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On injective homomorphisms for pure braid groups, and associated Lie algebras

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Abstract

The purpose of this article is to record the center of the Lie algebra obtained from the descending central series of Artin's pure braid group, a Lie algebra analyzed in work of Kohno [T. Kohno, Linear representations of braid groups and classical Yang–Baxter equations, in: Contemp. Math., vol. 78, 1988, pp. 339–363; T. Kohno, Vassiliev invariants and the de Rham complex on the space of knots, in: Symplectic Geometry and Quantization, in: Contemp. Math., vol. 179, Amer. Math. Soc., Providence, RI, 1994, pp. 123–138; T. Kohno, Série de Poincaré–Koszul associée aux groupes de tresses pures, Invent. Math. 82 (1985) 57–75], and Falk and Randell [M. Falk, R. Randell, The lower central series of a fiber-type arrangement, Invent. Math. 82 (1985) 77–88]. The structure of this center gives a Lie algebraic criterion for testing whether a homomorphism out of the classical pure braid group is faithful which is analogous to a criterion used to test whether certain morphisms out of free groups are faithful [F.R. Cohen, J. Wu, On braid groups, free groups, and the loop space of the 2-sphere, in: Algebraic Topology: Categorical Decomposition Techniques, in: Progr. Math., vol. 215, Birkhäuser, Basel, 2003; Braid groups, free groups, and the loop space of the 2-sphere, math.AT/0409307]. However, it is as unclear whether this criterion for faithfulness can be applied to any open cases concerning representations of P_n such as the Gassner representation.

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1. Introduction

A classical construction due to Philip Hall dating back to 1933 gave a Lie algebra associated to any discrete group π [10] which is obtained from filtration quotients of the descending central series of π. That Lie algebra has admitted applications to the structure of certain discrete groups such as Burnside groups, as well as applications to problems in topology. The purpose of this article is to record some additional structure for this Lie algebra in case π is Artin's pure braid group P_n as described below.

That is, define the descending central series of a group π inductively by $\{\Gamma^k(\pi)\}_{k\geqslant 1}$ with

(1) $\Gamma^1(\pi)=\pi$,
(2) $\Gamma^k(\pi)$ is the subgroup generated by commutators $[\dots[\gamma_1,\gamma_2],\gamma_3],\dots],\gamma_t]$ for γ_i in π with $t\geqslant k$,
(3) $\Gamma^{k+1}(\pi)$ is a normal subgroup of $\Gamma^k(\pi)$,
(4) $E_0^k(\pi)=\Gamma^k(\pi)/\Gamma^{k+1}(\pi)$, and
(5) $E_0^*(\pi)=\bigoplus_{k\geqslant 1}\Gamma^k(\pi)/\Gamma^{k+1}(\pi)$.

There is a bilinear homomorphism

$$[-,-]: E_0^p(\pi)\otimes_{\mathbb{Z}} E_0^q(\pi)\to E_0^{p+q}(\pi)$$

induced by the commutator map (not in general a homomorphism) $c:\pi\times\pi\to\pi$. Natural properties of the map $[-,-]$ due to P. Hall, and E. Witt give $E_0^*(\pi)$ the structure of a Lie algebra which was developed much further in work of W. Magnus, M. Lazard, A.I. Kostrikin, E. Zelmanov, T. Kohno [12,13], M. Falk with R. Randell [9], D. Cohen [5], and others.

One standard notation for the Lie algebra attached to the descending central series is given by $gr_*(\pi)$. The notation $E_0^*(\pi)=gr_*(\pi)$ used below is adapted from the convention in [18] for the associated graded obtained from a decreasing filtration.

Let P_n denote the pure braid group on n strands with B_n the full braid group [4,17]. A choice of generators for P_n is $A_{i,j}$, $1\leqslant i<j\leqslant n$, subject to relations given in [17]. Choices of braids which represent the $A_{i,j}$ are given by a full twist of strand j around strand i. It is a classical fact using fibrations of Fadell and Neuwirth [8] that the choice of subgroup generated by $A_{i,n}$, $1\leqslant i\leqslant n-1$, denoted F_{n-1}, is free, and is the kernel of the homomorphism obtained by "deleting the last strand."

In the case of the pure braid group, the structure of the Lie algebra $E_0^*(P_n)$ is given in work of [12–14] and subsequently in [7,9]: This Lie algebra is generated by elements $B_{i,j}$ given by the classes of $A_{i,j}$ in $E_0^1(P_n)$, with $1\leqslant i<j\leqslant n$. Since $E_0^1(P_n)=H_1(P_n)$ is an abelian group, the sum of all of the $B_{i,j}$ given by

$$\Delta(n)=\sum_{1\leqslant i<j\leqslant n}B_{i,j}$$

is a well-defined element in $E_0^1(P_n)$. A complete set of relations for $E_0^*(P_n)$, the "infinitesimal braid relations," are listed in Section 3 here.

Properties required to state the main result are listed next. Consider the free group $F[S]$ generated by a set S with $L[S]$ the free Lie algebra over the integers $\mathbb{Z}$ generated by the set S. A classical fact due to P. Hall [10,19] is that the morphism of Lie algebras $e:L[S]\to E_0^*(F[S])$

which sends an element s in S to its equivalence class in $E_0^1(F[S]) = H_1(F[S])$ is an isomorphism of Lie algebras.

Restrict to the subgroup F_{n-1} the free group generated by $A_{i,n}$ for $1 \leqslant i < n$. Let $L[V_n]$ denote the free Lie algebra generated by $B_{i,n}$ with $1 \leqslant i < n$. Thus there is a morphism of Lie algebras

$$\Theta_n : L[V_n] \to E_0^*(P_n)$$

which sends $B_{i,n}$ to the class of $A_{i,n}$ in $E_0^*(F_{n-1})$. One feature of $E_0^*(P_n)$ is that Θ_n is an isomorphism onto its image [9,14]. From now on, $L[V_n]$ is identified with its image in $E_0^*(P_n)$.

Let $\mathcal{L}$ denote a Lie algebra with Lie ideal $\mathcal{W}$. The *centralizer* of $\mathcal{W}$ in $\mathcal{L}$ is defined by the equation

$$C_{\mathcal{L}}(\mathcal{W}) = \{x \in \mathcal{L} \mid [x, B] = 0, \text{ for all } B \in \mathcal{W}\}.$$

Theorem 1.1. *If $n > 2$,*

$$C_{E_0^*(P_n)}(L[V_n]) = C_{E_0^*(P_n)}(E_0^*(F_{n-1})) = L[\Delta(n)].$$

Remark 1.2. It is quite possible that Theorem 1.1 appears in the earlier work concerning the Lie algebra $E_0^*(P_n)$. The authors are unaware of a reference.

A direct corollary is stated next. Recall the classical construction of the adjoint representation

$$ad : L \to Der_*^{Lie}(L)$$

of a graded Lie algebra L for which $Der_*^{Lie}(L)$ denotes the graded Lie algebra of graded derivations of L. The map ad is defined by the equation $ad(x)(y) = [x, y]$ for x, and y in L. Regard $E_0^*(P_n)$ as a graded Lie algebra by the convention that $E_0^q(P_n)$ has degree $2q$. Restriction to the Lie ideal $L[V_n]$ gives an induced morphism of Lie algebras $ad|_{L[V_n]} : E_0^*(P_n) \to Der_*^{Lie}(L[V_n])$ defined by $ad|_{L[V_n]}(x)(y) = [x, y]$.

Corollary 1.3. *The kernel of the adjoint representation $ad : E_0^*(P_n) \to Der_*^{Lie}(E_0^*(P_n))$ as well as the kernel of the restriction of the adjoint representation $ad|_{L[V_n]} : E_0^*(P_n) \to Der_*^{Lie}(L[V_n])$ is given by the cyclic group generated by $\Delta(n)$ in $E_0^1(P_n)$. Thus there is a short exact sequence of Lie algebras*

$$0 \longrightarrow L[\Delta(n)] \longrightarrow E_0^*(P_n) \xrightarrow{ad|_{L[V_n]}} Image(ad|_{L[V_n]}) \longrightarrow 0.$$

Three remarks are given next:

(1) Theorem 1.1 provides a setting for testing whether certain group homomorphisms $f : P_n \to G$ are faithful by testing whether induced morphisms on the level of Lie algebras $E_0^*(f) : E_0^*(P_n) \to E_0^*(G)$ are faithful. A variation of this setting for homomorphisms $g : F_n \to G$ where F_n is a free group has been applied in one case related to the homotopy groups of the 2-sphere [1,6]. The utility of this setting is that working with free Lie algebras is sometimes more efficient than working directly with free groups.

(2) It is natural to ask whether Theorem 1.1 can be applied to various well-known representations such as the Gassner representation, the Burau representation for B_4 [4], or the Lawrence–Krammer representation [3,15]. It is also natural to ask whether there are similar structure theorems for representations of other related discrete groups such as those in [5], or variations which test whether a representation is both faithful as well as discrete. The authors have attempted to use the above Lie algebraic methods above to test whether the classical Burau representation $\beta_4 : B_4 \to GL(4, \mathbb{Z}[t^{\pm 1}])$ is faithful [4] by checking whether the restriction to P_4, $\beta_4|_{P_4}$, is faithful. There is an induced morphism of Lie algebras $E_0^*(\beta_4|_{P_4}) : E_0^*(P_4) \to E_0^*(\beta_4(P_4))$ obtained from the associated graded Lie algebras for the descending central series of both P_4, and $\beta_4(P_4)$. Since P_n is residually nilpotent [9], it follows that if the induced morphism of graded Lie algebras $E_0^*(\beta_4|_{P_4})$ is a monomorphism, then $\beta_4|_{P_4}$ and hence β_4 are faithful. Although there is substantial positive computer-based evidence that $E_0^*(\beta_4|_{P_4})$ is a monomorphism, the authors have been unable to verify this last statement in general.

(3) D.D. Long [16] has proven the beautiful theorem which states that if $\phi : B_n \to G$ is a morphism whose restriction to the free group F_{n-1} and to the center is injective, then ϕ is injective. Thus it has long been recognized that proving injections on the level of this free group is important. The content of Theorem 1.1 is a possibly more efficient way to attempt to verify such properties.

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2. On the Lie algebra for the pure braid group

The structure of $E_0^*(P_n)$ is given in [7,9,12–14]. Recall that $L[S]$ denotes the free Lie algebra over $\mathbb{Z}$ generated by a set S. Then $E_0^*(P_n)$ is the quotient of the free Lie algebra generated by $B_{i,j}$ for $1 \leqslant i < j \leqslant n$ modulo the *infinitesimal braid relations* (or horizontal 4T relations or Yang–Baxter–Lie relations)

$$E_0^*(P_n) = L[B_{i,j} \mid 1 \leqslant i < j \leqslant n]/I,$$

where I denotes the 2-sided (Lie) ideal generated by the infinitesimal braid relations as listed next:

(1) $[B_{i,j}, B_{s,t}] = 0$, if $\{i, j\} \cap \{s, t\} = \emptyset$.
(2) $[B_{i,j}, B_{i,s} + B_{s,j}] = 0$.
(3) $[B_{i,j}, B_{i,t} + B_{j,t}] = 0$.
(4) It follows from 2, and 3 above that $[B_{j,s}, B_{i,j} + B_{i,s}] = 0$.

In addition, it is convenient to introduce new generators $B_{j,i}$ for $i < j$ with the convention that

$$B_{j,i} = B_{i,j}, \quad \text{for } i < j.$$

Consider the abelianization homomorphism

$$P_n \to P_n/[P_n, P_n] = E_0^1(P_n) = H_1(P_n)$$

for which the image of $A_{i,j}$ is denoted $B_{i,j}$. The first homology group $H_1(P_n)$ is isomorphic to $\bigoplus_{(n-1)n/2}\mathbb{Z}$ with basis given by the $B_{i,j}$, for $1\leqslant i<j\leqslant n$.

Furthermore, there is an induced split short exact sequence of Lie algebras

$$0\to E_0^*(F_{n-1})\to E_0^*(P_n)\to E_0^*(P_{n-1})\to 0.$$

Thus for each $i>0$, there is a split short exact sequence of abelian groups

$$0\to E_0^i(F_{n-1})\to E_0^i(P_n)\to E_0^i(P_{n-1})\to 0,$$

and $E_0^i(P_n)$ is isomorphic, as an abelian group, to $\bigoplus_{1\leqslant j\leqslant n-1} E_0^i(F_j)$.

The structure of the Lie algebra $E_0^*(P_n)$ is given in more detail next via [9,12–14]. Let $L[V_q]$ denote the free Lie algebra (over $\mathbb{Z}$) generated by the set V_q with

$$V_q=\{B_{1,q},B_{2,q},\dots,B_{q-1,q}\},\quad \text{for } 2\leqslant q\leqslant n.$$

Furthermore, there are morphisms of Lie algebras

$$\Theta_q: L[V_q]\to E_0^*(P_n)\quad \text{given by } \Theta_q(B_{j,q})=B_{j,q}$$

for $1\leqslant j<q$ such that the additive extension of the Θ_q to

$$\Theta: L[V_2]\oplus L[V_3]\oplus\cdots\oplus L[V_n]\to E_0^*(P_n)$$

is an isomorphism of graded abelian groups. That is if $a_j(q)$ is an element of $E_0^q(F_{j-1})$ with $E_0^*(F_{j-1})=L[V_j]$ for $2\leqslant j\leqslant n$ and

$$x(q)=a_2(q)+a_3(q)+\cdots+a_n(q),$$

then

$$\Theta\big(x(q)\big)=\Theta_2\big(a_2(q)\big)+\Theta_3\big(a_3(q)\big)+\cdots+\Theta_n\big(a_n(q)\big).$$

The elements $a_j(q)$ will be identified below with the image $\Theta_j(a_j(q))$ unless otherwise noted. The isomorphism of graded abelian groups Θ is not an isomorphism of Lie algebras, but restricts to a morphism of Lie algebras $\Theta_q: L[V_q]\to E_0^*(P_n)$ for each $q\geqslant 2$. The infinitesimal braid relations give the "twisted" underlying Lie algebra structure of $E_0^*(P_n)$.

Lemma 2.1. *If $i,j,s<n$, then,*

$$[B_{i,j},B_{s,n}]\in L[B_{1,n},B_{2,n},\dots,B_{n-1,n}]=E_0^*(F_{n-1}).$$

Therefore, for each $X\in E_0^*(P_n)$, $[X,B_{s,n}]\in E_0^*(F_{n-1})$, and $E_0^*(F_{n-1})$ is a Lie ideal of $E_0^*(P_n)$.

Proof. This follows immediately from the infinitesimal braid relations. □

Centralizers in a free Lie algebra are the subject of the following exercise from "Groupes et algèbres de Lie, chapitres 2–3" [2, exercice 3, chapitre 2, section 3].

Lemma 2.2. *Let $L[S]$ be the free Lie algebra over the integers $\mathbb{Z}$ generated by a set S, and let a be an element of S with S of cardinality at least 2. Then the centralizer of a in $L[S]$ is the linear span of a.*

Proof. Let A_S denote the free abelian group generated by S with $a \in S$, and S of cardinality at least 2. The universal enveloping algebra of $L[S]$ is the tensor algebra $T[A_S]$ while the standard Lie algebra homomorphism

$$j : L[S] \to T[A_S]$$

is injective by the Poincaré–Birkhoff–Witt theorem ([2], and [11, p. 168]). Identify the elements of $L[S]$ with their images in $T[A_S]$. Thus if $x \in L[S]$ centralizes a, then a commutes with all x in $T[A_S]$.

Consider an element x of $(A_S)^{\otimes n}$ such that $a \otimes x = x \otimes a$. Notice that $x = a \otimes x'$ for some element x' in $(A_S)^{\otimes n-1}$. Thus by induction on n, x is a scalar multiple of $a^{\otimes n}$, and so x is in the subalgebra generated by a. The intersection of $L[S]$ with the subalgebra generated by a is precisely the linear span of a, thus proving the lemma. □

The proof of Theorem 1.1 is given next.

Proof. There are two parts to this proof. The first part is to show that the non-zero homogeneous elements of degree q in $C_{E_0^*(P_n)}(L[V_n])$ are concentrated in degree $q = 1$. The second part of the proof is to show that the homogeneous elements of degree 1 in $C_{E_0^*(P_n)}(L[V_n])$ are precisely scalar multiples of $\Delta(n)$.

As described above, a restatement of results of Kohno [12–14], and Falk and Randell [9] is that there is a splitting of $E_0^i(P_n)$ as an abelian group, for each $i > 0$:

$$E_0^i(P_n) = E_0^i\big(L[V_2]\big) \oplus E_0^i\big(L[V_3]\big) \oplus \cdots \oplus E_0^i\big(L[V_n]\big),$$

where, for each $1 < m \leqslant n$, V_m is the linear span of the set $\{B_{1,m}, B_{2,m}, \ldots, B_{m-1,m}\}$.

Let $x(q)$ denote an element in $E_0^q(P_n)$. Thus $x(q)$ is a linear combination given by $x(q) = a_2(q) + a_3(q) + \cdots + a_n(q)$, $a_j(q) \in E_0^*(L[V_j])$ for which all $a_j(q)$ have the same degree q.

Assume that $x(q)$ is in the centralizer of $L[V_n]$. Thus

$$\big[x(q), \Gamma\big] = 0, \quad \text{for all } \Gamma \in L[V_n].$$

It will be shown below by downward induction on j that if $q > 1$, then $a_j(q) = 0$.

The first case to be checked is that the "top component" $a_n(q)$ vanishes for $q > 1$. Assume that $q > 1$. Let $\mathcal{B}(n) = B_{1,n} + B_{2,n} + \cdots + B_{n-1,n}$. The infinitesimal braid relations

$$[B_{i,j}, B_{s,t}] = 0, \quad \text{if } \{i, j\} \cap \{s, t\} = \emptyset$$

and

$$[B_{i,n} + B_{j,n}, B_{i,j}] = 0$$

imply that, for $j < n$, $[a_j(q), \mathcal{B}(n)] = 0$. It follows that

$$\bigl[x(q), \mathcal{B}(n)\bigr] = \bigl[a_n(q), \mathcal{B}(n)\bigr] = 0.$$

Thus $a_n(q)$ belongs to the centralizer of the element $\mathcal{B}(n)$ and both are in $L[V_n]$, which is a free Lie algebra.

By a direct change of basis, there is an equality

$$L[V_n] = L\bigl[\mathcal{B}(n), B_{2,n}, \ldots, B_{n-1,n}\bigr].$$

In addition, notice that Lemma 2.2 implies that $a_n(q)$ is a scalar multiple of $\mathcal{B}(n)$ contradicting the assumption that $q > 1$, and $n > 2$.

Recall that

(1) P_n is a normal subgroup of B_n,
(2) there is an isomorphism $q : B_n/P_n \to \Sigma_n$ where Σ_n is the symmetric group on n-letters,
(3) B_n acts on P_n by conjugation, with
(4) the induced action on $P_n/[P_n, P_n]$ factoring through the natural quotient map $q : B_n \to \Sigma_n$.

Thus there is an induced action of Σ_n on the Lie algebra $E_0^*(P_n)$. Note that this action does not preserve the top free Lie algebra: If σ is an element in Σ_n, then

$$\sigma(B_{i,j}) = B_{\sigma(i),\sigma(j)} = B_{\sigma(j),\sigma(i)}.$$

By downward induction, assume that

$$a_{s+1}(q) = a_{s+2}(q) = \cdots = a_n(q) = 0.$$

Thus $x(q) = a_2(q) + a_3(q) + \cdots + a_s(q)$ for $s < n$. Then

$$0 = \bigl[x(q), B_{s,n}\bigr] = \bigl[a_s(q), B_{s,n}\bigr]$$

as $x(q)$ is assumed to be in the centralizer of $L[V_n]$, and $[a_i(q), B_{s,n}] = 0$ for $i < s$ by the infinitesimal braid relations.

Let τ_s denote the element in Σ_n which interchanges s, and n leaving the other points fixed. Regard τ_s as a Lie algebra automorphism applied to the previous equation to obtain

$$0 = \bigl[\tau_s\bigl(a_s(q)\bigr), \tau_s(B_{s,n})\bigr] = \bigl[\tau_s\bigl(a_s(q)\bigr), B_{s,n}\bigr].$$

Observe that $\tau_s(a_s(q))$ is an element of $L[V_n]$ as $s < n$, and $\tau_s(a_s(q))$ commutes with $B_{s,n}$ with $q > 1$. Hence $\tau_s(a_s(q)) = 0$ by Lemma 2.2. Thus, $a_s(q) = 0$ as τ_s is an automorphism of Lie algebras.

The second part of the proof is an inspection of the homogeneous elements of degree 1 in $C_{E_0^*(P_n)}(L[V_n])$, and consists of showing that these are precisely scalar multiples of $\Delta(n)$ as is given next. Consider the element $x(1) = a_2(1) + a_3(1) + \cdots + a_n(1)$ in $C_{E_0^1(P_n)}(L[V_n])$. Then

$$x(1) = \sum_{1 \leqslant i < j \leqslant n} \alpha_{i,j} B_{i,j},$$

where $a_m(1) = \sum_{1 \leqslant i < m} \alpha_{i,m} B_{i,m}$ for some choice of integers $\alpha_{i,j}$. Furthermore,

$$\big[x(1), B_{p,n}\big] = 0$$

for every $1 \leqslant p < n$ as $x(1)$ is in $C_{E_0^1(P_n)}(L[V_n])$. It will be checked below that $\alpha_{i,j} = \alpha_{s,t}$ for all $i < j$, and $s < t$, thus showing that $x(1)$ is a scalar multiple of $\Delta(n)$.

Notice that $[x(1), B_{p,n}]$ is equal to

$$\begin{aligned} &\sum_{i \neq p,n} \alpha_{i,p}[B_{i,p}, B_{p,n}] + \sum_{i \neq p,n} \alpha_{i,n}[B_{i,n}, B_{p,n}] \\ &\quad = \sum_{i \neq p,n} (-\alpha_{i,p})[B_{i,n}, B_{p,n}] + \sum_{i \neq p,n} \alpha_{i,n}[B_{i,n}, B_{p,n}] \end{aligned}$$

by the infinitesimal braid relations, and the convention that $B_{i,j} = B_{j,i}$ for $i < j$. It follows that $\alpha_{i,n} = \alpha_{i,p}$ as $[B_{i,n}, B_{j,n}]$ for $i < j$ form a basis for the homogeneous elements of degree 2 in $L[V_n]$. A similar computation of $[x(1), B_{p,j}]$ gives $\alpha_{j,n} = \alpha_{p,j}$ for $p < j < n$.

Thus any element $x(1)$ in the centralizer of $L[V_n]$ is a scalar multiple of the element $\Delta(n)$. That $\Delta(n)$ centralizes $E_0^*(P_n)$ follows by inspection. Thus the centralizer of $L[V_n]$ is given by $L[\Delta(n)]$, the free Lie algebra generated by a single element $\Delta(n)$, a copy of $\mathbb{Z}$ in degree 1, and Theorem 1.1 follows. □

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