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On the convergence of a fourth-order method for a class of singular boundary value problems

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ABSTRACT

In the present paper we extend the fourth order method developed by Chawla et al. [M.M. Chawla, R. Subramanian, H.L. Sathi, A fourth order method for a singular two-point boundary value problem, BIT 28 (1988) 88–97] to a class of singular boundary value problems

$$(p(x)y')' = p(x)f(x, y), \quad 0 < x \le 1$$
$$y'(0) = 0, \qquad \alpha y(1) + \beta y'(1) = \gamma$$

where $p(x) = x^{b_0} q(x)$, $b_0 \ge 0$ is a non-negative function. The order of accuracy of the method is established under quite general conditions on $f(x, y)$ and is also verified by one example. The oxygen diffusion problem in a spherical cell and a nonlinear heat conduction model of a human head are presented as illustrative examples. For these examples, the results are in good agreement with existing ones.

1. Introduction

Consider a class of scalar singular boundary value problems

$$Ly \equiv (p(x)y')' = p(x)f(x, y), \quad 0 < x \le 1 \tag{1}$$

$$y'(0) = 0, \qquad \alpha y(1) + \beta y'(1) = \gamma, \tag{2}$$

where $\alpha > 0$, $\beta \geqslant 0$ and γ is a finite constant. We assume that $p(x)$ satisfies the following conditions

(A) (i) $p(x) > 0$ on $(0, 1]$,
 (ii) $p(x) \in C^1(0, 1]$,
 (iii) $p(x) = x^{b_0} q(x)$ on $[0, 1]$, $b_0 \ge 0$ and for some $r > 1$
 $Q(x) = 1/q(x)$ is analytic in $\{x : |x| < r\}$.

Further we assume that

(B) $f(x, y)$ is continuous, $\partial f/\partial y$ exists, it is continuous and $\partial f/\partial y \ge 0 \forall\ 0 \le x \le 1$ and all real y.

The existence-uniqueness of the solution of the boundary value problem (1) and (2) is established in [16,17] for xp'/p analytic in $\{x : |x| < r\}$ for some $r > 1$ and for the more general problem in [8] with nonlinear boundary conditions.

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Table 1
Maximum absolute errors

N/b_0	0.25	Order	1.0	Order	2.0	Order	8.0	Order
8	2.99(−3)[a]		8.21(−4)		6.72(−4)		1.15(−3)	
16	1.46(−4)	4.35	3.98(−5)	4.37	4.30(−5)	3.97	9.81(−5)	3.56
32	7.50(−6)		2.08(−6)		2.68(−6)		6.96(−6)	
64	3.95(−7)	4.25	1.16(−7)	4.16	1.66(−7)	4.01	4.61(−7)	3.92
128	2.11(−8)		6.99(−9)		1.04(−8)		2.96(−8)	
256	1.13(−9)	4.23	4.23(−10)	4.05	6.47(−10)	4.00	1.87(−9)	3.98

[a] $2.99(-3) = 2.99 \times 10^{-3}$.

The boundary value problem (1) and (2) with $b_0 = 0, 1, 2$ and $q(x) = 1$ arises in the study of various tumor growth problems [1–3,9] with linear $f(x, y)$ and also with non-linear $f(x, y)$ of the form

$$f(x, y) \equiv f(y) = \theta y/(y + \kappa), \quad \theta > 0,\ \kappa > 0. \tag{3}$$

The problem with $b_0 = 2$, $q(x) = 1$ arises in the study of steady state oxygen-diffusion in a cell with Michaelis-Menten uptake kinetics and in the study of the distribution of heat sources in the human head [4,14,15], in which

$$f(x, y) \equiv f(y) = -\delta e^{-\theta y}, \quad \theta > 0,\ \delta > 0. \tag{4}$$

There is a considerable literature on numerical methods for such problems e.g. [5,6,10,11,13,18,19]. Ciarlet et al. [6] and Jamet [13] discussed numerical methods for $q(x) = 1$, $b_0 \in (0, 1)$. Russel and Shampine [18] discussed numerical methods for linear case with $p(x) = x^{b_0}$, $b_0 = 1, 2$ while in [11] Gustafsson considered a linear problem in $(\delta, 1]$ instead of $(0, 1]$ and constructed compact second order, fourth order and non-compact fourth order methods for its solution. In [5,10] fourth order methods are described for the problem (1) in the case $p(x) = x^{b_0}$, $b_0 \geq 1$ and boundary conditions $y'(0) = 0, y(1) = B$.

In the present work, using Chawla's identity [19], we extend the fourth order finite difference method developed in [5] for $p(x) = x^{b_0}$, $b_0 \geq 1$ and boundary conditions $y'(0) = 0$, $y(1) = B$ to the problem (1) and (2) where the non-negative function $p(x)$ satisfies the conditions given in A(i)–(iii). In Section 3 we establish the order of accuracy of the method for a non-negative function $p(x)$ and under quite general conditions on $f(x, y)$. In the case $p(x) = x^{b_0}$, this work provides a fourth order method for $b_0 \geq 0$, while in most of the work methods are given for $b_0 \geq 1$. To illustrate the convergence and to corroborate the order of accuracy of the method, we apply it to one example. The results are displayed in Table 1. Two physiological problems, the oxygen diffusion problem in a spherical cell and the nonlinear heat-conduction model of a human head, have also been solved and the results are in good agreement with those of [4,7].

2. Finite difference method

This section is divided in two parts (i) Description of the method and (ii) Construction of the method. All coefficients not specified explicitly in this section can be found in the Appendix.

2.1. Description of the method

In this section we first state the method and then its construction process is explained in Section 2.2.

For a positive integer $N \geq 2$, we consider a uniform mesh $w_h = \{x_k\}_{k=0}^{N}$ over $[0, 1]$, where $x_k = kh$, $h = 1/N$. Let $g(x) := f(x, y(x))$ ($y(x)$ is the solution), $g_k = g(x_k)$, $y_k = y(x_k)$ etc. Now we approximate the differential operator Ly on the grid w_h by the difference operator

$$(L^h\widetilde{y})_1 = -\widetilde{y}_1/J_1 + \widetilde{y}_2/J_1,$$
$$(L^h\widetilde{y})_k = \widetilde{y}_{k-1}/J_{k-1} - (1/J_k + 1/J_{k-1})\widetilde{y}_k + \widetilde{y}_{k+1}/J_k, \quad k = 2(1)(N-1),$$
$$(L^h\widetilde{y})_N = \widetilde{y}_{N-1}/J_{N-1} - (1/J_{N-1} + \alpha/(\beta Q_N))\widetilde{y}_N + \gamma/(\beta Q_N)$$

where $L^h y$ is an approximation for a locally integrated version of $(py')'/p$ (defined in Section 2.2.1), $\tilde{y} = (\widetilde{y}_k)$ denotes the approximate solution, $\widetilde{y}_k \approx y_k$ and $Q_k = Q(x_k)$,

$$J_k = \int_{x_k}^{x_{k+1}} (p(\tau))^{-1} d\tau$$

then the difference scheme for the boundary value problem (1) and (2) is given by Eqs. (5)–(7)

$$(L^h\widetilde{y})_1 = b_{0,1}\widetilde{g}_1 + b_{1,1}\widetilde{g}_2 + b_{-1,1}\widetilde{g}_0, \tag{5}$$

$$(L^h\widetilde{y})_k = b_{-1,k}\widetilde{g}_{k-1} + b_{0,k}\widetilde{g}_k + b_{1,k}\widetilde{g}_{k+1}, \quad k = 2(1)(N-1), \tag{6}$$

$$(L^h\widetilde{y})_N = a_{-1,N}\widetilde{g}_{N-1} + a_{0,N}\widetilde{g}_N + (64/9)a_{1,N}\widetilde{g}_{N-\frac{3}{8}} \tag{7}$$

where

$$\widetilde{g}_k = f(x_k, \tilde{y}_k), \qquad \widetilde{g}_0 = f(x_0, \widetilde{y}_0), \qquad \widetilde{y}_0 = \tilde{y}_1 - \frac{1}{2(b_0+1)} x_1^2 \widetilde{g}_1,$$

$$\widetilde{g}_{N-\frac{3}{8}} = f\left(x_N - \frac{3}{8}h,\ \widetilde{y}_{N-\frac{3}{8}}\right),$$

$$\widetilde{y}_{N-\frac{3}{8}} = [9\tilde{y}_{N-1} + (55 + 15h\alpha/\beta)\tilde{y}_N - 15h\gamma/\beta]/64$$

and the coefficients $a_{i,j}$, $b_{i,j}$ are given in the Appendix.

2.2. Construction of the method

In this section we first describe the derivation of (6) and then using this, the construction of discretization (7) at $x = 1$ and (5) at $x = 0$ are given and estimates for truncation errors are given in Section 3 (without proof).

2.2.1. Derivation of the discretization (6)

By taking $z(x) = p(x)\,y'(x)$, the differential equation (1) can be written as $z' = p(x) f(x, y(x))$. Then an approximation for the differential operator Ly on the uniform mesh w_h is obtained as follows:

We integrate $z' = p(x) f(x, y(x))$ twice, first from x_k to τ and then from x_k to x_{k+1} and change the order of integration to get the following

$$y_{k+1} - y_k = z_k J_k + \int_{x_k}^{x_{k+1}} \left(\int_t^{x_{k+1}} (p(\tau))^{-1} \mathrm{d}\tau \right) p(t) g(t) \mathrm{d}t, \tag{8}$$

where $z_k = z(x_k)$ and $J_k = \int_{x_k}^{x_{k+1}} (p(\tau))^{-1} \mathrm{d}\tau$. In an analogous way, we get

$$y_k - y_{k-1} = z_k J_{k-1} - \int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} (p(\tau))^{-1} \mathrm{d}\tau \right) p(t) g(t) \mathrm{d}t. \tag{9}$$

Eliminating z_k from (8) and (9) we obtain Chawla's identity:

$$\frac{y_{k+1} - y_k}{J_k} - \frac{y_k - y_{k-1}}{J_{k-1}} = \frac{I_k^+}{J_k} + \frac{I_k^-}{J_{k-1}}, \quad k = 1\,(1)\,N-1 \tag{10}$$

where I_k^+ and I_k^- are as follows

$$I_k^+ = \int_{x_k}^{x_{k+1}} \left(\int_t^{x_{k+1}} (p(\tau))^{-1} \mathrm{d}\tau \right) p(t) g(t) \mathrm{d}t, \tag{11}$$

$$I_k^- = \int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^{t} (p(\tau))^{-1} \mathrm{d}\tau \right) p(t) g(t) \mathrm{d}t. \tag{12}$$

Now using Taylor's expansion for Q (as defined by condition A(iii) in Section 1) and g about x_k in $I_k^{\pm}$, the approximation (6) is obtained for a smooth solution $y(x)$ as

$$(L^h y)_k = b_{-1,k} g_{k-1} + b_{0,k} g_k + b_{1,k} g_{k+1} + t_k, \quad k = 2(1)(N-1) \tag{13}$$

where t_k is the truncation error and the coefficients can be found in the Appendix.

2.2.2. Derivation of the discretization (7)

For this, we use the Eq. (9) for $k = N$, the boundary condition at $x = 1$ and

$$g_N' = \frac{1}{15h}\left[55 g_N + 9 g_{N-1} - 64 g_{N-\frac{3}{8}}\right] + \frac{h^2}{16} g'''(\eta_N), \quad x_{N-1} < \eta_N < x_N,$$

$$g_N'' = \frac{16}{15h^2}\left[5 g_N + 3 g_{N-1} - 8 g_{N-\frac{3}{8}}\right] + \frac{11h}{24} g'''(\eta_N'), \quad x_{N-1} < \eta_N' < x_N,$$

to get

$$(L^h y)_N = a_{-1,N} g_{N-1} + a_{0,N} g_N + \frac{64}{9} a_{1,N} g_{N-\frac{3}{8}} + t_N \tag{14}$$

where $g_{N-\frac{3}{8}} = g(x_N - \frac{3}{8}h)$, t_N is the truncation error and the coefficients can be found in the Appendix.

Now discretization (14) involves the unknown $y_{N-\frac{3}{8}}$ which we approximate in the following way:

$$y_{N-\frac{3}{8}} = \bar{y}_{N-\frac{3}{8}} + \frac{15h^3}{1024} y_N'''$$

where

$$\bar{y}_{N-\frac{3}{8}} = \frac{1}{64}\left[9y_{N-1} + \left(55 + \frac{15h\alpha}{\beta}\right) y_N - \frac{15h\gamma}{\beta}\right].$$

Now, let $\overline{g}_{N-\frac{3}{8}} = f\left(x_N - \frac{3}{8}h,\ \bar{y}_{N-\frac{3}{8}}\right)$, then replacing $g_{N-\frac{3}{8}}$ by $\overline{g}_{N-\frac{3}{8}}$ in Eq. (14) we obtain the following discretization for a smooth solution $y(x)$ at $k = N$

$$(L^h y)_N = a_{-1,N} g_{N-1} + a_{0,N} g_N + \frac{64}{9} a_{1,N} \overline{g}_{N-\frac{3}{8}} + \bar{t}_N \tag{15}$$

where

$$\bar{t}_N = t_N + \frac{5h^3}{48} a_{1,N} y_N''' \frac{\partial f}{\partial y}\left(x_N - \frac{3}{8}h,\ y^*_{N-\frac{3}{8}}\right),$$

$$y^*_{N-\frac{3}{8}} \in \left(\min\left\{y_{N-\frac{3}{8}},\ \bar{y}_{N-\frac{3}{8}}\right\}, \max\left\{y_{N-\frac{3}{8}},\ \bar{y}_{N-\frac{3}{8}}\right\}\right),$$

and coefficients can be found in the Appendix.

2.2.3. Derivation of the discretization (5)

We first consider the Eq. (8) for $k = 1$. This involves z_1 which is obtained by integrating $z'(x) = p(x)f(x, y(x))$ from 0 to x_1 and using the boundary condition $y'(0) = 0$ and is given by

$$z_1 = \int_0^{x_1} p(t)g(t)\mathrm{d}t.$$

Now replacing z_1 in Eq. (8) we obtain the following identity

$$\frac{y_2 - y_1}{J_1} = I_1 + \frac{I_1^+}{J_1}$$

where $I_1 = \int_0^{x_1} p(t)g(t)\mathrm{d}t$.

Now using Taylor's expansion for $Q(x)$ and g about x_1 in I_1 and I_1^+ in the above equation, we get the following discretization for a smooth solution $y(x)$

$$(L^h y)_1 = b_{0,1} g_1 + b_{1,1} g_2 + b_{-1,1} g_0 + t_1 \tag{16}$$

where t_1 is truncation error and coefficients can be found in the Appendix. Since the discretization (16) involves the unknown value y_0 which we approximate in the following way:

$$y_0 = \bar{y}_0 + \tau_0$$

where

$$\bar{y}_0 = y_1 - \frac{x_1^2}{2(b_0 + 1)} g_1$$

$$\text{and}\quad \tau_0 = \frac{h^3}{6(b_0 + 1)(b_0 + 2)}\left\{(b_0 + 4) g'(\xi_0) - \frac{2Q_1'}{Q_1} g_1\right\}, \quad 0 < \xi_0 < x_1.$$

Now, let $\overline{g}_0 = f(x_0,\ \bar{y}_0)$, then replacing g_0 by $\overline{g}_0$ in Eq. (16) we obtain the following discretization for $k = 1$:

$$(L^h y)_1 = b_{0,1} g_1 + b_{1,1} g_2 + b_{-1,1} \overline{g}_0 + \bar{t}_1 \tag{17}$$

where $\bar{t}_1 = t_1 + b_{-1,1}\tau_0 \frac{\partial f}{\partial y}\left(x_0,\ y_0^*\right)$, $y_0^* \in (\min\{y_0,\ \bar{y}_0\}, \max\{y_0,\ \bar{y}_0\})$.

2.2.4. Computation of y_0

To compute y_0, we integrate $z' = pf$ twice, first from 0 to τ and then from 0 to x_1, and using boundary condition $y'(0) = 0$ we obtain the following

$$y_1 - y_0 = \int_0^{x_1} \left(\int_t^{x_1} (p(\tau))^{-1} \mathrm{d}\tau \right) p(t) g(t) \mathrm{d}t.$$

Now using Taylor's expansion for Q and g about x_1 it is easy to establish the following

$$y_0 = y_1 + a_{0,0} g_1 + a_{-1,0} \overline{g}_0 + a_{1,0} g_2 + O\left(h^5\right)$$

where

$$a_{\pm 1,0} = -\frac{1}{2h}\left[\pm A_{10,0} + \frac{1}{h} A_{20,0} \mp (A_{20,0} - A_{11,0}) Q_1'/Q_1\right],$$

$$a_{0,0} = -\left[A_{00,0} - \frac{1}{h^2} A_{20,0} + (-A_{10,0} + A_{01,0}) Q_1'/Q_1 + (A_{20,0} - A_{11,0})(Q_1'/Q_1)^2 + (-A_{20,0} + A_{02,0}) Q_1''/(2Q_1)\right],$$

$$A_{00,0} = \frac{1}{2\psi(1)} x_1^2; \qquad A_{10,0} = -\frac{(b_0+4)}{6\psi(2)} x_1^3; \qquad A_{11,0} = \frac{(b_0+3)}{12\psi(2)} x_1^4; \qquad A_{01,0} = -\frac{1}{6\psi(1)} x_1^3;$$

$$A_{20,0} = \frac{(b_0^2 + 7b_0 + 18)}{12\psi(3)} x_1^4; \qquad A_{02,0} = \frac{1}{12\psi(1)} x_1^4 \quad \text{with } \psi(i) = \prod_{j=1}^{i} (b_0 + j).$$

3. Convergence of the method

In this section we establish the fourth order accuracy of the method developed in the previous section for the boundary value problems (1) and (2). Let $G(\tilde{Y}) = (\widetilde{g}_1, \ldots, \widetilde{g}_N)^{\mathrm{T}}$, $\tilde{Y} = (\tilde{y}_1, \ldots, \tilde{y}_N)^{\mathrm{T}}$, $Q = (0, 0, \ldots, 0, \frac{\gamma}{\beta Q_N})^{\mathrm{T}}$ and $H(\widetilde{Y}) = (b_{-1,1}\widetilde{g}_0, 0, \ldots, 0, (64/9) a_{1,N}\, \widetilde{g}_{N-\frac{3}{8}})^{\mathrm{T}}$, then the difference scheme (5)–(7) can be expressed in matrix form as

$$D\tilde{Y} + PG(\tilde{Y}) + H(\tilde{Y}) = Q \tag{18}$$

where $D = (d_{ij})$ and $P = (p_{ij})$ are $(N \times N)$ tridiagonal matrices with

$$d_{k,k+1} = -1/J_k, \quad k = 1(1)(N-1); \qquad d_{k,k} = (1/J_k + 1/J_{k-1}), \quad k = 2(1)(N-1)$$
$$d_{N,N} = (1/J_{N-1} + \alpha/(\beta Q_N)), \qquad d_{1,1} = -1/J_1, \qquad d_{k,k-1} = -1/J_{k-1}, \quad k = 2\,(1)\,N$$

and

$$p_{k,k} = b_{0,k}, \qquad p_{k,k+1} = b_{1,k}, \quad k = 1\,(1)\,(N-1), \qquad p_{k,k-1} = b_{-1,k}, \quad k = 2\,(1)\,(N-1),$$
$$p_{N,N} = a_{0,N}, \qquad p_{N,N-1} = a_{-1,N}.$$

Let $Y = (y_1, \ldots, y_N)^{\mathrm{T}}$, $T = (\bar{t}_1, t_2, \ldots, t_{N-1}, \bar{t}_N)^{\mathrm{T}}$, $E = \tilde{Y} - Y = (e_1, \ldots, e_N)^{\mathrm{T}}$, then the Eqs. (14)–(16) can be written as

$$DY + PG(Y) + H(Y) + T = Q. \tag{19}$$

We may write

$$\widetilde{g}_0 - \overline{g}_0 = \bar{U}_0(\widetilde{y}_0 - \bar{y}_0), \qquad \widetilde{g}_{N-\frac{3}{8}} - \overline{g}_{N-\frac{3}{8}} = \bar{U}_{N-\frac{3}{8}}\left(\widetilde{y}_{N-\frac{3}{8}} - \bar{y}_{N-\frac{3}{8}}\right)$$

for some $\bar{U}_0, \bar{U}_{N-\frac{3}{8}}$ and then from Eqs. (18) and (19) we get the error equation

$$(D + PM + W)E = T \tag{20}$$

where

$$G(\tilde{Y}) - G(Y) = ME, \quad M = \operatorname{diag}\{U_1, \ldots, U_N\}\ (U_k = \partial f_k/\partial y_k \geq 0);$$
$$H(\tilde{Y}) - H(Y) = WE,$$

where $W = (w_{ij})$ is a matrix with

$$w_{1,1} = b_{-1,1}\bar{U}_0\left\{1 - \frac{x_1^2 U_1}{2(b_0+1)}\right\}, \qquad w_{N,N-1} = a_{1,N}\bar{U}_{N-\frac{3}{8}}, \qquad w_{N,N} = \frac{5}{9} a_{1,N}\left(11 + \frac{3h\alpha}{\beta}\right)\bar{U}_{N-\frac{3}{8}}$$

and all its other entries are zeros.

Now for fixed x_k and for $h \to 0$, J_k is asymptotically equal (written as $\backsim$) to $hx_k^{-b_0}Q_k$ and is written by $J_k \backsim hx_k^{-b_0}Q_k$.

Similarly

$$B_{00,1} \backsim 3hx_1^{b_0}/(2Q_1), \qquad B_{10,1} \backsim -h^2x_1^{b_0}/(3Q_1), \qquad B_{20,1} \backsim 5h^3x_1^{b_0}/(12Q_1),$$
$$B_{00,k} \backsim hx_k^{b_0}/Q_k, \qquad B_{10,k} \backsim -b_0h^3x_k^{b_0-1}/(12Q_k), \qquad B_{20,k} \backsim h^3x_k^{b_0}/(6Q_k),$$
$$B_{01,k} \backsim -b_0h^3x_k^{b_0-1}/(4Q_k), \qquad B_{02,k} \backsim h^3x_k^{b_0}/(2Q_k), \qquad B_{11,k} \backsim h^3x_k^{b_0}/(4Q_k),$$
$$A^-_{00,N} \backsim h^2/2, \qquad A^-_{10,N} \backsim -h^3/6, \qquad A^-_{01,N} \backsim -h^3/3, \qquad A^-_{20,N} \backsim h^4/12,$$
$$A^-_{02,N} \backsim h^4/4, \qquad A^-_{11,N} \backsim h^4/8.$$

Since

$$d_{i,i+1} \le 0, d_{i,i-1} \le 0 \quad \text{and} \quad \sum_{j=1}^{N} d_{i,j} \begin{cases} \ge 0, & i = 2, 3, \ldots, (N-1) \\ > 0, & i = 1, N \end{cases}$$

the matrix D is irreducible and monotonic (from the corollary of Theorems 7.2 and 7.4 of [12]). Similarly it is easy to see that $D + PM + W$ is also irreducible and monotonic as $PM + W \ge 0$. Now from Theorem 7.1 of [12], D^{-1}, *and* $(D + PM + W)^{-1}$ exist and are nonnegative, and from theorem 7.5 of [12] we get

$$(D + PM + W)^{-1} \le D^{-1}.$$

Let $Z = (1, \ldots, 1)^T$ and $S = (S_1, \ldots, S_N)^T = DZ$, denote the vector of row-sums of D, and let $V = (V_1, \ldots, V_N)^T$ where $V_j = (2\beta/\alpha) + 2 - \frac{1}{2}(x_j + 1)^2$ and $R = (R_1, \ldots, R_N)^T = DV$. Since $R_N > 0$ and for sufficiently small h

$$R_1 > h^{b_0}/Q_1,$$
$$R_k > (b_0h/(2Q_k))\, x_k^{b_0-1}, \quad k = 2\,(1)\,(N-1)$$

then from $D^{-1}R = V$ we get

$$h^{b_0}d^{-1}_{i,1}/Q_1 < (2\beta/\alpha) + (3/2), \quad i = 1\,(1)\,N \tag{21}$$

and

$$(b_0h/2)\sum_{k=2}^{N-1}(d^{-1}_{i,k}/Q_k)x_k^{b_0-1} \le V_i < (2\beta/\alpha) + (3/2), \quad i = 1\,(1)\,N. \tag{22}$$

Let $g^{(i)}$ for $i = 0(1)3$, $xg^{(iv)}$ and y''' be bounded on $(0, 1]$ then for sufficiently small h

$$\left|\bar{t}_1\right| \le \hat{C}h^{4+b_0}/\,|Q_1|\,, \tag{23}$$

$$|t_k| \le Ch^5x_k^{b_0-1}/\,|Q_k|\,, \quad k = 1\,(1)\,(N-1) \tag{24}$$

and

$$\left|\bar{t}_N\right| \le \overline{C}h^4/\,|Q_N| \tag{25}$$

for suitable constants $\hat{C}$, C and $\overline{C}$.

Now since $S_N = \alpha/(\beta Q_N)$ then from $D^{-1}S = Z$ we obtain

$$d^{-1}_{i,N} = \frac{\beta}{\alpha}Q_N, \quad i = 1\,(1)\,N \tag{26}$$

and thus from Eqs. (20)–(26) we get

$$|e_i| \le d^{-1}_{i,1}\left|\bar{t}_1\right| + \sum_{k=1}^{N} d^{-1}_{i,k}\,|t_k| + d^{-1}_{i,N}\left|\bar{t}_N\right| \le C^*h^4$$

where $C^* = [(\hat{C}/2 + C/b_0)(4\beta/\alpha + 3) + \bar{C}(\beta/\alpha)]$ and hence it follows that

$$\|E\|_\infty = O\left(h^4\right).$$

In the case of $b_0 = 1, 2, 3$ the convergence of the method can be established by taking the limits $b_0 \to 1^+, 2, 3$ respectively.

Thus we have established the following result:

Theorem 1. *Assume that $f(x, y)$ satisfies* (B) *and $p(x)$ satisfies the conditions in* (A)*.Then, the finite difference method* (5)–(7) *based on a uniform mesh applied to the boundary value problem* (1) *and* (2) *with $b_0 \ge 0$ is of fourth order accuracy for sufficiently small h provided $g^{(i)}$ for $i = 0(1)3$, $xg^{(iv)}$ and y''' are bounded on* $(0, 1]$, *where $g(x) := f(x, y(x))$.*

Remark 1. Estimates for truncation errors (as given in (23)–(25)) are not proved but can be established on the basis of explicit specification of coefficients given in the Appendix and under the assumptions that $g^{(i)}$ for $i = 0(1)3$, $xg^{(iv)}$ and y''' are bounded on $(0, 1]$.

Table 2
Numerical solution of physiological problems

x	First problem		Second problem			
	Numerical solution		Numerical solution		Numerical solution	
			$\alpha = \beta = 1$		$\alpha = 0.1, \beta = 1$	
	Fourth order	Solution in [4]	Fourth order	Solution in [4]	Fourth order	Solution in [7]
0.0	0.82848	0.82848	0.36752	0.36734	1.14704	1.14700
0.5	0.85906	0.85906	0.33841	0.33828	1.13375	1.13373
1.0	0.95095	0.95094	0.24793	0.24783	1.09325	1.09323

4. Numerical illustration

To illustrate the convergence and to corroborate the order of convergence of the method we consider the following singular two point boundary value problem

$$(x^{b_0} e^x y')' = 5e^x x^{b_0+3}(5x^5 e^y - (b_0 + 4) - x)/(4 + x^5)$$
$$y'(0) = 0, \qquad y(1) + 5y'(1) = \ln(1/5) - 5$$

with exact solution $y(x) = \ln(1/(4 + x^5))$. Maximum absolute errors and order of convergence (accuracy) for this problem have been displayed in Table 1 which show that the method works well for all finite values of b_0 and is of fourth order accuracy.

Two physiological problems are solved using this fourth order method. The first problem is an example of oxygen diffusion corresponding to (1) and (2) with $f(x, y)$ given in (3) and $p(x) = x^2$, $\theta = 0.76129$, $\kappa = 0.03119$, $\beta = 1$ and $\alpha = \gamma = 5$ and the results are displayed in Table 2 (First Problem).

The second problem is an example of a non-linear heat conduction model of the human head, which corresponds to (1) and (2) with $f(x, y)$ given in (4) and $p(x) = x^2$, $\delta = 1$, $\theta = 1$ and $\gamma = 0$. We have performed calculations for the following two cases

(i) $\alpha = \beta = 1$
(ii) $\alpha = 0.1, \beta = 1$

for comparison purposes and display the results in Table 2 (Second Problem). These results are in good agreement with those of [4,7].

Remark 2. For physiological problems the true solution is not available.

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Appendix

The coefficients $a_{i,j}$, $b_{i,j}$ used in the Eqs. (5)–(7) are given as follows:

$$b_{\pm1,1} = \frac{1}{2h}[\pm B_{10,1} + B_{20,1}/h \mp (B_{20,1} - A^{+}_{11,1}/J_1)Q'_1/Q_1],$$

$$b_{0,1} = B_{00,1} - B_{20,1}/h^2 + (-B_{10,1} + A^{+}_{01,1}/J_1)Q'_1/Q_1 + (B_{20,1} - A^{+}_{11,1}/J_1)(Q'_1/Q_1)^2 + (-B_{20,1} + A^{+}_{02,1}/J_1)Q''_1/(2Q_1),$$

$$b_{\pm1,k} = \frac{1}{2h}[\pm B_{10,k} + B_{20,k}/h \mp (B_{20,k} - B_{11,k})(Q'_k/Q_k)],$$

$$b_{0,k} = B_{00,k} - B_{20,k}/h^2 + (B_{01,k} - B_{10,k})(Q'_k/Q_k) + (B_{20,k} - B_{11,k})(Q'_k/Q_k)^2 + (B_{02,k} - B_{20,k})(Q''_k/(2Q_k))$$

$$a_{\pm1,N} = \mp\frac{3}{5hJ_{N-1}}\left[A^{-}_{10,N} + \frac{(11 \mp 5)}{6h}A^{-}_{20,N} + (-A^{-}_{20,N} + A^{-}_{11,N})Q'_N/Q_N\right],$$

$$a_{0,N} = \frac{1}{J_{N-1}}\left[A^{-}_{00,N} + \frac{11}{3h}A^{-}_{10,N} + \frac{8}{3h^2}A^{-}_{20,N} + \left(-A^{-}_{10,N} + A^{-}_{01,N} - \frac{11}{3h}A^{-}_{20,N} + \frac{11}{3h}A^{-}_{11,N}\right)Q'_N/Q_N \right.$$
$$\left. + (A^{-}_{20,N} - A^{-}_{11,N})(Q'_N/Q_N)^2 + (A^{-}_{02,N} - A^{-}_{20,N})Q''_N/(2Q_N)\right],$$

where

$$B_{m0,1} = (A^{+}_{m0,1}/J_1 + A^{-}_{m0,1}/Q_1), \quad m = 0\,(1)\,2;$$
$$B_{0m,k} = (A^{+}_{0m,k}/J_k + A^{-}_{0m,k}/J_{k-1}), \quad m = 0\,(1)\,2;$$
$$B_{1m,k} = (A^{+}_{1m,k}/J_k + A^{-}_{1m,k}/J_{k-1}), \quad m = 0, 1;$$
$$B_{20,k} = (A^{+}_{20,k}/J_k + A^{-}_{20,k}/J_{k-1}),$$

and

$$A^{\pm}_{00,k} = [-(x^2_{k\pm1} - x^2_k)/2 + \mu^{\pm}_{00,k}]/\phi\,(1)\,,$$
$$A^{\pm}_{10,k} = \left[\pm\frac{h^3}{6} - \frac{h^2}{2}x_{k\pm1} \pm \frac{h}{\psi\,(1)}x^2_{k\pm1} - \mu^{\pm}_{10,k}\right]\Big/\phi\,(1)\,,$$
$$A^{\pm}_{20,k} = \left[\frac{h^4}{12} \mp \frac{h^3}{3}x_{k\pm1} + \frac{h^2}{\psi\,(1)}x^2_{k\pm1} \mp \frac{2h}{\psi\,(2)}x^3_{k\pm1} + \mu^{\pm}_{20,k}\right]\Big/\phi\,(1)\,,$$
$$A^{\pm}_{01,k} = \left[\pm\frac{h^3}{6} - \frac{h^2}{2}x_{k\pm1} \pm h\mu^{\pm}_{00,k}\right]\Big/\phi\,(1) + \left[\frac{1}{3}(x^3_{k\pm1} - x^3_k) - \mu^{\pm}_{01,k}\right]\Big/\phi\,(2)\,,$$
$$A^{\pm}_{02,k} = \left[\frac{h^4}{12} \mp \frac{h^3}{3}x_{k\pm1} + h^2\mu^{\pm}_{00,k}\right]\Big/\phi\,(1) + \left[\frac{h^4}{6} \mp \frac{2h^3}{3}x_{k\pm1} + h^2x^2_{k\pm1} \mp 2h\mu^{\pm}_{01,k}\right]\Big/\phi\,(2) + \left[-\frac{1}{2}(x^4_{k\pm1} - x^4_k) + \mu^{\pm}_{02,k}\right]\Big/\phi\,(3)\,,$$
$$A^{\pm}_{11,k} = \left[\frac{h^4}{12} \mp \frac{h^3}{3}x_{k\pm1} + \frac{h^2}{\psi\,(1)}x^2_{k\pm1} \mp h\mu^{\pm}_{10,k}\right]\Big/\phi\,(1) + \left[\frac{h^4}{12} \mp \frac{h^3}{3}x_{k\pm1} + \frac{h^2}{2}x^2_{k\pm1} \mp \frac{h}{\psi\,(1)}x^3_{k\pm1} + \mu^{\pm}_{11,k}\right]\Big/\phi\,(2)\,,$$
$$A^{-}_{00,1} = x_1^{b_0+1}/\psi\,(1)\,, \qquad A^{-}_{10,1} = -x_1^{b_0+2}/\psi\,(2)\,, \qquad A^{-}_{20,1} = 2x_1^{b_0+3}/\psi\,(3)\,,$$

with

$$\mu^{\pm}_{ij,k} = i!j!x^{j+1-b_0}_{k\pm1}(x^{b_0+1+i}_{k\pm1} - x^{b_0+1+i}_k)/\psi\,(1+i),$$
$$\phi(i) = \prod_{j=1}^{i}(j - b_0) \quad \text{and} \quad \psi(i) = \prod_{j=1}^{i}(b_0 + j).$$

A.1. Coefficients in the case $b_0 = 1$

The difference scheme for $b_0 = 1$ can be obtained by taking limit $b_0 \to 1^+$ and the coefficients for this case are same as given above except for the following which are obtained by taking the limit $b_0 \to 1^+$.

$$A^{\pm}_{00,k} = \frac{1}{4}[(x^2_{k\pm1} - x^2_k) - 2x^2_k \ln(x_{k\pm1}/x_k)],$$
$$A^{\pm}_{10,k} = \frac{1}{36}[\pm h(4x^2_{k\pm1} - 5x_{k\pm1}x_k - 5x^2_k) + 6x^3_k \ln(x_{k\pm1}/x_k)],$$
$$A^{\pm}_{01,k} = \frac{1}{12}[\pm h(2x^2_{k\pm1} - x_{k\pm1}x_k - 7x^2_k) + 6x^3_k \ln(x_{k\pm1}/x_k)],$$
$$A^{\pm}_{20,k} = \frac{1}{72}\left[18h^2x^2_{k\pm1} \mp 20hx^3_{k\pm1} + \frac{13}{2}(x^4_{k\pm1} - x^4_k) - 6x^4_k \ln(x_{k\pm1}/x_k)\right],$$
$$A^{\pm}_{11,k} = \frac{1}{36}[-3h^4 \pm 12h^3x_{k\pm1} - h^2(12x^2_{k\pm1} + 4x_{k\pm1}x_k + 5x^2_k) \pm 6hx^3_{k\pm1} - 6x^4_k \ln(x_{k\pm1}/x_k)],$$
$$A^{\pm}_{02,k} = \frac{1}{24}[-4h^4 \pm 16h^3x_{k\pm1} - 24h^2x^2_{k\pm1} \pm 3h(5x^3_{k\pm1} - 3x^2_{k\pm1}x_k - 3x_{k\pm1}x^2_k + 5x^3_k) - 12x^4_k \ln(x_{k\pm1}/x_k)],$$
$$A^{-}_{00,1} = \frac{1}{2}x_1^2; \qquad A^{-}_{10,1} = -\frac{1}{6}x_1^3; \qquad A^{-}_{20,1} = \frac{1}{12}x_1^4.$$

A.2. Coefficients in the cases $b_0 = 2$ and $b_0 = 3$

In a similar fashion the difference scheme for $b_0 = 2$ can be obtained by taking the limit $b_0 \to 2$ and the coefficients different from the general case are as follows:

$$A^{\pm}_{01,k} = \frac{1}{18}[\mp 3h^3 + 9h^2 x_{k\pm1} \mp 6h(x^2_{k\pm1} - x^3_k/x_{k\pm1}) + 2(x^3_{k\pm1} - x^3_k) - 6x^3_k \ln(x_{k\pm1}/x_k)],$$

$$A^{\pm}_{11,k} = \frac{1}{72}\left[-6h^4 \pm 24h^3 x_{k\pm1} - 24h^2 x^2_{k\pm1} \pm 2h(7x^3_{k\pm1} - 3x^4_k/x_{k\pm1}) - \frac{7}{2}(x^4_{k\pm1} - x^4_k) + 6x^4_k \ln(x_{k+1}/x_k)\right],$$

$$A^{\pm}_{02,k} = \frac{1}{36}[-3h^4 \pm 12h^3 x_{k\pm1} - 12h^2(x^2_{k\pm1} - x^3_k/x_{k\pm1}) \pm 8h(x^3_{k\pm1} - 3x^3_k) - 2(x^4_{k\pm1} - x^4_k) + 24x^4_k \ln(x_{k\pm1}/x_k)].$$

Similarly for $b_0 = 3$ only the coefficient $A^{\pm}_{02,k}$ is different from the general case and is given as:

$$A^{\pm}_{02,k} = \frac{1}{48}[2h^4 \mp 8h^3 x_{k\pm1} + 6h^2(3x^2_{k\pm1} + x^4_k/x^2_{k\pm1}) \mp 12h(x^3_{k\pm1} - x^4_k/x_{k\pm1}) + 3(x^4_{k\pm1} - x^4_k) + 12x^4_k \ln(x_{k\pm1}/x_k)].$$

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