$ where $L: N \times N$ is a positive definite hermitian matrix. This paper deals with a representation for the density function of $Z$ in terms of a ratio of determinants. This representation also yields a compact form for the distribution of the generalized variance $|Z|$. 


Key words and phrases: quadratic form, complex normal vector, hypergeometric functions, distributions.

1. INTRODUCTION

If the complex random matrix $X: M \times N$, $M < N$, is distributed as Gaussian whose density is given by

$$
\pi^{-MN} |\Sigma|^{-N} |B|^{-M} \exp(-\text{tr } \Sigma^{-1}XB^{-1}X^H) \quad (1)
$$

where $\Sigma$ and $B$ are hermitian positive definite, then the density function of $Z = XLX^H$ ($L$ being a hermitian positive definite matrix) is given by Khatri [1] as

$$
\begin{align*}
  f(Z) &= (\hat{F}_{md}(N) |LB|^M |\Sigma|^N)^{-1} |Z|^{N-M} \\
  &\quad \times \exp(-q^{-1} \text{tr } \Sigma^{-1}Z) \ {}_0F_0(T^*, q^{-1} \Sigma^{-1}Z) \quad (2)
\end{align*}
$$

where

$$
q > 0, \quad T^* = I_N - qL^{-1/2}B^{-1}L^{-1/2},
$$

$$
\hat{F}_{md}(N) = \pi^{-1/2M(M-1)} \prod_{i=1}^{M} \Gamma(N-i+1)
$$
Although compact and simple to state, the density function given in (2) is extremely difficult to compute due to the hypergeometric function in matrix argument. A straightforward power series expansion of $_0F_0( . )$ in terms of zonal polynomials is unlikely to produce a satisfactory numerical procedure due to the slow convergence of the series and the difficulty of working with partitions of large integers [2, 3]. For example the algorithm due to McLaren [4] for computing the coefficients of zonal polynomials is restricted to partitions of the integers up to 13 and published tables go as high as 12 [5]. Alternative representations for hypergeometric functions are available in terms of partial differential equations [6], series of Laguerre polynomials [2, 7, 8], series of chi-square distributions [8] and Wishart type representations [7]. None of these approaches is satisfactory in general for numerical work and so in this paper we derive yet another representation based on the work of Gross and Richards [9]. Our result expresses the density function of $Z$ in terms of a ratio of determinants. For $M = 1$ this collapses to the simple scalar distribution discussed for example in [10]. For small $M \geq 2$ the expression can be expanded to give a numerically practical formulation.

The outline of the paper is as follows. In Section 2 we state and prove the new representation for the density of $Z$. In Section 3 we discuss the use of this result in computational work, give some special cases and provide an integral for the distribution of the generalized variance $|Z|$.

## 2. A DETERMINANT REPRESENTATION FOR THE DENSITY OF Z

**Lemma.** Let $x_1, x_2, ..., x_M$ be nonzero eigenvalues of $\Sigma^{-1}Z$, and $\gamma_1, \gamma_2, ..., \gamma_N$ be eigenvalues of the matrix $L^{1/2}BL^{1/2}$, then the density function, $f(Z)$, of $Z$ is given by

$$f(Z) = \pi^{M(M-1)/2} |\Sigma|^{-M} \frac{A_1}{A_2 A_3}$$

where

$$A_1 = 
\begin{vmatrix}
\gamma_1 & \ldots & \gamma_1^{N-M-1} & \gamma_1^{N-M-1} \exp(-x_1 \gamma_1) & \ldots & \gamma_1^{N-M-1} \exp(-x_M \gamma_1) \\
\gamma_2 & \ldots & \gamma_2^{N-M-1} & \gamma_2^{N-M-1} \exp(-x_1 \gamma_2) & \ldots & \gamma_2^{N-M-1} \exp(-x_M \gamma_2) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\gamma_N & \ldots & \gamma_N^{N-M-1} & \gamma_N^{N-M-1} \exp(-x_1 \gamma_N) & \ldots & \gamma_N^{N-M-1} \exp(-x_M \gamma_N)
\end{vmatrix}$$

$$A_2 = 
\begin{vmatrix}
x_1 & x_1^2 & \ldots & x_1^{M-1} \\
x_2 & x_2^2 & \ldots & x_2^{M-1} \\
\vdots & \vdots & \ddots & \vdots \\
x_M & x_M^2 & \ldots & x_M^{M-1}
\end{vmatrix} = \prod_{i>j} (x_i - x_j)$$

$$A_3 = 
\begin{vmatrix}
x_1 & x_1^2 & \ldots & x_1^{M-1} \\
x_2 & x_2^2 & \ldots & x_2^{M-1} \\
\vdots & \vdots & \ddots & \vdots \\
x_M & x_M^2 & \ldots & x_M^{M-1}
\end{vmatrix}$$
Proof. First we set up some notation which is necessary for the proof.

\[ 0 < x_1 < x_2 < \cdots < x_M, \quad \gamma = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_N), \]

\[ T = \text{diag}(t_1, t_2, \ldots, t_N) = I_N - qN^{-1} \]

\[ = \text{diag}(1 - q\gamma_1^{-1}, 1 - q\gamma_2^{-1}, \ldots, 1 - q\gamma_N^{-1}) \]

\[ S = \text{diag}(s_1, \ldots, s_N) \]

\[ = \text{diag}(0, q^{-1}e_1, q^{-1}e_2, \ldots, q^{-1}e_{N-M-1}, q^{-1}x_1, q^{-1}x_2, \ldots, q^{-1}x_M) \]

and

\[ 0 < q^{-1}e_1 < q^{-1}e_2 < \cdots < q^{-1}e_{N-M-1} < q^{-1}x_1 < q^{-1}x_2 < \cdots < q^{-1}x_M \]

\[ h_i(x) = \exp[-q^{-1}x_i], \quad g_i(x) = x_i^{e_i-1}, \quad i = 1, 2, \ldots, N \]

\[ h(x) = (h_1(x), h_2(x), \ldots, h_N(x))^T, \quad g(x) = (g_1(x), g_2(x), \ldots, g_N(x))^T \]

\[ h^{(n)}(x) = \frac{d^n h_i(x)}{dx^n} = \left( \frac{d^n h_1(x)}{dx^n}, \frac{d^n h_2(x)}{dx^n}, \ldots, \frac{d^n h_N(x)}{dx^n} \right)^T \]

\[ g^{(n)}(x) = \frac{d^n g_i(x)}{dx^n} = \left( \frac{d^n g_1(x)}{dx^n}, \frac{d^n g_2(x)}{dx^n}, \ldots, \frac{d^n g_N(x)}{dx^n} \right)^T \]

The algebra behind the proof is cumbersome but the steps are simple. First we note that the exact result in (2) is simple to compute except for the hypergeometric function. Hence we appeal to a result due to Gross and Richard's [9] who give a representation for the hypergeometric function in terms of a ratio of determinants. Unfortunately their representation requires all the eigenvalues of the matrix arguments to be unequal. This is not the case for the matrix argument \( q^{-1} \Sigma^{-1} Z \) since we must inflate this \( M \times M \) matrix to an \( N \times N \) matrix by adding a border of zero elements. This results in \( N-M \) eigenvalues equal to zero. Hence we perturb the zero diagonal elements by \( \epsilon_1, \epsilon_2, \ldots \) in such a way that the perturbed matrix \( S \) has \( q^{-1} \Sigma^{-1} Z \) as the non-zero principal minor in the limit as \( \{ \epsilon_i \} \to 0 \). From the properties of the hypergeometric function we have

\[ _0F_0(T, q^{-1} \Sigma^{-1} Z) = _0F_0(T, q^{-1} \Sigma^{-1} Z). \]

Also we can choose \( q > 0 \) and a set \( \{ \epsilon_i \} \) such that \( ||T|| ||S|| < 1 \), thus \( _0F_0(T, S) \) converges absolutely. Hence all the conditions required by Gross and Richard's [9] result are satisfied and we have

\[ _0F_0(T, q^{-1} \Sigma^{-1} Z) = \lim_{\{ \epsilon_i \} \to 0} _0F_0(T, S) = \lim_{\{ \epsilon_i \} \to 0} \beta_N \frac{\det(0,F_0(s,t_j))}{V(S) V(T)} \]
where
\[ \beta_N = \prod_{j=1}^{N} \Gamma(j), \quad V(S) = (-1)^{N(N-1)/2} \prod_{1 \leq i < j \leq N} (s_j - s_i), \]
\[ V(T) = (-1)^{N(N-1)/2} \prod_{1 \leq i < j \leq N} (t_j - t_i) \]
\[ \det(\mathcal{F}_0(s,t)) = \begin{vmatrix} \mathcal{F}_0(s_1,t_1) & \mathcal{F}_0(s_2,t_1) & \cdots & \mathcal{F}_0(s_N,t_1) \\ \mathcal{F}_0(s_1,t_2) & \mathcal{F}_0(s_2,t_2) & \cdots & \mathcal{F}_0(s_N,t_2) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{F}_0(s_1,t_N) & \mathcal{F}_0(s_2,t_N) & \cdots & \mathcal{F}_0(s_N,t_N) \end{vmatrix} \]
and the functions \( \mathcal{F}_0(.) \) are the standard scalar hypergeometric functions. Using the fact that \( V(S) \) and \( V(T) \) are determinants of Vandermonde matrices the three determinants in (4) can be expressed as
\[ V(S) = (-1)^{N(N-1)/2} q^{-N(N-1)/2} \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 0 & e_1 & \cdots & x_{N-1} & x_1 & \cdots & x_M \\ 0 & e_1^2 & \cdots & x_{N-1}^2 & x_1^2 & \cdots & x_M^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & e_1^{N-1} & \cdots & x_{N-1}^{N-1} & x_1^{N-1} & \cdots & x_M^{N-1} \end{vmatrix} \]
\[ V(T) = (-q)^{N(N-1)/2} (\prod_{i=1}^{N} \tau_i)^{-N+1} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \tau_1 & \tau_2 & \cdots & \tau_N \\ \tau_1^2 & \tau_2^2 & \cdots & \tau_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ \tau_1^{N-1} & \tau_2^{N-1} & \cdots & \tau_N^{N-1} \end{vmatrix} \]
\[ \det(\mathcal{F}_0(s,t)) = \exp \left[ \sum_{i=1}^{M} x_i \right] \]
\[ = \exp \left[ \sum_{i=1}^{M} x_i \right] \begin{vmatrix} \exp(-\gamma_1 e_1) & \exp(-\gamma_1 e_2) & \exp(-\gamma_1 e_3) & \cdots & \exp(-\gamma_1 e_M) \\ \exp(-\gamma_2 e_1) & \exp(-\gamma_2 e_2) & \exp(-\gamma_2 e_3) & \cdots & \exp(-\gamma_2 e_M) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \exp(-\gamma_M e_1) & \exp(-\gamma_M e_2) & \exp(-\gamma_M e_3) & \cdots & \exp(-\gamma_M e_M) \end{vmatrix} \]
Note that we have omitted the term \( \exp\left[ q^{-1} \sum_{i=1}^{M-1} e_i \right] \) in the above as it disappears in the limit. The difficult part of (4) is the ratio of
The determinant \( \det(\mathcal{F}(x,t)) \) to \( \mathcal{I}(S) \) since both vanish as \( \{e_i\} \to 0 \). We evaluate this ratio using Cauchy's mean value theorem as below:

\[
\begin{align*}
\det[h(0), h(e_1), \ldots, h(e_{N-M-1}), h(x_1), \ldots, h(x_M)] \\
\det[g(0), g(e_1), \ldots, g(e_{N-M-1}), g(x_1), \ldots, g(x_M)] \\
= \frac{\det[h(0), h(1)(e_1), h(e_2), \ldots, h(e_{N-M-1}), h(x_1), \ldots, h(x_M)]}{\det[g(0), g(1)(e_1), g(e_2), \ldots, g(e_{N-M-1}), g(x_1), \ldots, g(x_M)]} \\
\vdots \\
= \frac{\det[h(0), h(1)(e_1), h(1)(e_2), \ldots, h(1)(e_{N-M-1}), h(x_1), \ldots, h(x_M)]}{\det[g(0), g(1)(e_1), g(1)(e_2), \ldots, g(1)(e_{N-M-1}), g(x_1), \ldots, g(x_M)]} \\
\vdots \\
= \frac{\det[h(0), h(1)(e_1), h(2)(e_2), \ldots, h(1)(e_{N-M-1}), h(x_1), \ldots, h(x_M)]}{\det[g(0), g(1)(e_1), g(2)(e_2), \ldots, g(1)(e_{N-M-1}), g(x_1), \ldots, g(x_M)]} \\
\vdots \\
= \frac{\det[h(0), h(1)(e_1), h(2)(e_2), \ldots, h(N-M-1)(e_{N-M-1}), h(x_1), \ldots, h(x_M)]}{\det[g(0), g(1)(e_1), g(2)(e_2), \ldots, g(N-M-1)(e_{N-M-1}), g(x_1), \ldots, g(x_M)]}
\end{align*}
\]

where \( 0 \leq e_i \leq e_j, \ i = 1, 2, \ldots, N - M - 1 \). Repeated application of the mean value theorem is required since the \( j \)th column requires \( j-1 \) differentiations before it gives a column which in the limit does not cause the determinants to vanish.

Since \( h^{(i)}(x) \) and \( g^{(i)}(x) \) are vectors which are continuous functions at \( x = 0 \), we have

\[
\lim_{\{e_i\} \to 0} \frac{\det[h(0), h(e_1), \ldots, h(e_{N-M-1}), h(x_1), \ldots, h(x_M)]}{\det[g(0), g(e_1), \ldots, g(e_{N-M-1}), g(x_1), \ldots, g(x_M)]} = \lim_{\{e_i\} \to 0} \frac{\det[h(0), h(1)(e_1), h(e_2), \ldots, h(e_{N-M-1}), h(x_1), \ldots, h(x_M)]}{\det[g(0), g(1)(e_1), g(e_2), \ldots, g(e_{N-M-1}), g(x_1), \ldots, g(x_M)]} \\
= \frac{\det[h(0), h(1)(e_1), h(2)(e_2), \ldots, h(N-M-1)(e_{N-M-1}), h(x_1), \ldots, h(x_M)]}{\det[g(0), g(1)(e_1), g(2)(e_2), \ldots, g(N-M-1)(e_{N-M-1}), g(x_1), \ldots, g(x_M)]} \\
= \frac{\det[h(0), h^{(1)}(0), h^{(2)}(0), \ldots, h^{(N-M-1)}(0), h(x_1), \ldots, h(x_M)]}{\det[g(0), g^{(1)}(0), g^{(2)}(0), \ldots, g^{(N-M-1)}(0), g(x_1), \ldots, g(x_M)]}
\]
From (4) and the above we obtain

\[ gF_0(T, q^{-1}Z) \]

\[ = \lim_{(x_\delta) \to 0} \frac{\beta_N \det((\mathcal{B}_0(x_{\delta}, t)))}{V(S) V(T)} \]

\[ = \frac{\beta_N}{V(T)} \lim_{(x_\delta) \to 0} \left( \exp[q^{-1} \sum_{i=1}^{M} x_i] \times \det\left[\begin{array}{c} h(0), h(x_1), ..., h(x_N) \end{array}\right] \right) \]

\[ = \frac{\beta_N}{V(T)} \lim_{(x_\delta) \to 0} \left( (-1)^{N-1} q^{-N(N-1)/2} \times \det\left[\begin{array}{c} g(0), g(x_1), ..., g(x_N) \end{array}\right] \right) \]

\[ = \left( \frac{\beta_N \exp[q^{-1} \sum_{i=1}^{M} x_i]}{(q^{N(N-1)/2})^{N+1} A_3 q^{-1} A_1} \times \det\left[\begin{array}{c} h(0), h(x_1), ..., h(x_N) \end{array}\right] \right) \]

That is

\[ gF_0(T, q^{-1}Z) \]

\[ = \frac{(\beta_N \exp[q^{-1} \text{tr}(Z)]}{(LB)^{N+1} A_3} \times \det\left[\begin{array}{c} h(0), h(x_1), ..., h(x_N) \end{array}\right]} \]

\[ = \left( \frac{\beta_N \exp[q^{-1} \sum_{i=1}^{M} x_i]}{(q^{N(N-1)/2})^{N+1} A_3 q^{-1} A_1} \times \det\left[\begin{array}{c} g(0), g(x_1), ..., g(x_N) \end{array}\right] \right) \]

Now it is necessary to evaluate the matrices in (5) which now contain derivatives of \( h(\cdot) \) and \( g(\cdot) \). Since \( h^{(1)}(0) = (\gamma_1^{-1}, \gamma_2^{-1}, ..., \gamma_N^{-1}) \), we have in the numerator

\[ \det(h(0), h^{(1)}(0), h^{(2)}(0), ..., h^{(N-M-1)}(0), h(x_1), ..., h(x_M)) \]

\[ = \prod_{i=1}^{N-1} \left( \begin{array}{c} 1 \gamma_1^{-1} \cdots \gamma_{i-1}^{-1} \gamma_{i+1}^{-1} \cdots \gamma_N^{-1} \\ \gamma_1^{-1} \cdots \gamma_{i-1}^{-1} \gamma_{i+1}^{-1} \cdots \gamma_N^{-1} \\ \vdots \\ 1 \gamma_1^{-1} \cdots \gamma_{N-1}^{-1} \gamma_N^{-1} \end{array} \right) \]

\[ = \prod_{i=1}^{N-1} \left( \begin{array}{c} 1 \gamma_1^{-1} \cdots \gamma_{i-1}^{-1} \gamma_{i+1}^{-1} \cdots \gamma_N^{-1} \\ \gamma_1^{-1} \cdots \gamma_{i-1}^{-1} \gamma_{i+1}^{-1} \cdots \gamma_N^{-1} \\ \vdots \\ 1 \gamma_1^{-1} \cdots \gamma_{N-1}^{-1} \gamma_N^{-1} \end{array} \right) \]

\[ \times \left( \begin{array}{c} 1 \gamma_1^{-1} \cdots \gamma_{i-1}^{-1} \gamma_{i+1}^{-1} \cdots \gamma_N^{-1} \\ \gamma_1^{-1} \cdots \gamma_{i-1}^{-1} \gamma_{i+1}^{-1} \cdots \gamma_N^{-1} \\ \vdots \\ 1 \gamma_1^{-1} \cdots \gamma_{N-1}^{-1} \gamma_N^{-1} \end{array} \right) \]
Hence
\[ \det[h(0), h^{(1)}(0), h^{(2)}(0), \ldots, h^{(N-M-1)}(0), h(x_1), \ldots, h(x_M)] = |LB|^{-N+M+1} A_1 \] (6)

Also in the denominator we have
\[ \det[g(0), g^{(1)}(0), \ldots, g^{(N-M-1)}(0), g(x_1), \ldots, g(x_M)] =
\begin{vmatrix}
1 & 0 & 0 & \ldots & 0 & 1 & \ldots & 1 \\
0 & 1! & 0 & \ldots & 0 & x_1 & \ldots & x_M \\
0 & 0 & 2! & \ldots & 0 & x_1^2 & \ldots & x_M^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & (N-M-1)! & x_1^{N-M-1} & \ldots & x_M^{N-M-1} \\
0 & 0 & 0 & \ldots & 0 & x_1^{N-M} & \ldots & x_M^{N-M} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 & x_1^{N-1} & \ldots & x_M^{N-1} \\
\end{vmatrix}
\]

\[ = \prod_{j=1}^{N-M} (j-1)! \prod_{i=1}^{M} x_i^{N-M} \begin{vmatrix}
1 & 1 & \ldots & 1 \\
x_1 & x_2 & \ldots & x_M \\
x_1^2 & x_2^2 & \ldots & x_M^2 \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{N-M-1} & x_2^{N-M-1} & \ldots & x_M^{N-M-1} \\
\end{vmatrix} \]

Hence
\[ \det[g(0), g^{(1)}(0), \ldots, g^{(N-M-1)}(0), g(x_1), \ldots, g(x_M)] = A_2 |\Sigma^{-1}Z|^{N-M} \prod_{j=1}^{N-M} I(f(j)) \] (7)

Substituting (6) and (7) into (5) gives
\[ _0F_0(T, q^{-1}\Sigma^{-1}Z) = \frac{\beta_N \exp[q^{-1} tr(\Sigma^{-1}Z)] |LB|^{-N+M+1} A_1}{M \prod_{j=1}^{N-M} I(f(j))} \] (8)

Hence
\[ _0F_0(T, q^{-1}\Sigma^{-1}Z) = \frac{\beta_N \exp[q^{-1} tr(\Sigma^{-1}Z)] |LB|^M A_1}{A_2 A_3 |\Sigma^{-1}Z|^{-M} \prod_{j=1}^{N-M} I(f(j))} \] (9)
From Eqs. (2) and (9) we have
\[
\begin{align*}
f(Z) &= (\hat{f}_M(N) |L|B|^M| |\Sigma|^N)^{-1} |Z|^{N-M} \\
&\times \exp(-q^{-1} tr\Sigma^{-1}Z) \sigma F_q(T^*, q^{-1}\Sigma^{-1}Z) \\
&= (\hat{f}_M(N) |L|B|^M| |\Sigma|^N)^{-1} |Z|^{N-M} \\
&\times \exp(-q^{-1} tr\Sigma^{-1}Z) \sigma F_q(T, q^{-1}\Sigma^{-1}Z) \\
&= (\hat{f}_M(N) |L|B|^M| |\Sigma|^N)^{-1} |Z|^{N-M} \exp(-q^{-1} tr\Sigma^{-1}Z) \\
&\times \beta_N \exp\left[ q^{-1} tr(\Sigma^{-1}Z) \right] |L|B|^M A_1 \\
&\times \beta_N \exp\left[ q^{-1} tr(\Sigma^{-1}Z) \right] |L|B|^M \prod_{j=1}^{N-M} I(f(j)) \\
&\times \beta_N A_1 \\
&\frac{1}{\pi^{2M(M-1)}} |\Sigma|^{-M} A_1 A_2 A_3 \prod_{j=1}^{N-M} I(f(j)) \\
&= \frac{1}{\pi^{2M(M-1)}} |\Sigma|^{-M} A_1 A_2 A_3 \prod_{j=1}^{N-M} I(f(j))
\end{align*}
\]

3. CONCLUSION

The quadratic form \( Z \) has been studied for many years and several representations are already available for the density of \( Z \). However, these representations are usually complex series solutions, often involving summations over partitions, which are difficult to use in numerical work. In Section 2 we have derived a new expression for density of \( Z \) in terms of a ratio of determinants based on the work of Gross and Richard [9]. This expression gives an alternative form of solution which is an appealing formulation in its own right and may be useful in numerical work for small values of \( M \). For example when

\[
M = 1, \quad B = I_N, \quad L = \text{diag}(\gamma_1, \gamma_2, ..., \gamma_N) \quad \text{and} \quad \Sigma = I_N
\]

we have the well known scalar quadratic form

\[
Z = \sum_{i=1}^{N} \gamma_i Z^2
\]
where $\gamma_i^2$ are iid chi-square random variables with 2 d.o.f. In this case Eq. (3) gives

$$f(z) = \frac{1}{\prod_{i=1}^{N} \gamma_i} \exp\left(-\frac{z^2}{\gamma_1}\right)$$

Using Laplace's expansion theorem to expand the numerator in (10) gives the density as a sum of exponentials and by mathematical induction we can show that

$$f(z) = (-1)^{N-1} \sum_{i=1}^{N} \frac{\gamma_i^{N-2} \exp(-z\gamma_i^{-1})}{\prod_{j \neq i} (\gamma_j - \gamma_i)}$$

This is the form given in [10] and [11] for example.

For small $M \geq 2$ a similar approach leads to reasonably simple expressions for the density. By repeated use of Laplace’s expansion theorem a recursion can be developed to compute $A_1/(A_2 \cdot A_3)$ which leads to finite double sums in exponentials ($M = 2$), triple sums in exponentials ($M = 3$), etc. The general recursion can be written in terms of the subdeterminants $A^{(N-\gamma_j)(N-j)}(N-j)$ of $A_1$, where $A^{(N-\gamma_j)(N-j)}$ is the determinant of the $N-j \times N-j$ submatrix of $A_1$ gained by removing the last $j$ columns of $A_1$ and the set of rows $\{r_1, ..., r_j\}$. We also use $A_2(M-j)$ and $A_3(N-j)$ to refer to the determinants of the leading principal minors, with sizes $M-j$ and $N-j$, of the matrices with determinants $A_2$ and $A_3$ respectively. The first step is to write

$$\frac{A_1}{A_2 \cdot A_3} = \frac{(-1)^N}{\prod_{i=1}^{N} \gamma_i} \prod_{i=1}^{N-M-1} \left(\frac{x_i - x_j}{\gamma_i}\right) \sum_{k=1}^{N} (-1)^k \cdot \gamma_k^{N-M-1}$$

$$\times \exp(-x_M \gamma_k^{-1}) \frac{A_1^{(N-1)}}{A_2(M-1) \cdot A_3(N-1)}$$
Subsequent recursions are given by
\[
A_{r_1, \ldots, r_j}^{(N-j+1)}(N-j+1) = A_2(M-j+1) A_3(N-j+1) - (1)^{N-j+1} \\
= \prod_{j=1}^{N-j+1} (\gamma_j(N-j+1-\gamma_j)) \prod_{k=1}^{M-j+1} (x_{M-j+1} - x_k) \\
\times \sum_{k=1}^{N-j+1} (-1)^k \gamma_j^{N-M-1} \exp(-x_{M-j+1} + 1) \\
\times A_{r_1, \ldots, r_j}^{(N-j+1)}(N-j) / A_2(M-j) A_3(N-j)
\]
where \(\{r_1, \ldots, r_j\} \cup \{s_{N-j+1} \ldots s_N\} = \{1, 2, \ldots, N\}\). The recursion stops after \(M\) steps since \(A_{1}^{(N-M)}(N-M), A_2(0)\) and \(A_3(N-M)\) are all known Vandermonde determinants. As an example of this approach for \(M=2\) we can write
\[
f(x_1, x_2) = \frac{\pi}{(x_2 - x_1) \prod_{i \geq j} (y_i - y_j)} \\
\times \sum_{k=1}^{N} \left\{ \sum_{h < k} (-1)^{h+k+1} (y_k y_h)^{N-3} \times \exp[-(x_2 y_k^{-1} + x_1 y_h^{-1})] \prod_{i \geq h} (y_i - y_h) \right\} \\
\times \sum_{k=1}^{N} \left\{ \sum_{h > k} (-1)^{h+k} (y_k y_h)^{N-3} \times \exp[-(x_2 y_k^{-1} + x_1 y_h^{-1})] \prod_{i > h} (y_i - y_h) \right\}
\]
Finally we can also use the new representation to give a new and compact expression for the distribution of the generalized variance \(|Z|\)
\[
F_{|Z|}(z) = P(|Z| \leq z) = P(|\Sigma^{-1} Z| \leq |\Sigma|^{-1} z) \\
= \pi^{1/2(M-1)} |\Sigma|^{-M} \int_{x_1 \ldots x_M \leq |\Sigma|^{-1} z} \frac{A_1}{A_2 A_3} d\Sigma \int_{x_1 \ldots x_M} d\Sigma
\]
where \(A_i, i = 1, 2, 3\) are shown in (3).

REFERENCES


