Differential Geometry and its Applications 11 (1999) 117–128
North-Holland
www.elsevier.com/locate/difgeo

Stationary Veselov–Novikov equation and isothermally asymptotic surfaces in projective differential geometry

E.V. Ferapontov [1]
Institute for Mathematical Modelling, Academy of Science of Russia, Miusskaya 4, 125047 Moscow, Russia

Communicated by B.A. Dubrovin
Received 26 March 1998

Abstract: It is demonstrated that the stationary Veselov–Novikov (VN) and the stationary modified Veselov–Novikov (mVN) equations describe one and the same class of surfaces in projective differential geometry: the so-called isothermally asymptotic surfaces, examples of which include arbitrary quadrics and cubics, quartics of Kummer, projective transforms of affine spheres and rotation surfaces. The stationary mVN equation arises in the Wilczynski approach and plays the role of the projective "Gauss–Codazzi" equations, while the stationary VN equation follows from the Lelieuvre representation of surfaces in 3-space. This implies an explicit Bäcklund transformation between the stationary VN and mVN equations which is an analog of the Miura transformation between their $(1+1)$-dimensional limits.

Keywords: Projective differential geometry, integrable systems, Bäcklund transformations.

MS classification: 53A40, 53A20, 35Q58, 35L60.

1. Introduction

The Veselov–Novikov (VN) equation

$$\begin{aligned} u_t &= \alpha u_{xxx} + \beta u_{yyy} - 3\alpha(vu)_x - 3\beta(wu)_y, \\ w_x &= u_y, \\ v_y &= u_x \end{aligned} \tag{1}$$

(here α, β are arbitrary constants) was introduced independently in [18, 19] for real and complex-conjugate x, y respectively and arises from the compatibility conditions of the linear problem

$$\begin{aligned} \nu_{xy} &= u\nu, \\ \nu_t &= \alpha\nu_{xxx} + \beta\nu_{yyy} - 3\alpha v\nu_x - 3\beta w\nu_y. \end{aligned} \tag{2}$$

[1] *E-mail*: fer@landau.ac.ru.

PII S0926-2245(99)00028-5

The modified Veselov–Novikov (mVN) equation

$$\begin{aligned}
p_t &= \alpha p_{xxx} + \beta p_{yyy} - 2\alpha V p_x - 2\beta W p_y - \alpha p V_x - \beta p W_y,\\
W_x &= \tfrac{3}{2}(p^2)_y,\\
V_y &= \tfrac{3}{2}(p^2)_x
\end{aligned} \tag{3}$$

was introduced in [1] and is associated with the two-dimensional Dirac operator

$$\begin{aligned}
\psi^1_x &= p\psi^2,\\
\psi^2_y &= p\psi^1,\\
\psi^1_t &= \alpha\psi^1_{xxx} + \beta\psi^1_{yyy} - 2\beta W\psi^1_y - 3\alpha p_x\psi^2_x - \beta W_y\psi^1 - 2\alpha p V\psi^2,\\
\psi^2_t &= \alpha\psi^2_{xxx} + \beta\psi^2_{yyy} - 3\beta p_y\psi^1_y - 2\alpha V\psi^2_x - \alpha V_x\psi^2 - 2\beta p W\psi^1.
\end{aligned} \tag{4}$$

In the $(1+1)$-dimensional limit $\alpha = \beta = \frac{1}{2}$, $y = x$ equations (1) and (3) reduce respectively to the KdV and mKdV equations

$$u_t = u_{xxx} - 6uu_x, \qquad p_t = p_{xxx} - 6p^2 p_x,$$

related by the Miura transformation $u = p^2 + p_x$.

In this paper we establish a similar link between the stationary VN and the stationary mVN equations, which in the case $\alpha = -\beta$ assume the following forms

– *Stationary VN*:

$$\begin{aligned}
u_{xxx} - 3(vu)_x &= u_{yyy} - 3(wu)_y,\\
w_x &= u_y,\\
v_y &= u_x.
\end{aligned} \tag{5}$$

– *Stationary mVN*:

$$\begin{aligned}
p_{xxx} - 2Vp_x - pV_x &= p_{yyy} - 2Wp_y - pW_y,\\
W_x &= \tfrac{3}{2}(p^2)_y,\\
V_y &= \tfrac{3}{2}(p^2)_x.
\end{aligned} \tag{6}$$

Let us introduce the linear system

$$\begin{aligned}
r_{xx} &= pr_y + \tfrac{1}{2}(V - p_y)r,\\
r_{yy} &= pr_x + \tfrac{1}{2}(W - p_x)r
\end{aligned} \tag{7}$$

the compatibility conditions of which coincide with (6). Bäcklund transformation between equations (5) and (6) is given by the formulae

$$\begin{aligned}
u &= p^2 - 2(\ln r_0)_{xy},\\
v &= \tfrac{2}{3}V - 2(\ln r_0)_{xx},\\
w &= \tfrac{2}{3}W - 2(\ln r_0)_{yy},
\end{aligned} \tag{8}$$

where r_0 is an arbitrary solution of (7). It can be viewed as the stationary analog of the Miura transformation. We emphasize that although the (1 + 1)- and the stationary limits of VN and mVN equations are related by Bäcklund transformations, there is no explicit link between the full (2+1)-dimensional equations.

The origin of this Bäcklund transformation is purely differential-geometric: both systems (5) and (6) arise in projective differential geometry and describe the so-called isothermally asymptotic surfaces which have been a subject of interest of differential geometers of the first half of the century. A number of intriguing properties and interesting examples of isothermally asymptotic surfaces can be found in the classical textbooks on projective differential geometry [8, 5, 6, 12, 2].

In Sect. 2 we recall the standard approach to projective differential geometry following Wilczynski [20] and derive the stationary mVN equations (6) playing the role of projective "Gauss–Codazzi" equations for isothermally asymptotic surfaces. In this approach linear system (7) specifies the radius-vector r of the surface. Particular examples of isothermally asymptotic surfaces and exact solutions of the stationary mVN equation corresponding to them are discussed in the end of Sect. 2. These include quadrics, cubics, quartics of Kummer, projective transforms of surfaces of rotation and affine spheres.

In Sect. 3 we recall the Lelieuvre representation of surfaces in 3-space and derive for isothermally asymptotic surfaces the stationary VN equation (5). Thus stationary VN and mVN equations are just two different parametrizations of one and the same class of isothermally asymptotic surfaces and the corresponding Bäcklund transformation (8) follows directly.

2. Surfaces in projective differential geometry

Following [20] we define a surface M^2 in projective space P^3 in terms of solutions of a linear system

$$\begin{aligned} r_{xx} &= pr_y + \tfrac{1}{2}(V - p_y)r, \\ r_{yy} &= qr_x + \tfrac{1}{2}(W - q_x)r \end{aligned} \tag{9}$$

where p, q, V, W are certain functions of x, y. Cross-differentiating (9) and assuming r, r_x, r_y, r_{xy} to be independent, we arrive at the compatibility conditions

$$\begin{aligned} p_{yyy} - 2p_yW - pW_y &= q_{xxx} - 2q_xV - qV_x, \\ W_x &= 2qp_y + pq_y, \\ V_y &= 2pq_x + qp_x \end{aligned} \tag{10}$$

(see also [12, p. 120] and [4]). For any fixed solution p, q, V, W of (10) the linear system (9) is compatible and possesses exactly four linearly independent solutions $r = (r^0, r^1, r^2, r^3)$ which may be regarded as homogeneous coordinates of a surface in projective space. For definiteness one can think of a surface M^2 in the ordinary 3-space with the radius-vector $R = (r^1/r^0, r^2/r^0, r^3/r^0)$. Choosing any other four solutions $\tilde{r} = (\tilde{r}^0, \tilde{r}^1, \tilde{r}^2, \tilde{r}^3)$ of the same system (9) we see that the corresponding surface $\tilde{M}^2$ with the radius-vector $\tilde{R} = (\tilde{r}^1/\tilde{r}^0, \tilde{r}^2/\tilde{r}^0, \tilde{r}^3/\tilde{r}^0)$ is a projective transform of M^2, so that any fixed solution p, q, V, W

of equations (10) defines a surface M^2 uniquely up to projective equivalence. Moreover, a simple calculation yields

$$R_{xx} = pR_y + aR_x,$$
$$R_{yy} = qR_x + bR_y \tag{11}$$

($a = -2(\ln r^0)_x$, $b = -2(\ln r^0)_y$) which implies that x, y are asymptotic coordinates on M^2. In what follows we assume that our surfaces are hyperbolic and the corresponding asymptotic coordinates x, y are real (it is not a problem to reformulate results that follow in the convex situation regarding x, y as complex conjugate). Since equations (10) specify a surface uniquely up to projective equivalence, they can be viewed as the "Gauss–Codazzi" equations in projective geometry.

The most important invariants in projective differential geometry are the projective metric

$$2pq\, dx\, dy \tag{12}$$

and the Darboux cubic form

$$p\, dx^3 + q\, dy^3 \tag{13}$$

which define "generic" surface uniquely up to projective equivalence. Projective metric (12) gives rise to the projective area functional

$$\iint pq\, dx\, dy. \tag{14}$$

Linear system (11) can be the starting point for developing affine differential geometry of surfaces referred to asymptotic coordinates x, y. Indeed, let $R = (R^1, R^2, R^3)$ be any three nonconstant solutions of (11) viewed as the radius-vector of a surface M^2 in 3-space. Since any other three nonconstant solutions $\tilde{R} = (\tilde{R}^1, \tilde{R}^2, \tilde{R}^3)$ are related to R through an affine transformation $\tilde{R} = AR + b$ where A is a constant 3×3 matrix and b is a constant 3-vector ($R =$ const is always a solution of (11)), system (11) specifies a surface M^2 uniquely up to affine equivalence. The compatibility conditions of system (11)

$$(q_x + aq + \tfrac{1}{2}b^2 - b_y)_x = 2qp_y + pq_y,$$
$$(p_y + bp + \tfrac{1}{2}a^2 - a_x)_y = 2pq_x + qp_x, \tag{15}$$
$$a_y = b_x$$

manifest the "Gauss–Codazzi" equations in affine geometry.

Remark. The map (9) $\to$ (11) from projective to affine geometry can be inverted. Let us consider linear system (11) satisfying the compatibility conditions (15). In view of $a_y = b_x$ there exists a function r^0 satisfying $a = -2(\ln r^0)_x$, $b = -2(\ln r^0)_y$. Introducing the 4-vector $r = (r^0, r^0R^1, r^0R^2, r^0R^3)$ we arrive at (9) with V and W given by

$$V = p_y + bp + \tfrac{1}{2}a^2 - a_x, \qquad W = q_x + aq + \tfrac{1}{2}b^2 - b_y. \tag{16}$$

Thus the "inverse" map (11) $\to$ (9) from affine to projective geometry is just the transformation from affine invariants (p, q, a, b) to projective invariants (p, q, V, W).

Isothermally asymptotic surfaces in projective differential geometry are specified by the constraint

$$p = q \tag{17}$$

in which case linear system (9) assumes the form (7) while its compatibility conditions (10) reduce to the stationary mVN equation (6). The name "isothermally asymptotic" which is due to Fubini [8] reflects the property that in asymptotic coordinates x, y Darboux's cubic form (13) becomes isothermic: $p(dx^3+dy^3)$. These surfaces are projective analogs of isothermic surfaces in conformal geometry characterized by the isothermicity of the metric in coordinates of lines of curvature. Fifty years ago the class of isothermally asymptotic surfaces was probably among the most popular ever discussed in the context of projective differential geometry. Isothermally asymptotic surfaces arise as the focal surfaces of special W-congruences preserving Darboux's curves (zero curves of the Darboux cubic form) and are uniquely specified by the requirement that the 3-web formed by asymptotic and Darboux's curves is hexagonal. We refer to standard textbooks [8, 5, 6, 12, 2] for further discussion.

Remark 1. Isothermally asymptotic surfaces may be defined by the equation

$$\left(\ln \frac{p}{q}\right)_{xy} = 0 \tag{18}$$

which is equivalent to (17) in view of the form-invariance of system (9) under the following transformations:

$$x^* = f(x), \quad y^* = g(y), \quad r^* = \sqrt{f'g'}\, r, \quad p^* = \frac{pg'}{f'^2}, \quad q^* = \frac{qf'}{g'^2},$$
$$V^*(f')^2 = V + S(f), \quad W^*(g')^2 = W + S(g)$$

where $S(\cdot)$ is the usual Schwarzian derivative, that is,

$$S(f) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

(transformation properties of V and W suggest their interpretation as projective connections along x- and y-asymptotic curves, respectively). So (18) is just the invariant form of (17). It is important that fixing the normalization $p = q$ we fix coordinates x, y up to affine transformations $x^* = ax + b$, $y^* = ay + c$, $a, b, c =$ const. Thus any isothermally asymptotic surface is endowed with a canonical affine structure. In these coordinates Darboux's curves are just straight lines $x + y =$ const. The lines $x + ky =$ const also make geometric sense: they can be defined in an invariant way as those curves on M^2 which have constant cross-ratio with the asymptotic and Darboux's directions.

Remark 2. Equations (6) are invariant under the discrete symmetry $(p, V, W) \to (-p, V, W)$. Geometrically, this means that the class of isothermally asymptotic surfaces is self-dual. Indeed, two surfaces (7) corresponding to (p, q, V, W) and $(-p, -q, V, W)$ are projective duals of each other.

For isothermally asymptotic surfaces projective metric assumes the form $2p^2\,dx\,dy$ with the corresponding area functional $\iint p^2\,dx\,dy$ which is the conserved quantity of the mVN equation (3).

Let us list some of the most important examples of isothermally asymptotic surfaces with the emphasize on solutions of the stationary mVN equation (6) corresponding to them.

Quadrics correspond to the trivial solution $p = 0$, $W = W(y)$, $V = V(x)$.

Projective transforms of rotation surfaces are specified by $p = p(x+y)$, $W = V = \frac{3}{2}p^2 + c$ where p is an arbitrary function of $(x + y)$ and c is an arbitrary constant. For $c > 0$ these are indeed projective transforms of surfaces $z = f(x^2 + y^2)$, while the cases $c = 0$ and $c < 0$ correspond to projective transforms of surfaces $z = f(x^2 + y)$ and $z = f(x^2 - y^2)$, respectively. Travelling-wave solutions $p(x + cy)$ of equation (6) correspond to surfaces, which are invariant under one-parameter groups of projective transformations. In the case $c \neq 1$ the function p is no longer arbitrary and can be expressed in elliptic functions (compare with [15]).

Cubic surfaces are specified by the following additional constraints in (6):

$$V = -\tfrac{1}{2}(\ln p)_{xx} + \tfrac{1}{8}(\ln p)_x^2 + \tfrac{5}{2}p_y, \qquad W = -\tfrac{1}{2}(\ln p)_{yy} + \tfrac{1}{8}(\ln p)_y^2 + \tfrac{5}{2}p_x \tag{19}$$

[13], see also [12, p. 131]. With these V, W equations (6) imply

$$\left(\frac{(\ln p)_{xy}}{\sqrt{p}} + 4p\sqrt{p}\right)_y = 5\,\frac{p_{xx}}{\sqrt{p}}, \qquad \left(\frac{(\ln p)_{xy}}{\sqrt{p}} + 4p\sqrt{p}\right)_x = 5\,\frac{p_{yy}}{\sqrt{p}}.$$

Integration of these equations for p would provide a 4-parameter family of exact solutions of equation (6): indeed, up to projective equivalence cubics in P^3 depend on 4 essential parameters.

The Roman surface of Steiner is a rational quartic in P^3 with the equation

$$(x^2 + y^2 + z^2 - 1)^2 = ((z-1)^2 - 2x^2)((z+1)^2 - 2y^2)$$

owing it's name to Steiner who investigated this surface in Rome in 1844. Besides quadrics and ruled cubic surfaces the Roman surface of Steiner is the only surface in P^3 possessing infinitely many conic sections through any of it's points. This result was announced several times: by Moutard in 1865, Darboux in 1880 and Wilczynski in 1908 (see [20], 1909 for historical remarks). The interest to the Roman surface of Steiner in projective differential geometry is due to the remarkable construction of Darboux, relating with an arbitrary surface M^2 in P^3 and an arbitrary point p on M^2 an osculating Roman surface of Steiner which has the fourth order of tangency with M^2 at this point. Analytically, the Roman surface of Steiner corresponds to the choice

$$\begin{aligned} V &= -\tfrac{1}{2}(\ln p)_{xx} + \tfrac{1}{8}(\ln p)_x^2 - \tfrac{5}{2}p_y, \\ W &= -\tfrac{1}{2}(\ln p)_{yy} + \tfrac{1}{8}(\ln p)_y^2 - \tfrac{5}{2}p_x, \\ (\ln p)_{xy} &= \tfrac{4}{9}p^2 \end{aligned} \tag{20}$$

([2, p. 149–150]) implying upon substitution in (6) the following equations for p:

$$p_{xx} = -\tfrac{4}{3}pp_y, \qquad p_{yy} = -\tfrac{4}{3}pp_x, \qquad (\ln p)_{xy} = \tfrac{4}{9}p^2. \tag{21}$$

These can be explicitly integrated:

$$p^2 = \frac{9}{4}\frac{f'g'}{(f+g)^2}$$

where the functions $f(x)$ and $g(y)$ satisfy the ODE's

$$f'^3 = (a_0 + a_1 f + a_2 f^2)^2, \quad g'^3 = (a_0 - a_1 g + a_2 g^2)^2.$$

Here a_i are arbitrary constants. Under the transformation $(p, V, W) \to (-p, V, W)$ equations (20) transform to (19). This means, that the dual of the Roman surface of Steiner is a cubic, and hence the Roman surface itself is a quartic of class 3 ([2, p. 150]).

The Roman surface of Steiner belongs to a broader class of isothermally asymptotic quartic surfaces known as

Quartics of Kummer investigated by Kummer as singular surfaces of quadratic line complexes. Around 1870 Kummer himself constructed plaster models of his quartics which now belong to the Göttingen collection (pictures of these models and the necessary explanations can be found in [7]). Analytically, the quartics of Kummer are specified by the conditions

$$\begin{aligned} &V = \tfrac{11}{8}(\ln p)_{xx} + 2(\ln p)_x^2, \quad W = \tfrac{11}{8}(\ln p)_{yy} + 2(\ln p)_y^2, \\ &(\ln p)_{xy} = \tfrac{4}{9}p^2 \end{aligned} \tag{22}$$

([2, p. 231]). Substituting these V, W in (6) we arrive at

$$\left(\frac{1}{p^2}(p^2(p^2)_y)_y\right)_y = \left(\frac{1}{p^2}(p^2(p^2)_x)_x\right)_x . \tag{23}$$

With

$$p^2 = \frac{9}{4}\frac{f'g'}{(f+g)^2}$$

equations (23) can be rewritten in the form

$$\begin{aligned} &\frac{1}{3}(f+g)^3\left(\frac{d^3(f'^3)}{df^3} - \frac{d^3(g'^3)}{dg^3}\right) - 4(f+g)^2\left(\frac{d^2(f'^3)}{df^2} - \frac{d^2(g'^3)}{dg^2}\right) \\ &\qquad + 20(f+g)\left(\frac{d(f'^3)}{df} - \frac{d(g'^3)}{dg}\right) - 40\left(f'^3 - g'^3\right) = 0 \end{aligned} \tag{24}$$

(here we used the identities $\partial_x = f'\partial_f$, $\partial_y = g'\partial_g$). Applying to (24) operator $\partial^6/\partial f^3 \partial g^3$ we arrive at

$$\frac{d^6(f'^3)}{df^6} - \frac{d^6(g'^3)}{dg^6} = 0$$

implying that f'^3 and g'^3 are polynomials of the 6th order in f and g, respectively. Coefficients of these polynomials are not independent and can be fixed upon substitution in (24):

$$f'^3 = P(f), \quad g'^3 = P(-g) \tag{25}$$

where P is an arbitrary polynomial of the 6th order. Calculations we have presented here follow [5, p. 66–69]. Formulae (25) reflect the uniformizability of Kummer's quartics via theta functions of genus 2 [9]. Since equations (22) are invariant under the transformation $(p, V, W) \to (-p, V, W)$ the class of Kummer's quartics is self-dual.

Quartics of Kummer constitute a subclass of *projectively applicable surfaces* which are characterized by a condition

$$(\ln p^2)_{xy} = cp^2 \tag{26}$$

for some constant c. Quartics of Kummer correspond to $c = \frac{8}{9}$. Condition (26) means that the projective metric $2p^2\,dx\,dy$ has constant Gaussian curvature $K = -c$. To investigate equations (6) with the additional constraint (26) we introduce the anzatz

$$\begin{aligned} V &= (\ln p)_{xx} + \tfrac{1}{2}(\ln p)_x^2 + A, \\ W &= (\ln p)_{yy} + \tfrac{1}{2}(\ln p)_y^2 + B, \end{aligned} \tag{27}$$

which implies upon substitution in (6) the following equations for A, B:

$$\begin{aligned} &A_y = \tfrac{3}{2}(1-c)(p^2)_x, \quad B_x = \tfrac{3}{2}(1-c)(p^2)_y, \\ &(p^2)_x A + p^2 A_x = (p^2)_y B + p^2 B_y. \end{aligned} \tag{28}$$

For $c = 1$ equations (28) are satisfied if $A = B = 0$. The corresponding surfaces are improper affine spheres; they will be discussed below. Here we consider the case $c \neq 1$. Introducing F by the formulae

$$A_x = -A(\ln p^2)_x + F, \quad B_y = -B(\ln p^2)_y + F \tag{29}$$

and writing down the compatibility conditions of (29) with $(28)_1$, $(28)_2$ we arrive at the equations for F

$$\begin{aligned} F_x &= 2cBp^2 + \tfrac{3}{2}(1-c)\frac{1}{p^2}(p^2(p^2)_y)_y, \\ F_y &= 2cAp^2 + \tfrac{3}{2}(1-c)\frac{1}{p^2}(p^2(p^2)_x)_x \end{aligned} \tag{30}$$

the compatibility conditions of which coincide with (23). Inserting in (23)

$$p^2 = \frac{1}{c}\frac{f'g'}{(f+g)^2} \tag{31}$$

we end up with the same f, g as in (25). For any p given by (31) the functions V and W can be recowered from (27) where A, B, F satisfy the compatible system (28), (29), (30). Thus for any such p there exists 3-parameter family of surfaces which have the same metric $2p^2\,dx\,dy$ and the same cubic form $p(dx^3 + dy^3)$ and are not projectively equivalent. In general, two projectively different surfaces in P^3 having the same projective metric (12) and the same cubic form (13) in a common asymptotic parametrization x, y are called projectively applicable. One can show that only for isothermally asymptotic surfaces M^2 satisfying (26) does there exist 3-parameter family of projectively different surfaces which are all projectively applicable to M^2 (the value 3 is the maximal possible). We point out that geometry of the surface M^2 depends crucially on the value of constant c. Isothermally asymptotic surfaces possessing only one-parameter families of projective applicabilities have been discussed in [16].

Affine spheres constitute an important subclass of isothermally asymptotic surfaces specified by the following reduction in (6):

$$V = \frac{p_{xx}}{p} - \frac{1}{2}\left(\frac{p_x}{p}\right)^2, \quad W = \frac{p_{yy}}{p} - \frac{1}{2}\left(\frac{p_y}{p}\right)^2. \tag{32}$$

After this ansatz the first equation in (6) will be satisfied identically while the last two imply the Tzitzeica equation for p:

$$(\ln p)_{xy} = p^2 + \frac{c}{p}, \qquad c = \text{const}. \tag{33}$$

The cases $c \neq 0$ and $c = 0$ correspond to proper and improper affine spheres, respectively. With V, W given by (32) equations (7) possess particular solution $r^0 = \sqrt{p}$. With $R = r/r^0$ equations (11) assume the form

$$R_{xx} = pR_y - \frac{p_x}{p}R_x,$$
$$R_{yy} = pR_x - \frac{p_y}{p}R_y$$

which become the familiar equations for the radius-vector of affine spheres after adding the compatible equation

$$R_{xy} = -\frac{c}{p}R.$$

The fact that Tzitzeica's equation solves the stationary mVN equation is also reflected in the following nonlocal representation of the mVN equation (in case $\alpha = -\beta = 1$):

$$p_t = \left(\frac{1}{p}\partial_x p^2 \partial_y^{-1} \frac{1}{p}\partial_x - \frac{1}{p}\partial_y p^2 \partial_x^{-1} \frac{1}{p}\partial_y\right)\left(p(\ln p)_{xy} - p^3\right).$$

In the recent paper [10] Konopelchenko and Pinkall introduced integrable dynamics of surfaces in 3-space governed by the VN equation. One can show that isothermally asymptotic surfaces can be interpreted as the stationary points of this evolution. So it is not surprising that they are described by the stationary mVN equation (which, as will be demonstrated below, is equivalent to the stationary VN).

Similar class of surfaces, described by the stationary mVN equation (with different real reduction) arises in Lie sphere geometry [3]. These are the so-called diagonally cyclidic surfaces which in Lie sphere geometry play a role similar to that of isothermally asymptotic surfaces in projective geometry.

3. The Lelieuvre representation of surfaces in 3-space

We consider a surface $M^2 \in E^3$ parametrized by asymptotic coordinates x, y with the radius-vector $R(x, y)$ satisfying equations (11)

$$R_{xx} = pR_y + aR_x,$$
$$R_{yy} = qR_x + bR_y$$

with the compatibility conditions (15).

The Lelieuvre representation of a surface M^2 [14] is defined by the formulae

$$R_x = \nu \times \nu_x, \quad R_y = -\nu \times \nu_y \tag{34}$$

where the 3-vector ν (called the affine conormal) satisfies the equation

$$\nu_{xy} = u\nu \tag{35}$$

for certain potential $u(x, y)$. In formulae (34) "$\times$" denotes vector product in E^3. Equations (34) are compatible in view of (35). Our aim is to show that in the case of isothermally asymptotic surfaces potential u satisfies the stationary VN equation (5).

Assuming ν, ν_x, ν_y to be independent we introduce the expansions

$$\nu_{xx} = c^1\nu_x + c^2\nu_y + c^3\nu,$$
$$\nu_{yy} = c^4\nu_x + c^5\nu_y + c^6\nu$$

where the coefficients c^i are uniquely expressible through p, q, a, b. Indeed, differentiating the first equation (34) with respect to x and the second with respect to y we arrive at $c^1 = a$, $c^2 = -p$, $c^4 = -q$, $c^5 = b$. The remaining coefficients can be obtained from the compatibility conditions of the equations for ν: $u = a_y + pq = b_x + pq$, $c^3 = p_y + pb$, $c^6 = q_x + qa$, so that the final equations for ν assume the form

$$\nu_{xy} = u\nu, \quad \nu_{xx} = a\nu_x - p\nu_y + (p_y + pb)\nu, \quad \nu_{yy} = -q\nu_x + b\nu_y + (q_x + qa)\nu \tag{36}$$

($u = a_y + pq = b_x + pq$). The compatibility conditions of equations (36) coincide with (15).

Remark. Introducing r^0 by the formulae $a = -2(\ln r^0)_x$, $b = -2(\ln r^0)_y$ one can easily check that $\nu = (r^0)^2(R_x \times R_y)$ satisfies (36).

In the case $p = q$ of isothermally asymptotic surfaces the radius-vector R satisfies the equations

$$R_{xx} = pR_y + aR_x,$$
$$R_{yy} = pR_x + bR_y$$

with the compatibility conditions

$$(p_x + ap + \tfrac{1}{2}b^2 - b_y)_x = \tfrac{3}{2}(p^2)_y,$$
$$(p_y + bp + \tfrac{1}{2}a^2 - a_x)_y = \tfrac{3}{2}(p^2)_x,$$
$$a_y = b_x$$

while equations for the conormal ν assume the form

$$\begin{aligned} \nu_{xy} &= u\nu, \\ \nu_{xx} &= a\nu_x - p\nu_y + (p_y + pb)\nu, \\ \nu_{yy} &= -p\nu_x + b\nu_y + (p_x + pa)\nu \end{aligned} \tag{37}$$

$(u = a_y + p^2 = b_x + p^2)$. It is a direct check that in this case equations (37) imply

$$\begin{aligned} \nu_{xy} &= u\nu, \\ \nu_{xxx} - 3v\nu_x &= \nu_{yyy} - 3w\nu_y, \\ v_y &= u_x, \\ w_x &= u_y \end{aligned} \tag{38}$$

where $v = \frac{2}{3}V + a_x$, $w = \frac{2}{3}W + b_y$ and V, W are projective invariants (16). Since (38) is just the stationary VN linear problem, the stationary VN equation (5) for u, v, w follows directly. Thus any solution p, V, W of the stationary mVN equation generates solution u, v, w of the stationary VN equation according to the formulae

$$u = p^2 + a_y = p^2 + b_x, \qquad v = \tfrac{2}{3}V + a_x, \qquad w = \tfrac{2}{3}W + b_y$$

where $a = -2(\ln r^0)_x$, $b = -2(\ln r^0)_y$ and r^0 is a solution of (7). This gives the desired Bäcklund transformation (8).

Let us apply transformation (8) to affine spheres choosing $r^0 = \sqrt{p}$ which is a solution of linear system (7). With this r^0 formulae (8), (32), (33) give the following expressions for u, v, w:

$$u = -\frac{c}{p}, \qquad v = -\frac{1}{3}\frac{p_{xx}}{p} + \frac{2}{3}\left(\frac{p_x}{p}\right)^2, \qquad w = -\frac{1}{3}\frac{p_{yy}}{p} + \frac{2}{3}\left(\frac{p_y}{p}\right)^2$$

implying that

$$(\ln u)_{xy} = u - \frac{c^2}{u^2}, \qquad v = \frac{1}{3}\frac{u_{xx}}{u}, \qquad w = \frac{1}{3}\frac{u_{yy}}{u}$$

solve (5). This is reflected in the nonlocal representation of the VN equation (in the case $\alpha = -\beta = 1$)

$$u_t = \left(\partial_x u \partial_y^{-1}\frac{1}{u^2}\partial_x - \partial_y u \partial_x^{-1}\frac{1}{u^2}\partial_y\right)\left(u^2(\ln u)_{xy} - u^3\right)$$

which was obtained in [11].

As it was pointed out in [17], VN equation is covariant under the Moutard transformation which can be specialized to the stationary case as well. Geometrically, this construction should give the known Bäcklund transformation for isothermally asymptotic surfaces which is generated by a W-congruence preserving Darboux's curves (see, e.g., [6]).

4. Acknowledgements

This research was supported by the RFFI grants 96-01-00166, 96-06-80104, INTAS 96-0770 and the Alexander von Humboldt Foundation. The author is thankful to B.G. Konopelchenko for useful discussions.

References

[1] L.V. Bogdanov, Veselov–Novikov equation as a natural two-dimensional generalization of the Korteweg-de Vries equation, *Theor. Math. Phys.* **70** (1987) 309.
[2] G. Bol, *Projektive Differentialgeometrie* (Vandenhoeck and Ruprecht, Göttingen, 1954).
[3] E.V. Ferapontov, Surfaces in Lie sphere geometry and the stationary Davey–Stewartson hierarchy, Preprint SFB 288 No. 287, Berlin, 1997 (DG 9710128).
[4] E.V. Ferapontov and W.K. Schief, Surfaces of Demoulin: differential geometry, Bäcklund transformation and integrability, *J. Geom. Phys.* **30** (1999) 343–363.
[5] S.P. Finikov, *Projective Differential Geometry* (Moscow–Leningrad, 1937).
[6] S.P. Finikov, *Theory of Congruences* (Moscow–Leningrad, 1950).
[7] G. Fischer, *Mathematische Modelle, I, II* (Braunschweig, Wiesbaden, 1986).
[8] G. Fubini and E. Čech, *Geometria Proiettiva Differenziale* (Zanichelli, Bologna, 1926).
[9] R.W.H.T. Hudson, *Kummer's Quartic Surface* (Cambridge Univ. Press 1990; first published 1905).
[10] B.G. Konopelchenko and U. Pinkall, Integrable deformations of affine surfaces via Nizhnik–Veselov–Novikov equation, preprint, 1998.
[11] B.G. Konopelchenko and C. Rogers, On generalized Loewner systems: novel integrable equations in $2+1$ dimensions, *J. Math. Phys.* **34** (1993) 214–242.
[12] E. Lane, *A Treatise on Projective Differential Geometry* (The Univ. of Chicago Press, 1942).
[13] E. Lane, The contact of a cubic surface with an analytic surface, *Trans. Amer. Math. Soc.* **29** (1927) 471–480.
[14] M. Lelieuvre, Sur les lignes asymptotiques et leur représentation sphérique, *Bull. Sci. Mathematiques* **12** (1888) 126–128.
[15] W. Lingenberg, Zur Differentialgeometrie der Flächen, die eine eingliedrige projektive Gruppe in sich gestatten und über allgemeine Projektivrotationsflächen, *Math. Z.* **66** (1957) 409–446.
[16] W. Lingenberg, Über die projektivabvickelbaren isotherm-asymptotischen Flächen, *Archiv der Mathematik* **16** (1965) 274–287.
[17] V.B. Matveev and M.A. Salle, *Darboux Transformations and Solitons* (Springer, Berlin, 1991).
[18] L.P. Nizhnik, Integration of multidimensional nonlinear equations by the method of inverse problem, *Dokl. Akad. Nauk. SSSR* **254** (1980) 332–335.
[19] A.P. Veselov and S.P. Novikov, Finite-gap two-dimensional potential Schrödinger operators. Explicit formulae and evolution equations, *Dokl. Akad. Nauk. SSSR* **279** (1984) 20–24.
[20] E.I. Wilczynski, Projective-differential geometry of curved surfaces, *Trans. Amer. Math. Soc.* **8** (1907) 233–260; **9** (1908) 79–120, 293–315; **10** (1909) 176–200, 279–296.