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The Grassmann Space of a Planar Space

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In this paper we give a characterization of the Grassmann space of a planar space.

1. INTRODUCTION

A *linear space* is a pair $(S, \mathcal{L})$, where S is a set of *points* and $\mathcal{L}$ is a family of proper subsets of S, called *lines*, each one with at least two points, such that

1.1 any two distinct points are on exactly one line.

A subset H of S is a *subspace* of $(S, \mathcal{L})$ if it contains the line $[x, y]$ through any pair x, y of distinct points of H.

A *planar space* is a triple $(S, \mathcal{L}, \mathcal{P})$, where $(S, \mathcal{L})$ is a linear space and $\mathcal{P}$ is a family of proper subspaces of $(S, \mathcal{L})$, called *planes*, such that

1.2 three points not on a line are contained in a unique plane;
1.3 every plane contains at least three non-collinear points.

Let $(S, \mathcal{L}, \mathcal{P})$ be a finite planar space. For every plane π in $\mathcal{P}$, denote by $\mathcal{L}_\pi$ the set of the lines of $\mathcal{L}$ contained in π.

A planar space $(S, \mathcal{L}, \mathcal{P})$ is *embeddable* in a projective space $\mathbf{P}$ if there is an injection from S into $\mathbf{P}$ which preserves collinearities, complanarities, non-collinearities, and non-complanarities.

If $(S, \mathcal{L}, \mathcal{P})$ is a planar space, for every point p, a *star* with center p is the set of the lines through p; if π is a plane and p a point of π, a *pencil of lines* of center p is the set of the lines through p contained in π.

A *partial line space* is a pair $(S_0, \mathcal{R}_0)$, where S_0 is a set of *points* and $\mathcal{R}_0$ is a family of proper subsets of S_0, called *lines*, such that:

(1) *$\mathcal{R}_0$ is a covering of S_0;*
(2) *every line has at least two points;*
(3) *any two distinct points are on at most one line.*

If two different points p, p' of S_0 are collinear we write $p \sim p'$. If p and p' are not collinear, we write $p \not\sim p'$.

A subset H of S_0 is a *subspace* of $(S_0, \mathcal{R}_0)$ if for every $x, y \in H$, x and y are collinear and the line $[x, y]$ is contained in H.

Two partial line spaces $(S, \mathcal{R})$ and $(S'\, \mathcal{R}')$ are **isomorphic** if there is a bijection $\phi : \mathcal{S} \to \mathcal{S}'$ such that both ϕ and ϕ^{-1} preserve collinearities.

The *Grassmann space* of a planar space $(S, \mathcal{L}, \mathcal{P})$ is the partial line space $\mathbf{G}(S) = (S_0, \mathcal{R}_0)$, whose *points* are the lines of $(S, \mathcal{L}, \mathcal{P})$ and whose *lines* are the pencils of lines of $(S, \mathcal{L}, \mathcal{P})$. The lines of a star T_x with center x of $(S, \mathcal{L}, \mathcal{P})$ are the points of a maximal subspace T of $(S_0, \mathcal{R}_0)$, again called a *star*.

Let $\mathcal{T}$ be the family of the stars of $(S_0, \mathcal{R}_0)$. Then it is easy to see that the following properties hold.

I. Through any point of $(S_0, \mathcal{R}_0)$ there are at least two distinct stars. Moreover, let T be a star and p be a point not on T. Then, each star T' through p meets T in a single point p', the set $\{p' = T \cap T'\}_{T' \ni p} = \ell_p(T)$ is contained in a line $L_p(T)$, and $\ell_p(T)$ consists of all the points of T collinear with p.

II. Let T and T' be two different stars, with $T \cap T' = \{p\}$. If $x \in T, x \neq p$, then for any $y \in L_x(T'), y \neq p$, we have $L_y(T) = [p, x]$.

If $(S, \mathcal{L}, \mathcal{P})$ is a projective space, then $\mathbf{G}(S)$ is called a *projective* Grassmann space. In such a case, for every star T and for every point $p \notin T$, we have $\ell_p(T) = L_p(T)$.

If $\ell_p(T) = L_p(T)$, then Property I implies Property II, and Melone and Olanda [2], have shown that this characterizes the Grassmann space $\mathbf{G}(S)$ of a projective space $\mathbf{P} = (S, \mathcal{L}, \mathcal{P})$. Their theorem can be rephrased as follows.

THEOREM 1.1 (MELONE AND OLANDA [2]). *Let $(S, \mathcal{R})$ be a proper partial line space whose lines are not maximal subspaces. If $(S, \mathcal{R})$ has a covering $\mathcal{T}$ of maximal subspaces with the property*

> *for each $T \in \mathcal{T}$ and each $p \in S - T$, every element of $\mathcal{T}$ through p intersects T in a unique point, and these points trace out a line $L_p(T)$ formed by all the points of T collinear with p.*

Then $(S, \mathcal{R})$ is isomorphic to the Grassmann space of a projective space.

In their proof, Melone and Olanda constructed another family Π of maximal subspaces in $(\mathcal{S}, \mathcal{R})$ and showed that both the families $\mathcal{T}$ and Π fullfil the following properties.

(a) Three pairwise collinear points of S, not on a common line, are contained in a unique element of $\mathcal{T} \cup \Pi$;

(b) $T, T' \in \mathcal{T}, T \neq T' \Longrightarrow |T \cap T'| = 1$;

(c) $T \in \mathcal{T}, \pi \in \Pi \Longrightarrow T \cap \pi = \emptyset$ or $T \cap \pi \in \mathcal{R}$;

(d) $\forall \ell \in \mathcal{R} \Longrightarrow \exists! T \in \mathcal{T}, \exists! \pi \in \Pi : T \cap \pi = \ell$.

These properties, as shown by Tallini in the celebrated paper [3], allow us to construct a projective space $\mathbf{P}$ whose Grassmann space is isomorphic to $(\mathcal{S}, \mathcal{R})$. Hence, using this result, Melone and Olanda obtained their result.

In this paper we show that Properties I and II characterize the Grassmann space $\mathbf{G}(S)$ of a planar space $(S, \mathcal{L}, \mathcal{P})$. Our main result is the following.

THEOREM 1.2. *Let $(S, \mathcal{R})$ be a partial line space whose lines are not maximal subspaces. If $(S, \mathcal{R})$ has a family $\mathcal{T}$ of maximal subspaces, whose elements are called stars, with Properties I and II, then there exists a planar space $\mathbf{P_0} = (S_0, \mathcal{L}_0, \mathcal{P}_0)$ such that $\mathbf{G}(\mathbf{P_0})$ is isomorphic to $(S, \mathcal{R})$.*

In the proof of the theorem we show that the family $\mathcal{T}$ enables us to construct the planar space $\mathbf{P_0}$. Moreover, we show that, if in Property I $\ell_p(T) = L_p(T)$ holds, then Property I implies Property II, and furthermore the planar space $\mathbf{P_0}$ is a projective space (possibly reducible).

Hence, the theorem of Melone and Olanda follows from Theorem 1.2 as a corollary. On the other hand, we obtain a new direct proof of Theorem 1.1 without using the result of Tallini.

2. Construction of the Planar Space $\mathbf{P_0}$

Let us first show some properties of a partial line space, whose lines are not maximal subspaces, and with a family $\mathcal{T}$ of maximal subspaces, called *stars*, satisfying Properties I and II.

From Property I it easily follows:

PROPOSITION 2.1. *Two distinct stars intersect at a single point.*

Moreover, the following holds.

PROPOSITION 2.2. *Every line r is contained in a unique star $T(r)$.*

PROOF. From Proposition 2.1 it follows that there is at most one star containing r. Let p be a point on r and let T be a star through p. We may assume that T does not contain r. Put $y \in r - \{p\}$. Since $p \in T$ and p is collinear with y, there exists a star T_y through y which intersects T in p. Hence $r \subset T_y$. □

We now construct a planar space $(S_0, \mathcal{L}_0, \mathcal{P}_0)$ by using the family $\mathcal{T}$.

The set S_0 of *points* is defined to be $\mathcal{T}$. A *line* is a subset of S_0 consisting of all members of $\mathcal{T}$ through a fixed point $p \in \mathcal{S}$, which we denote by L_p. Denote by $\mathcal{L}_0$ the family $\{L_p\}_{p\in\mathcal{S}}$. Since two stars intersect in a single point, then $(\mathcal{S}_0, \mathcal{L}_0)$ is a linear space. For every line $r \in \mathcal{R}$, a *plane* generated by r is the following subset π_r of S_0

$$\pi_r = \{T \in \mathcal{T} : T \cap r \neq \emptyset\}.$$

The following propositions describe the subsets π_r.

PROPOSITION 2.3. *Every plane π_r is a subspace of $(S_0, \mathcal{L}_0)$ and it contains at least three non-collinear points.*

PROOF. We first show that π_r is a subspace. Let $T = T(r)$ be the unique star containing the line r, and let T' and T'' be two points of π_r. Let p be the point $T' \cap T''$. If $p \in r$, then the line L_p through the points T' and T'' is contained in π_r. If $p \notin r$, then T' and T'' are different from T and, since $r = L_p(T)$, it follows that the line L_p through the points T' and T'' is contained in π_r. Hence, π_r is a subspace.

Let a and b be two points of r and let T_a and T_b be two stars through a and b resp., both different from T. The three points T_a, T_b, T of π_r are not collinear. □

PROPOSITION 2.4. *Let $r, r' \in \mathcal{R}$. If r and r' are contained in a star T, then either $\pi_r \cap \pi_{r'} = \{T\}$ or $\pi_r \cap \pi_{r'}$ is a line.*

PROOF. Let r and r' be two lines of $(S, \mathcal{R})$ and let T be the star containing both r and r'. If $r \cap r' = \emptyset$ then, since two stars intersect at a single point, it follows that if $T_0 \in \pi_r - \{T\}$, then $T_0 \notin \pi_{r'}$. Hence $\pi_r \cap \pi_{r'} = \{T\}$.

If $r \cap r' = \{p\}$, then clearly $\pi_r \cap \pi_{r'} = L_p$. □

PROPOSITION 2.5. *Let $r, r' \in \mathcal{R}$. If $r \cap r' = \{p\}$ and there is no star containing both r and r', then*

$$\pi_r = \pi_{r'} \Leftrightarrow \exists a \in r - \{p\} \text{ and } \exists a' \in r' - \{p\} \text{ such that } a \sim a'.$$

PROOF. $\Leftarrow$. Let $a \in r - \{p\}$ and let a' be a point of $r' - \{p\}$ such that $a \sim a'$. Let T and T' be two stars containing r and r', respectively, and let T_0 be the star containing the line $[a, a']$. The stars T, T' and T_0 are pairwise different. Moreover $L_a(T') = [p, a'] = r'$ and $L_{a'}(T) = [p, a] = r$. It follows that, if $y \in r = L_{a'}(T)$, then $a' \in L_y(T') = r'$.

Thus, if T'' is a star through y, then $T'' \cap T'$ is a point of r'. Hence $\pi_r \subseteq \pi_{r'}$. Similarly one can show $\pi_{r'} \subseteq \pi_r$. Then $\pi_r = \pi_{r'}$.

$\Rightarrow$. Let T be the star containing r. If $a \in r$, then $L_a \subseteq \pi_r$. Since $|L_a| \geq 2$, there exists $T_0 \in L_a$, $T_0 \neq T$ and $T_0 \in \pi_{r'}(= \pi_r)$. Hence, T_0 meets r' in a point $a' \neq p$ and so $a' \sim a$. □

PROPOSITION 2.6. *Let $r, r' \in \mathcal{R}$. If $r \cap r' = \emptyset$ and there is no star containing both r and r', then $\pi_r \neq \pi_{r'}$.*

PROOF. Let T and T' be the two stars containing r and r', respectively, and let $\{p\} = T \cap T'$. If $\pi_r = \pi_{r'}$, then from $T \in \pi_r$ it follows that $T \cap r' \neq \emptyset$, hence $p \in r'$; similarly, since $T' \in \pi_{r'}$, then $T' \cap r \neq \emptyset$ and $p \in r$. It follows $r \cap r' = \{p\}$, a contradiction. □

Denote by $\mathcal{P}_0$ the family of distinct planes π_r, $r \in \mathcal{R}$. The following holds.

PROPOSITION 2.7. *Let π_r and $\pi_{r'}$ be two distinct elements of $\mathcal{P}_0$. Then π_r and $\pi_{r'}$ have at most one line in common.*

PROOF. Let π_r and $\pi_{r'}$ be two distinct planes generated by r and r', respectively. If r and r' lie in a common star T, then from Proposition 2.4 it follows that either $\pi_r \cap \pi_{r'} = \{T\}$ or $\pi_r \cap \pi_{r'}$ is a line.

We may therefore assume that r and r' lie in different stars T and T', respectively. Let $\{p\} = T \cap T'$. If $r \cap r' \neq \emptyset$, then $r \cap r' = \{p\}$ and $L_p \subseteq \pi_r \cap \pi_{r'}$. Since r and r' generate two distinct planes, from Proposition 2.5 it follows that for every $a \in r - \{p\}$ and for every $a' \in r' - \{p\}$, it is $a \not\sim a'$. Hence $\pi_r \cap \pi_{r'} = L_p$.

If $r \cap r' = \emptyset$ and one of them, say r', contains p, then the two planes π_r and $\pi_{r'}$ have at most one line in common. In fact, if $\pi_r \cap \pi_{r'}$ contains three non-collinear points T_1, T_2 and T_3, then at least two of them, say T_1 and T_2, are different from T. Then $p \notin T_1$. Let $a = T_1 \cap r$ and $a' = T_1 \cap r'$. Since T_1, T_2 and T_3 are three non-collinear points, at least one of T_2 and T_3, say T_2, does not contain a. Let $c = T_2 \cap r$ and $c' = T_2 \cap r'$. Since $c' \in L_a(T')$, then $a \in L_{c'}(T)$. But $c \in L_{c'}(T)$, so $L_{c'}(T) = r$ and since $p \in L_{c'}(T)$, we have $p \in r$, contradicting $r \cap r' = \emptyset$.

If $r \cap r' = \emptyset$ and no one of them contains p, let T_1 and T_2 be two points of $\pi_r \cap \pi_{r'}$ and let $\{y\} = T_1 \cap T_2$. Since $\pi_r \cap \pi_{r'}$ is a subspace, it contains the line L_y, so $y \notin T \cup T'$. Put $a = T_1 \cap r$, $a' = T_1 \cap r'$, $b = T_2 \cap r$, and $b' = T_2 \cap r'$. If $s = [y, a] \neq [y, a'] = s'$, then from Proposition 2.5 it follows that $\pi_s = \pi_r$ and $\pi_{s'} = \pi_{r'}$. Hence, $L_y = \pi_s \cap \pi_{s'} = \pi_r \cap \pi_{r'}$. We may therefore assume that $s = s'$ and $[y, b] = [y, b']$. From Proposition 2.5 it follows $\pi_s = \pi_r$ and $\pi_s = \pi_{r'}$, hence $\pi_r = \pi_{r'}$, contradicting Proposition 2.6. □

Now we prove that:

PROPOSITION 2.8. *The triple $(S_0, \mathcal{L}_0, \mathcal{P}_0)$ is a planar space.*

PROOF. We only have to show that every three non-collinear points lie in a plane, which is unique by Proposition 2.7. Let T, T', T'' be three non-collinear points. Put $p = T' \cap T''$, $p' = T \cap T''$, $p'' = T \cap T''$ and $r = [p', p'']$, $r' = [p, p'']$, $r'' = [p, p']$. The plane $\pi_r (= \pi_{r'} = \pi_{r''})$ contains the three points T, T', T'' and this completes the proof. □

If $T \in \mathcal{T}$ and r is a line contained in T, then all lines L_p, with $p \in r$, form the pencil of lines, $\mathcal{F}_T$ with center T, in the plane π_r.

Conversely it is not difficult to show, by using Propositions 2.2 and 2.5 and II, that every pencil of lines of the planar space can be obtained in this way.

To complete the proof of Theorem 1.2, we only have to show that the Grassmann space of $(S_0, \mathcal{L}_0, \mathcal{P}_0)$ is isomorphic to $(S, \mathcal{R})$.

Let $(S', \mathcal{R}') = \mathbf{G}(S_0)$ be the Grassmann space of $(S_0, \mathcal{L}_0, \mathcal{P}_0)$. We recall that the points of $(S', \mathcal{R}')$ are the lines of $(S_0, \mathcal{L}_0, \mathcal{P}_0)$ and the lines of $(S', \mathcal{R}')$ are the pencils of lines of $(S_0, \mathcal{L}_0, \mathcal{P}_0)$.

The map $f : L_p \in S' \mapsto p \in S$ is a bijection. It remains to show that f maps lines to lines. Let $L' \in \mathcal{R}'$. Then L' is a pencil of lines of $(S_0, \mathcal{L}_0, \mathcal{P}_0)$, so are all the lines contained in a plane π_L and containing the common point $T = T(L)$. Hence, every line of the pencil L' determines a point of L, then f maps the line L' to the line $L \in \mathcal{R}$.

Conversely, if $p \in L$, $L \in \mathcal{R}$, the line L_p contains $T = T(L)$ and it is contained in π_L. So, it is a line of the pencil L'. It follows that both f and its inverse map lines to lines and so f is an isomorphisms between $(S', \mathcal{R}')$ and $(S, \mathcal{R})$. Theorem 1.2 is completely proved.

We conclude with the following:

PROPOSITION 2.9. *If $\ell_p(T) = L_p(T)$, $p \in S$ and $T \in \mathcal{T}$, then the planar space $(S_0, \mathcal{L}_0, \mathcal{P}_0)$ is a projective space (possibly reducible). Moreover, $(S_0, \mathcal{L}_0, \mathcal{P}_0)$ is irreducible if and only if $(S, \mathcal{R})$ is irreducible.*

PROOF. It is enough to show that the planes of $\mathcal{P}_0$ are projective planes (possibly reducible). Let π_r be a plane of $\mathcal{P}_0$ and let $L_y, L_{y'}$ be two of its lines. Denote by T the star $T(r)$. We show that $L_y \cap L_{y'} \neq \emptyset$. If $y, y' \in r$, then $L_y \cap L_{y'} = \{T\}$. If $y \in r$, $y' \notin r$, then $L_{y'}(T) = r$ so $y \sim y'$. Hence $L_y \cap L_{y'} = \{T'\}$, where T' is the unique star containing the line $[y, y']$. Finally, if $y, y' \notin r$, then let T_1 be a star through y and let T_2 be a star through y'. Put $a = T_1 \cap T$, $a' = T_2 \cap T$ and $z = T_2 \cap T_1$. Then $L_{a'}(T_1) = [a, y]$, so T_2 meets T_1 in a point of $[a, y]$. It follows that $L_{y'}(T_1) = [a, y]$, so $y \sim y'$. Then, the unique star T' containing the line $[y, y']$ is the point of intersection of L_y and $L_{y'}$.

Since it is easy to see that $(S_0, \mathcal{L}_0, \mathcal{P}_0)$ is irreducible if and only if $(S, \mathcal{R})$ is so, the proof is complete. □

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