On the Stochastic Korteweg–de Vries Equation

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We consider a stochastic Korteweg-de Vries equation forced by a random term of white noise type. This can be a model of water waves on a fluid submitted to a random pressure. We prove existence and uniqueness of solutions in $H^1(R)$ in the case of additive noise and existence of martingales solutions in $L^2(R)$ in the case of multiplicative noise.

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1. INTRODUCTION

The motion of long, unidirectional, weakly nonlinear water waves on a channel can be described, as is well known, by the Korteweg–de Vries (KdV) equation [20]. When the surface of the fluid is submitted to a non constant pressure, or when the bottom of the layer is not flat, a forcing term has to be added to the equation [1, 23, 29]. This term is given by the gradient of the exterior pressure or of the function whose graph defines the bottom.

In this paper, we are interested in the case when the forcing term is random, which is a very natural approach if it is assumed that the exterior pressure is generated by a turbulent velocity field for instance. We also assume that this random force is of white noise type. Then, we are lead to study the following stochastic partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u \frac{\partial u}{\partial x} = f + \Phi(u) \frac{\partial^2 B}{\partial t \partial x},$$

(1.1)
where \( u \) is a random process defined on \((x, t) \in \mathbb{R} \times \mathbb{R}^+\), \( f \) is a deterministic forcing term, \( \Phi(u) \) is a linear operator depending on \( u \) and \( B \) is a two parameter brownian motion on \( \mathbb{R} \times \mathbb{R}^+ \). We will be interested in both cases of additive noise (when \( \Phi(u) \equiv 0 \) does not depend on \( u \)) and of multiplicative noise.

We recall that \( B(x, t), \ t \geq 0, \ x \in \mathbb{R} \), is a zero mean gaussian process whose covariance function is given by

\[
E(B(t, x) B(s, y)) = (t \land s)(x \land y),
\]

for \( t, s \geq 0, \ x, y \in \mathbb{R} \). Alternatively, we can consider a cylindrical Wiener process on \( L^2(\mathbb{R}) \) by setting

\[
W(t) = \frac{\partial B}{\partial x} = \sum_{i=1}^{\infty} \beta_i e_i,
\]

where \((e_i)\) is an orthonormal basis of \( L^2(\mathbb{R}) \) and \((\beta_i)\) is a sequence of mutually independent real brownian motions in a fixed probability space (see [10]).

We shall write (1.1) in the Itô form:

\[
du + \left( \frac{\partial^3 u}{\partial x^3} + u \frac{\partial u}{\partial x} \right) dt = f dt + \Phi(u) dW. \tag{1.2}
\]

This is supplemented with an initial condition

\[
u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \tag{1.3}
\]

Our aim in this paper is to study the existence and uniqueness of solutions to (1.2), (1.3). Before describing our work, we recall some facts about the theory on the deterministic equation. The first results were obtained in the framework of the usual Sobolev spaces \( H^s(\mathbb{R}) \). In these spaces, the linear part of the KdV equation generates a unitary group \((S(t))_{t \in \mathbb{R}}\). Using the well known invariant quantities of this equation, global existence was proved for initial data in \( H^s(\mathbb{R}) \), \( s \geq 1 \), while uniqueness results were restricted to \( s > 3/2 \) (see [6, 7, 16, 26, 27]). Also, T. Kato [15] discovered that \((S(t))_{t \in \mathbb{R}}\) possesses a local smoothing effect, namely if \( u_0 \in L^2(\mathbb{R}) \), then for \( t \in [0, T] \), \( S(t) u_0 \), is in \( L^2([0, T], H^1_{loc}(\mathbb{R})) \), and this enabled him to prove global existence in \( L^2(\mathbb{R}) \). Later, many other smoothing properties were discovered, and C. Kenig, G. Ponce and L. Vega [17] have been able to prove local existence and uniqueness for initial data in \( H^s(\mathbb{R}) \), \( s > 3/4 \) and global existence for \( s \geq 1 \). More recently, by introducing new classes of Sobolev type spaces, J. Bourgain [9] has been able to prove well posedness
in $L^2(\mathbb{R})$ and his approach has been generalized in [18, 19] to obtain local existence and uniqueness in $H^\sigma(\mathbb{R}), \sigma > -3/4$.

We will first consider the case of additive noise, $\Phi(u) \equiv \Phi$. In this case, when $\Phi$ defines a Hilbert-Schmidt operator in $H^4(\mathbb{R})$, we are able to generalize the approach of [17] and to prove global existence and uniqueness of solutions of (1.2), (1.3). We use a fixed point argument in the space $X_\sigma(T) = \{ u \in C(0, T; H^\sigma(\mathbb{R})) \cap L^2([0, T]) \},$

$$D^\nu \partial_x u \in L^\sigma(\mathbb{R}, L^2([0, T])), \partial_x u \in L^4([0, T]; L^\infty(\mathbb{R})), $$

for some $\sigma < 1$. Then, using a priori estimates, we prove global existence in $\Phi(0, T; H^4(\mathbb{R}))$. The main difficulty is to prove that the solution of the linear equation

$$\begin{cases}
\partial_t u + \partial_x^3 u + \partial_x u = \partial_x W, \\
\partial_t u(0) = 0
\end{cases}$$

is in $X_{\sigma}(T)$. For instance, we have to estimate the quantity

$$E \left( \sup_{t \in [0, T]} \left| \int_0^t S(t-s) \Phi dW(s) \right|^2 \right).$$

Several difficulties arise from this unusual type of norm in which the time variable $t$ and space variable $x$ are not in the usual order. If we use results from the deterministic theory, we are able to estimate quite easily. This is clearly not sufficient. The problem is due to the supremum inside the expectation. We overcome this difficulty by using the Sobolev embedding $W^p([0, T]) \subset L^\infty([0, T]),$ if $xp > 1$, then we can generalize the deterministic proof which uses Fourier analysis. Similar difficulties are encountered when we estimate norms involving a supremum in the space variable. Here we had to use the same type of argument as above together with interpolation theory.
It should be pointed out that here, Bourgain’s approach does not seem to apply. In [8], it was used to prove existence and uniqueness of solutions for the KdV equation forced by a term in a negative Sobolev space (in space variable). However, this forcing term cannot be irregular in time and this is typically the case for stochastic equations. Roughly speaking, in this method, the lack of smoothness in space is compensated by some smoothness in time through a clever choice of the function space in which the solution is sought. It can be checked through a direct computation that $u$, the solution of the linear stochastic KdV equation is not smooth enough (in time) and is not in that space.

In the third section, we consider the case of a multiplicative noise. We assume that the mapping $\Phi(\cdot)$ takes values in the space of Hilbert–Schmidt operators in $L^2(\mathbb{R})$ and we want solutions in $L^3(\mathbb{R})$. Since Bourgain’s method does not seem to be useful here, we have very little hope to get uniqueness of solutions. We only consider martingale solutions (or weak solutions in the stochastic differential equations language). We use Kato’s smoothing property and a recent method developed by F. Flandoli and D. Gatarek [11] to construct martingale solutions to stochastic partial differential equations. This method is a stochastic generalization of compactness methods for deterministic partial differential equations.

Although we restricted our attention to the KdV equation, the methods we have used in this paper are general and we believe that similar ideas can be applied to the generalized KdV equation or to the Benjamin–Ono equation.

We will treat the case $f \equiv 0$, in (1.2); all the results can easily be extended if $f \in L^1(0, T; H^1(\mathbb{R}))$ in Section 3, and if $f \in L^\infty(0, T; L^2(\mathbb{R}))$ in Section 4.

To our knowledge, only the case of the simplest stochastic KdV equations where the noise is the derivative of a single brownian motion in time (i.e.: white noise in time) and does not depend on the space variable has been studied [2, 28]. In this case, it is possible to use the inverse scattering method and to get informations on the diffusion of a soliton by the external noise. A similar work has been done on the Benjamin–Ono equation forced by the same kind of noise in [24].

2. NOTATIONS

When $X$ is a Banach space and $I$ an interval in $\mathbb{R}$, $L^p(I; X)$, $1 < p < \infty$, denotes the space of functions with integrable $p$th power on $I$, with values in $X$. In the case of $X = \mathbb{R}$, we simply write $L^p(I)$. The power $p = 2$ plays a central role, the norm on $L^2(\mathbb{R})$ will be denoted by $|\cdot|$ and the inner product by $(\cdot, \cdot)$. Also, $C([0, T]; X)$ (resp. $C^\beta([0, T]; X)$) is the space of continuous (resp. Hölder continuous with exponent $\beta$) functions from
\[ [0, T] \] into \( X \). The Sobolev space \( \mathcal{H}^\sigma(\mathbb{R}; X) \) is the space of those functions \( u \) such that
\[
|u|^2_{\mathcal{H}^\sigma(\mathbb{R}; X)} = \int_\mathbb{R} (1 + |\xi|^2)^{\sigma/2} |\hat{u}(\xi)|^2 \, d\xi < \infty
\]
where \( \mathcal{F} = \hat{\cdot} \) denotes the Fourier transform. The linear operators \( J_u, D, \mathcal{H} \) are defined by
\[
J_u u = \mathcal{F}^{-1}((1 + |\xi|^2)^{\sigma/2} \hat{u}(\xi)), \\
Du = \mathcal{F}^{-1}(|\xi| \hat{u}(\xi)), \\
\mathcal{H} u = \mathcal{F}^{-1}\left( \frac{\zeta}{|\zeta|} \hat{u}(\xi) \right),
\]
\( \mathcal{H} \) is the Hilbert transform.

Note that the solution of the linear KdV equation
\[
\begin{cases}
\frac{\partial u}{\partial t} + 3 \frac{\partial^3 u}{\partial x^3} = 0, & (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\
u(x, 0) = u_0(x), & x \in \mathbb{R}
\end{cases}
\]
when \( u_0 \in L^2(\mathbb{R}) \), is given by
\[ u(t) = S(t) u_0 = \mathcal{F}^{-1}(e^{it\partial_x^3} \hat{u}_0(\xi)). \]
We will also use the Sobolev space \( W^\infty(\mathbb{R}; X) \), with \( \alpha > 0 \) and \( 1 < p < +\infty \), which is the space of functions \( u \in L^\infty([0, T]; X) \) such that
\[
\left( \int_0^T \int_0^T \frac{|u(t) - u(s)|_X^p}{|t-s|^{1+\alpha}} \, dt \, ds \right)^{1/p} < \infty.
\]
When considering Sobolev spaces in the space variable \( x \), we will use the characterization
\[
W^\infty(\mathbb{R}; X) = \{ u \in \mathcal{S}'(\mathbb{R}; X), \mathcal{F}^{-1}((1 + |\xi|^2)^{\sigma/2} \mathcal{F} u) \in L^\infty(\mathbb{R}, X) \}
\]
for \( \sigma \in \mathbb{R} \) and \( 1 \leq p \leq +\infty \). Note that we use the same notation for both types of Sobolev spaces, although the characterizations are not equivalent. This causes no confusion, since one of them is used only in time variable, and the other one in space variable.

In Section 3, when the intervals where the different variables leave are specified, we will use shorter notations. For instance, if \( x \in \mathbb{R} \) and \( t \in [0, T] \), \( L^\infty([0, T]; L^p(\mathbb{R})) \) is the usual space \( L^\infty([0, T]; L^p(\mathbb{R})) \). Similarly, \( L^\infty([0, T]; L^p(\mathbb{R})) = L^\infty(\mathbb{R}; L^p([0, T])); \) (\( 0 \leq T < +\infty \)). In the appendix, we also use the space \( \text{BMO}(A) \)
of measurable functions $u$ of $x \in \mathbb{R}$ with values in a Banach space $A$, such that

$$|u|_{BMO(A)} \equiv \sup_{x \in \mathbb{R}} \frac{1}{2d} \int_{[-a,a]} |u(x) - m_x(u)|_A dx < +\infty$$

where $m_x(u) = \frac{1}{2a} \int_{[-a,a]} u(x) \, dx$.

We will work with stochastic processes; $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in \mathbb{R}})$ is a stochastic basis if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $(\mathcal{F}_t)_{t \in \mathbb{R}}$ a filtration and $(\mathcal{W}(t))_{t \in \mathbb{R}}$ a cylindrical Wiener process on $L^2(\mathbb{R})$ associated to this filtration. We define similarly as above the spaces $L^p(\mathbb{R}, X)$, and we will use notations like $L^p_0(L^q(X), H)$, $L^p_t(L^q_0(X), H)$, ...

Given $H$ a Hilbert space, we denote by $L^0_0(L^2(\mathbb{R}), H)$ the space of Hilbert-Schmidt operators from $L^2(\mathbb{R})$ into $H$. Its norm is given by

$$\|A\|_{L^0_0(L^2(\mathbb{R}), H)} = \sqrt{\sum_{i \in \mathbb{N}} \|\phi_i\|^2_H}$$

where $(\phi_i)_{i \in \mathbb{N}}$ is any orthonormal basis of $L^2(\mathbb{R})$. When $H = H^*(\mathbb{R})$, we will write

$$L^0_0(L^2(\mathbb{R}), H^*(\mathbb{R})) = L^0_2.$$

In Section 4, we will use the local $L^2$ and $H^\sigma$ spaces. The space $L^p_0(\mathbb{R})$ is the space of real valued functions $u$ such that for any compact interval $[a, b]$, it is in $L^p([a, b])$. It is a Fréchet space when endowed with the family of seminorms

$$p_k(u) = \left( \int_{-k}^k |u|^2(x) \, dx \right)^{1/2}, \quad k \in \mathbb{N}.$$  

Also, when $\sigma$ is a positive integer, for any interval $[a, b]$, $H^\sigma([a, b])$ is the space of functions whose derivatives up to order $\sigma$ are in $L^2([a, b])$, $H^\sigma_0([a, b])$ is the closure of $\mathcal{D}([a, b])$ in $H^\sigma([a, b])$, and $H^{-\sigma}([a, b])$ is the dual space of $H^\sigma_0([a, b])$. If we identify functions of $H^\sigma_0([a, b])$ with their extensions by zero outside $[a, b]$, we have the embedding

$$H^\sigma_0([a, b]) \subset H^\sigma_0([c, d]),$$

$$H^{-\sigma}([c, d]) \subset H^{-\sigma}([a, b])$$

when $[a, b] \subset [c, d]$.

Also, $H^\sigma_0(\mathbb{R})$ consists of the functions whose restrictions to $[a, b]$ are in $H^\sigma([a, b])$ for any $[a, b]$, it is a Fréchet space for the seminorms

$$p_k^\sigma(u) = |u|_{H^\sigma([-k, k])}, \quad k \in \mathbb{N}^*.$$
and

\[ H_{\text{loc}}^{-\sigma} = \bigcap_{k \geq 1} H^{-\sigma}_{k}\{[-k, k]\} \]

is also a Fréchet space for the seminorms

\[ p_k^{\sigma}(u) = |u|_{H^{-\sigma}_{[-k, k]}} \], \quad k \in \mathbb{N}^*. \]

We will use the following lemma whose proof relies on a classical compact embedding theorem (see [22]), Ascoli-Arzelà theorem and diagonal extraction.

**Lemma 2.1.** Let \( \sigma > 0, \beta > 0, T \geq 0 \) and a sequence \((\epsilon_k)_{k \in \mathbb{N}^*}\) of positive numbers. The set \( A(\{\epsilon_k\}) \) of functions \( u \in \Theta([0, T]; H_{\text{loc}}^{-\sigma}(\mathbb{R})) \cap L^2([0, T]; L^2_{\text{loc}}(\mathbb{R})) \) such that for any \( k \geq 1 \), \( |u|_{L^2([0, T]; H^\sigma([-k, k]))}^2 + |u|_{L^2([0, T]; H^\sigma([-k, k]))}^2 + |u|_{L^2([0, T]; \mathcal{C}([-\infty, k]))}^2 \leq \epsilon_k \) is compact in \( \Theta([0, T]; H_{\text{loc}}^{-\sigma}(\mathbb{R})) \cap L^2([0, T]; L^2_{\text{loc}}(\mathbb{R})) \).

In all the article, \( C, C_i, i \in \mathbb{N}, K, \ldots \) will denote various constants. The same symbol will be used for different constants. When it is important to precise that a constant depends on a parameter, we will use the notation \( C(\cdot), C_i(\cdot) \ldots \). For instance, \( C(T, m) \) is a constant depending on \( T \) and \( m \) but not on the other parameters or variables.

In Section 4, we will use Itô formula although its use is not rigorously justified due to some lack of regularity (in the proof of Lemma 4.2 for instance). However, the results are correct and could easily be justified by using some regularization.

### 3. ADDITIVE NOISE

In this section, we study the stochastic KdV equation with an additive noise:

\[
du + \left( \partial_x^3 u + u \partial_x u \right) dt = \Phi dW \tag{3.1}
\]

for \( x \in \mathbb{R}, t \geq 0 \). It is supplemented with an initial condition:

\[
u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \tag{3.2}\]
Here, $W$ is a cylindrical Wiener process on $L^2(\mathbb{R})$ adapted to a filtration $(\mathcal{F}_t)_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, P)$; $\Phi$ is a linear mapping and is assumed to be Hilbert–Schmidt from $L^2(\mathbb{R})$ into $H^\sigma(\mathbb{R})$, $\sigma \geq 0$,

$$|\Phi|_{L^2_{\mathcal{F}}} = |\Phi|_{L^2_{\mathcal{F}}(L^2(\mathbb{R}), H^\sigma(\mathbb{R}))} < \infty;$$

(3.3)

$u_0$ is $\mathcal{F}_0$-measurable.

We will solve (3.1), (3.2) by writing its mild form

$$u(t) = S(t) u_0 + \int_0^t S(t-s) \left( \frac{\partial u}{\partial x} \right) ds + \int_0^t S(t-s) \Phi dW(s).$$

(3.4)

We will work pathwise (i.e. $\omega \in \Omega$ is fixed) and use a fixed point argument in $X(\sigma)$ for some $\sigma$, where we recall that

$$X(\sigma) = \{ u \in \mathcal{C}(0, T; H^\sigma(\mathbb{R})) \cap L^2(\mathbb{R}; L^\infty([0, T])) ,$$

$$D^\sigma \partial_x u \in L^\infty(\mathbb{R}, L^2([0, T])); \partial_x u \in L^4([0, T]; L^2(\mathbb{R}))) \}.$$

We obtain the following result.

**Theorem 3.1.** Assume that $u_0 \in L^2(\Omega; H^1(\mathbb{R})) \cap L^4(\Omega; L^2(\mathbb{R}))$ and is $\mathcal{F}_0$-measurable, and $\Phi \in L^0_2(L^2(\mathbb{R}); H^\sigma(\mathbb{R}))$; then there exists a unique solution of (3.4) in $X(\sigma)(T_0)$ almost surely, for any $T_0 > 0$ and for any $\sigma$ with $3/4 < \sigma < 1$. Moreover, $u \in L^2(\Omega; L^\infty([0, T_0]; H^\sigma(\mathbb{R})))$.

**Remark 3.1.** If we only assume that $u_0 \in L^2(\Omega; H^\sigma(\mathbb{R})) \cap L^4(\Omega; L^2(\mathbb{R}))$, with $3/4 < \sigma < 1$, and is $\mathcal{F}_0$-measurable, then we cannot construct a solution on a fixed interval, even a finite one of the form $[0, T_0]$. But we could construct a solution on a random interval $[0, T(\omega))$.

**Remark 3.2.** Using a standard truncation argument, we could extend our result under the only assumption that $u_0 \in H^{\sigma + 1}(\mathbb{R})$ almost surely.

**Remark 3.3.** It follows from the proof that for any $\sigma \in (3/4, 1)$, $u \in \mathcal{C}(\mathbb{R} \cup [0, T_0]; H^\sigma(\mathbb{R}))$ almost surely, thus $u$ is weakly continuous with values in $H^\sigma(\mathbb{R})$ almost surely.

To prove Theorem 3.1, we first consider the linear equation

$$\begin{cases}
\frac{d^3 u}{dt^3} + \frac{\partial^3 u}{\partial x^3} dt = \Phi dW, \\
u(0) = 0
\end{cases}$$

(3.5)
whose solution is given by
\[ \tilde{u}(t) = \int_0^t S(t-s) \Phi \, dW(s) \] (3.6)
and first prove the following Theorem 3.2.

**Theorem 3.2.** Assume that \( \Phi \in L_2^0(L^1(\mathbb{R}), H^\tilde{\sigma}(\mathbb{R})) \) for some \( \tilde{\sigma} > 3/4 \).
Then \( \tilde{u} \) is almost surely in \( X_\sigma(T) \) for any \( T > 0 \) and any \( \sigma \) such that \( 3/4 < \sigma < \tilde{\sigma} \).
Moreover
\[ \mathbb{E}(|\tilde{u}|^2_{X_\sigma(T)}) \leq C(\sigma, \tilde{\sigma}, T) |\Phi|^2_{L_2^0}. \]

This theorem allows us to solve (3.4) locally in time by using arguments from [17]. Finally, when \( \tilde{\sigma} = 1 \), we obtain global existence thanks to an a priori estimate, when \( u_0 \) has values in \( H^1 \).

### 3.1. The Linear Equation

We now derive several regularity properties of \( \tilde{u} \) defined by (3.6), in order to prove Theorem 3.2. Hence, we assume in all the section that \( \Phi \in L_2^0(L^2(\mathbb{R}), H^\sigma(\mathbb{R})) \) for some \( \sigma > 3/4 \); also, in all the section, \( T > 0 \) is fixed and we use the contracted notations \( L^p_t, L^q_x, H^\sigma_x \), etc.

**Proposition 3.1.** We have for any \( \sigma \leq \tilde{\sigma} \),
\[ \tilde{u} \in L^2(\Omega; L^\sigma_t(H^\sigma_x)) \]
and
\[ \mathbb{E} \left( \sup_{t \in [0,T]} |\tilde{u}|^2_{L^\sigma_t} \right) \leq 38T |\Phi|^2_{L_2^0}. \]

**Proof.** The proof is quite standard. We use Itô formula on \( |\tilde{u}|^2_{L^\sigma_t} \) and deduce:
\[ |\tilde{u}(t)|^2_{L^\sigma_t} = 2 \left[ \int_0^t (J_\sigma \tilde{u}, J_\sigma \Phi \, dW(s)) + \int_0^t \text{Tr}(J_\sigma^2 \Phi \Phi^*) \, ds \right]. \]
We now write
\[ \text{Tr}(J_\sigma^2 \Phi \Phi^*) = |\Phi|^2_{L_2^0}. \]
and use a martingale inequality (see e.g. [10], Theorem 3.14),

\[
E \left( \sup_{t \in [0, T]} \left( J_u \Phi dW(s) \right) \right) \leq \frac{1}{\epsilon} \left( \int_0^T \left| \Phi^* \tilde{u} \right|^2 \, ds \right)^{1/2} 
\]

\[
\leq \frac{1}{\epsilon} \left( \int_0^T \left| \tilde{u}(t) \right|^2 \, ds \right)^{1/2} + 9T |\Phi|_{L^2}^2. \quad \blacksquare
\]

**Remark 3.4.** Using the same argument as in [10], Theorem 6.10, it can be proved that \( \tilde{u} \) is in \( L^2(\Omega; \mathcal{H}(\{0, T\}; H^s_x)) \).

**Proposition 3.2.**

\[ \tilde{u} \in L^2(\Omega; L^2_x(L^\infty_t)) \]

and

\[
E \left( \int_\mathbb{R} \sup_{t \in [0, T]} \left| \tilde{u}(t) \right|^2 \, dx \right) \leq C(\tilde{\sigma}, T) |\Phi|_{L^2_x}^2,
\]

where \( C(\tilde{\sigma}, T) \) depends on \( \tilde{\sigma} \) and \( T \).

**Proof.** Let \( (e_i)_{i \in \mathbb{N}} \) be an orthonormal basis of \( L^2(\mathbb{R}) \) and \( (\beta_i)_{i \in \mathbb{N}} \) a sequence of mutually independent brownian motions such that

\[ W = \sum_{i \in \mathbb{N}} \beta_i e_i. \]

We take \( (\psi_k)_{k \in \mathbb{N}} \) a partition of unity on \( \mathbb{R}^+ \) such that

\[
\text{supp } \psi_k \subset [2^{k-2}, 2^k], \quad \text{for } k \geq 1, \]

\[
\text{supp } \psi_0 \subset [-1, 1], \quad \text{and}
\]

\[
\psi_k(\xi) = \psi_1 \left( \frac{\xi}{2^k - 1} \right), \quad \xi \in \mathbb{R}^+, \quad k \geq 1.
\]

We also consider \( \tilde{\psi}_k \in C^\infty_c(\mathbb{R}) \) with \( \text{supp } \tilde{\psi}_k \subset [2^{k-3}, 2^{k+1}] \) such that \( \tilde{\psi}_k \geq 0 \) and \( \tilde{\psi}_k \equiv 1 \) on \( \text{supp } \psi_k \). For \( k \in \mathbb{N} \), we define the group \( (S_k(t))_{t \in \mathbb{R}} \) by

\[
S_k(t) u_0(\xi) = \psi_k(|\xi|) \tilde{S}(t) u_0(\xi) = e^{it^2/2} \psi_k(|\xi|) \tilde{u}_0(\xi)
\]

and the operator \( \Phi_k \) by

\[
\Phi_k e_i(\xi) = \psi_k(|\xi|) \Phi e_i, \quad i \in \mathbb{N}.
\]
Note that
\[ S_k(t) \Phi = S_k(t) \Phi_k \]
and
\[ S(t) \Phi = \sum_{k=1}^{\infty} S_k(t) \Phi_k. \]

We prove below that for any \( k \in \mathbb{N} \) and \( \sigma > 3/4 \)
\[
E \left( \int_{\mathbb{R}} \sup_{t \in [0, T]} \left| \int_0^t S_k(t-s) \Phi_k \, dW(s) \right|^2 \, dx \right)^{1/2} \leq C(T, \tilde{\sigma}) 2^{|2\tilde{\sigma}|} |\Phi_k|_{L^2}^{2 \tilde{\sigma}}. \tag{3.7}
\]

Then, by Minkowski inequality we will get
\[
E \left( \int_{\mathbb{R}} \sup_{t \in [0, T]} \left| \int_0^t S(t-s) \Phi \, dW(s) \right|^2 \, dx \right)^{1/2} \\
\leq \sum_{k} E \left( \int_{\mathbb{R}} \sup_{t \in [0, T]} \left| \int_0^t S_k(t-s) \Phi_k \, dW(s) \right|^2 \, dx \right)^{1/2} \\
\leq C(T, \tilde{\sigma}) \sum_{k} 2^{|2\tilde{\sigma}|} |\Phi_k|_{L^2}^{2 \tilde{\sigma}} \\
\leq C(T, \sigma, \tilde{\sigma}) |\Phi|_{L^2}^{2 \tilde{\sigma}}
\]
by choosing \( \tilde{\sigma} > \sigma > 3/4 \). We have used
\[
\sum_k 2^{|2\tilde{\sigma}|} |\Phi_k|_{L^2}^{2 \tilde{\sigma}} = \sum_k 2^{|2\tilde{\sigma}|} |\Phi_k \sigma|_{L^2}^{2 \tilde{\sigma}}
\]
and
\[
\sum_k 2^{|2\tilde{\sigma}|} |\Phi_k \sigma|_{L^2}^{2 \tilde{\sigma}} \leq C(\tilde{\sigma}) |\Phi \sigma|_{L^2}^{2 \tilde{\sigma}}
\]
for any \( \tilde{\sigma} \geq 0 \). Hence, it suffices to prove (3.7). We thus choose \( \sigma \) such that \( \tilde{\sigma} > \sigma > 3/4 \) and then set \( \tau = \inf \{ 1/4, 2(\sigma - 3/4)/15 \} > 0 \). Let \( p \geq 2 \) be such that \( \tau p > 1 \). Then, by the Sobolev embedding
\[ W^0, p \subset L^\tau. \]
we have
\[
\mathbb{E} \left( \int_{\mathbb{R}} \sup_{t, s \in [0, T]} \left| \int_0^t S_k(t-s) \Phi_k \, dW(s) \right|^2 \, dx \right) \\
\leq CE \left( \int_{\mathbb{R}} \left| \int_0^T \frac{\left( \int_0^t S_k(t-s) \Phi_k \, dW(s) \right) - \left( \int_0^t S_k(t-s) \Phi_k \, dW(s) \right)}{|t-s|^{1+np}} \, dt \right|^{2/p} \, dx \right) \\
\leq CE \left( \int_{\mathbb{R}} \left| \int_0^T \left( \int_0^t S_k(t-s) \Phi_k \, dW(s) \right) \, dt \right| \, dx \right) \\
+ CE \left( \int_{\mathbb{R}} \left| \int_0^T S_k(t-s) \Phi_k \, dW(s) \right|^2 \, dx \right) \\
\leq I_1 + I_2. \quad (3.8)
\]

We first estimate \( I_1 \). We have by Fubini’s Theorem and Hölder’s inequality
\[
I_1 \leq C \left( \int_{\mathbb{R}} \left| \int_0^T \frac{\mathbb{E} \left( \int_0^t S_k(t-s) \Phi_k \, dW(s) \right) - \mathbb{E} \left( \int_0^t S_k(t-s) \Phi_k \, dW(s) \right)}{|t-s|^{1+np}} \, dt \right|^{2/p} \, dx \right) \quad (3.9)
\]

Now, using the Gaussianity of
\[
\int_0^t S_k(t-s) \Phi_k \, dW(s) - \int_0^t S_k(s-s) \Phi_k \, dW(s),
\]
we have
\[
\mathbb{E} \left( \left| \int_0^t S_k(t-s) \Phi_k \, dW(s) - \int_0^t S_k(s-s) \Phi_k \, dW(s) \right|^p \right) \\
\leq CE \left( \left| \int_0^t S_k(t-s) \Phi_k \, dW(s) - \int_0^t S_k(s-s) \Phi_k \, dW(s) \right|^{2p/2} \right). \quad (3.10)
\]

We first consider, in the double integral in \( t \) and \( s \), the domain where \( t \) and \( s \) are close enough to satisfy \( |t-s| \leq 2^{1+1/p} \), i.e. we set
\[
\int_0^T \int_0^T \frac{\mathbb{E} \left( \left| \int_0^t S_k(t-s) \Phi_k \, dW(s) - \int_0^t S_k(s-s) \Phi_k \, dW(s) \right|^2 \right)^{p/2}}{|t-s|^{1+np}} \, dt \, ds = I_{1,1} + I_{1,2} \quad (3.11)
\]
where

\[
I_{k,1} = \int_0^T \int_0^T \frac{TZ_{1\{t-s \geq 2^{n(k)1}\}}(t,s)}{|t-s|^1 + \nu} \frac{\nu}{t-s} \int_0^t S_k(t-\tau) \Phi(\Phi_k^e_\delta) dW(\tau) - \int_0^s S_k(s-\tau) \Phi dW(\tau) \bigg)^2}^{\nu/2} \, dt \, ds
\]

(3.12)

and we proceed with the estimate of \( I_{k,1} \). Assuming thus that \( |t-s| \leq 2^{n_k} \leq 1 \), and furthermore that \( t > s \), we have

\[
\mathbb{E} \left( \left| \int_0^t S_k(t-\tau) \Phi dW(\tau) - \int_0^s S_k(s-\tau) \Phi dW(\tau) \right|^2 \right) = \sum_{i} \int_0^t |S_k(t-\tau) \Phi_k e_i|^2 \, dt
\]

\[
+ \sum_{i} \int_0^s |S_k(s-\tau) \Phi_k e_i|^2 \, dt.
\]

(3.13)

We now estimate separately the two terms in (3.13). For the first one, we write:

\[
S_k(t-\tau) \Phi_k e_i(x) = C \int_\mathbb{R} \mathcal{F}(S_k(t-\tau) \Phi_k e_i(x)) \ e^{ix_\xi} \, d_\xi
\]

\[
= C \int_\mathbb{R} e^{it_\xi} e^{ix_\xi} \Phi_k e_i(x) \, d_\xi = C \mathcal{F}^{-1}(e^{it_\xi} e^{ix_\xi} \Phi_k e_i(x)).
\]

By Lemma 3.1 below,

\[
|\mathcal{F}^{-1}(e^{it_\xi} e^{ix_\xi} \Phi_k e_i(x))| \leq C |\Phi_k|_{L^2(\mathbb{R})} + |\Phi_k^e|_{L^2(\mathbb{R})} + |\psi_k^e|_{L^2(\mathbb{R})} H_1^2(x).
\]

It is easily seen that for \( k \gg 1 \),

\[
|\Phi_k|_{L^2(\mathbb{R})} + |\Phi_k^e|_{L^2(\mathbb{R})} + |\psi_k^e|_{L^2(\mathbb{R})} + |\psi_k^e|_{L^2(\mathbb{R})} + \frac{1}{2^{n_k-1}} |\psi_k^e|_{L^2(\mathbb{R})}
\]

thus

\[
|\mathcal{F}^{-1}(e^{it_\xi} e^{ix_\xi} \Phi_k e_i(x))| \leq C H_1^2(x).
\]
and
\[ \int_s^t |S_h(t - \tau) \Phi_k e_i|^2 \, d\tau \leq C |t - s| (H^T_k * |\Phi_k e_i|)^2. \tag{3.14} \]

We have used the following result taken from [17] (see also [21]).

**Lemma 3.1.** There exists \( C_1, C_2, C_3 \) constants such that if for \( T > 0 \) we define \( H^T_k \) by
\[
H^T_k(x) = \begin{cases} 
2^k - 1 & \text{if } |x| < C_1(T + 1) \\
2^k - 1/2 |x|^{1/2} & \text{if } C_1(T + 1) < |x| \leq C_2(T + 1) 2^{2(k - 1)} \\
1/(1 + x^2) & \text{if } C_2(T + 1) 2^{2(k - 1)} \leq |x|
\end{cases}
\]

when \( k \geq 1 \) and
\[
H^0_k(x) = \begin{cases} 
1 & \text{if } |x| < C_1(T + 1) \\
1/(1 + x^2) & \text{if } |x| \geq C_1(T + 1)
\end{cases}
\]

then for any \( t \in [0, T] \) and \( \psi \in \mathbb{R} \) being supported in \( [2^{k-2}, 2^{k+1}] \) if \( k \geq 1 \) or \([ -1, 1] \) if \( k = 0 \) we have
\[
\left| \int_{\mathbb{R}} e^{i t \xi^2 + i \xi \tilde{\psi}(\xi)} \, d\xi \right| \leq C_3 (|\psi|_{L^2(\mathbb{R})} + |\psi|_{L^1(\mathbb{R})} + |\psi|_{L^1(\mathbb{R})}) H^T_k(x).
\]

For the second term in (3.13), we write
\[
S_h(t - \tau) \Phi_k e_i - S_h(s - \tau) \Phi_k e_i
\]
\[
= \int_{\mathbb{R}} e^{i(t-s) \xi^2 + i \xi \tilde{\psi}(\xi)} \, d\xi
\]
\[
= \int_{\mathbb{R}} e^{i(t-s) \xi^2} (1 - e^{i(s-t) \xi^2}) \tilde{\psi}_k(\xi) \, d\xi
\]
\[
= \mathcal{F}^{-1} e^{i(t-s) \xi^2} (1 - e^{i(s-t) \xi^2}) \tilde{\psi}_k * \Phi_k e_i.
\]

By Lemma 3.1, we have
\[
\left| \mathcal{F}^{-1} e^{i(t-s) \xi^2} (1 - e^{i(s-t) \xi^2}) \tilde{\psi}_k \right|
\]
\[
\leq C (|1 - e^{i(s-t) \xi^2}) \psi_k|_{L^2(\mathbb{R})} + |((1 - e^{i(s-t) \xi^2}) \psi_k)'|_{L^2(\mathbb{R})} + |((1 - e^{i(s-t) \xi^2}) \psi_k)'|_{L^2(\mathbb{R})}) H^T_k(x)
\]
\[
\leq C (|t - s| 2^{3k} + |t - s|^2 2^{5k}) H^T_k(x).
\]
It follows
\[
\int_0^\tau |S_k(t-\tau) \Phi_k e_i - S_k(s-\tau) \Phi_k e_i|^2 d\tau
\leq C |t-s| 2^{3k} + |t-s|^2 2^{5k} \sum_i (H_k^T * |\Phi_k e_i|)^2.
\]

This inequality and (3.14) in (3.13) give:
\begin{align*}
\mathbb{E} \left( \left| \int_0^\tau S_k(t-\tau) \Phi_k dW(\tau) - \int_0^\tau S_k(s-\tau) \Phi_k dW(\tau) \right|^2 \right) \\
&\leq C |t-s| + |t-s|^2 2^{6k} + |t-s|^4 2^{10k} \sum_i (H_k^T * |\Phi_k e_i|)^2.
\end{align*}

Since \( \alpha \leq 1/4 \) and \( |t-s| 2^{15k/4} \leq 1 \), we have
\[
|t-s| + |t-s|^2 2^{6k} + |t-s|^4 2^{10k} \leq C |t-s|^{4\alpha} 2^{15k}
\]
and therefore,
\begin{align*}
\mathbb{E} \left( \left| \int_0^\tau S_k(t-\tau) \Phi_k dW(\tau) - \int_0^\tau S_k(s-\tau) \Phi_k dW(\tau) \right|^2 \right) \\
&\leq C |t-s|^{4\alpha} 2^{15k - 3\alpha} \sum_i (H_k^T * |\Phi_k e_i|)^2. \quad (3.15)
\end{align*}

We now consider the case when \( |t-s| 2^{15k/4} \gg 1 \), i.e. we estimate \( I_{1,2} \). In this case, we simply have
\begin{align*}
\mathbb{E} \left( \left| \int_0^\tau S_k(t-\tau) \Phi_k dW(\tau) - \int_0^\tau S_k(s-\tau) \Phi_k dW(\tau) \right|^2 \right) \\
&\leq 2 \mathbb{E} \left( \left| \int_0^\tau S_k(t-\tau) \Phi_k dW(\tau) \right|^2 \right) + 2 \mathbb{E} \left( \left| \int_0^\tau S_k(s-\tau) \Phi_k dW(\tau) \right|^2 \right) \\
&\leq 2 \sum_i \left( \int_0^\tau |S_k(t-\tau) \Phi_k e_i|^2 d\tau + \int_0^\tau |S_k(s-\tau) \Phi_k e_i|^2 d\tau \right) \\
&\leq C |t-s|^{4\alpha} 2^{15k} \sum_i \left( \int_0^\tau |S_k(t-\tau) \Phi_k e_i|^2 d\tau + \int_0^\tau |S_k(s-\tau) \Phi_k e_i|^2 d\tau \right).
\end{align*}
Collecting (3.9), (3.10) and (3.15), we deduce

\[
I_1 \leq C 2^{15\epsilon} - \frac{3\epsilon}{2} \left( \int_0^T \int_0^T \frac{Z_{|t-s| \leq \eta}}{|t-s|^{1-\eta}} \, dt \, ds \right)^{2/p} \sum_i |H_k e_i|_{L^2(\mathbb{R})}^2 \\
+ C 2^{15\epsilon} \int_{\mathbb{R}} \left( \int_0^T \int_0^T \frac{Z_{|t-s| \leq \eta}}{|t-s|^{1-\eta}} \, dt \, ds \right)^{2/p} \, dx
\]

\[
\leq C(T) 2^{15\epsilon} - \frac{3\epsilon}{2} |H_k|_{L^2(\mathbb{R})} \sum_i |\Phi_k e_i|_{L^2(\mathbb{R})}^2 \\
+ C(T) 2^{15\epsilon} \sum_i \left( \left( \int_0^T \sup_{t \in [0, T]} |S_k(t) \Phi_k e_i|^2 \, dx \right) dt \right)^{2/p}
\]

\[
\leq C(T) 2^{15\epsilon} + \frac{3\epsilon}{2} \sum_i |\Phi_k e_i|_{L^2(\mathbb{R})}^2
\]

(3.16)

where we have used \(|H_k|_{L^2(\mathbb{R})} \leq C 2^{3\epsilon/2}\) and Theorem 2.7 in [17].

We estimate \(I_2\) in the same way:

\[
I_2 = \mathbb{E} \left( \left( \int_{\mathbb{R}} \left( \int_0^T \left( \int_0^t S_k(t-s) \Phi_k W(s) \right)^p \, dt \right)^{2/p} \, dx \right) \right)^{2/p}
\]

\[
\leq C \int_{\mathbb{R}} \left( \int_0^T \mathbb{E} \left( \left( \int_0^t S_k(t-s) \Phi_k W(s) \right)^p \, dt \right)^{2/p} \, dx \right)^{2/p}
\]

\[
\leq C \int_{\mathbb{R}} \left( \int_0^T \left( \sum_i \left( \int_0^t |S_k(t-s) \Phi_k e_i|^2 \, dt \right)^{p/2} \, dx \right) \right)^{2/p}
\]

\[
\leq C(T) \sum_i \left( \int_{\mathbb{R}} \sup_{t \in [0, T]} |S_k(t) \Phi_k e_i|^2 \, dx \right)
\]

\[
\leq C(T) 2^{3\epsilon/2} \sum_i |\Phi_k e_i|_{L^2(\mathbb{R})}^2
\]

again by Theorem 2.7 in [17]. This together with (3.16) gives exactly (3.7), since \(\sigma \geq 3/4 + 15\epsilon/2\).

**Proposition 3.3.** Let \(0 < \epsilon < \inf \{ \delta, 2 \} \), then

\[
D^\theta \partial_u \bar{u} \in L^2(\Omega; L^\infty(\mathbb{R}))
\]
and

\[ E \left( \sup_{x \in \mathbb{R}} \int_0^T |D^{\delta - 1} \partial_x u|^2 \, dt \right) \leq C(\varepsilon, T) |\Phi|_{L^2_x}^2. \]

**Proof.** Let us set \( q = 4/\varepsilon \), we will prove that

\[ E \left( \left( \sup_{x \in \mathbb{R}} \int_0^T |D^{\delta - 1} \partial_x u|^2 \, dt \right)^{\frac{q}{2}} \right) \leq C(\varepsilon) |\Phi|_{L^2_x}^2 \]  

(3.17)

which implies the result by Hölder inequality. We will use interpolation arguments and Sobolev embeddings. We first estimate \( |D^{1 + \alpha}u|_{L^q_t(L^2)} \).

We use successively Hölder inequality, Fubini Theorem and properties of gaussian random variables as in the proof of Proposition 3.2, to get

\[ |D^{1 + \alpha}u|_{L^q_t(L^2)} = \sup_{x \in \mathbb{R}} \mathbb{E} \left( \left( \int_0^T \left| \int_0^T D^{1 + \alpha}S(t - \tau) \Phi \, dW(\tau) \right|^2 \, dt \right)^{\frac{q}{2}} \right) \]

\[ \leq C \sup_{x \in \mathbb{R}} \left( \mathbb{E} \left( \left( \int_0^T D^{1 + \alpha}S(t - \tau) \Phi \, dW(\tau) \right)^2 \right)^{\frac{q}{2}} \right) \, dt \]

\[ \leq C \sup_{x \in \mathbb{R}} \left( \int_0^T \sum_{i=0}^\infty |D^{1 + \alpha}S(t - \tau) \Phi \varepsilon_i|^2 \, dt \right)^{\frac{q}{2}} \]

\[ \leq C \int_0^T \left( \sum_{i=0}^\infty \mathbb{E} \left( |D^{1 + \alpha}S(t - \tau) \Phi \varepsilon_i|^2 \right)^{\frac{q}{2}} \right) \, dt. \]

We know by Lemma 2.1 in [17] that

\[ \sup_{x \in \mathbb{R}} \int_0^T |D^{1 + \alpha}S(t - \tau) \Phi \varepsilon_i|^2 \, d\tau \leq C |D^{1 + \alpha} \Phi \varepsilon_i|_{L^2} \]

\[ \leq C |\Phi \varepsilon_i|_{H^\alpha(R)}, \]

therefore

\[ |D^{1 + \alpha}u|_{L^q_t(L^2)} \leq C \left( \sum_{i=0}^\infty \mathbb{E} \left( |\Phi \varepsilon_i|^2 \right)^{\frac{q}{2}} \right)^{\frac{1}{2}} \, dt \]

\[ \leq C(T) |\Phi|_{L^2_x}^{\frac{q}{2}} \varepsilon. \]  

(3.18)
We now consider the quantity $|D^u|_{L^2_t(L^2_x)}$. We derive similarly

$$|D^u|_{L^2_t(L^2_x)}^2 = \int_R \mathbb{E}\left(\left(\int_0^T \left|\int_0^\tau D^u S(t - \tau) \Phi \, dW(\tau)\right|^2 \, dt\right)^{\frac{q}{2}}\right) \, dx$$

$$\leq C \int_R \mathbb{E}\left(\left(\int_0^T \left|\int_0^\tau D^u S(t - \tau) \Phi \, dW(\tau)\right|^2 \, dt\right)^{\frac{q}{2}}\right) \, dx$$

$$\leq C \int_R \mathbb{E}\left(\sum_{\tau} \left(\int_0^T \left|\int_0^\tau \Phi_{e_i} \, dW(\tau)\right|^2 \, dt\right)^{\frac{q}{2}}\right) \, dx$$

$$\leq C \sum_{\tau} \int_R \left|\int_0^T \left|\int_0^\tau \Phi_{e_i} \, dW(\tau)\right|^2 \, dt\right|^{\frac{q}{2}} \, dx$$

$$\leq C |\Phi|_{L^2}^{2^\frac{q}{2}}$$

since $S(\tau)$ is an isometry in $L^2_x$ for any $\tau$. We now deduce from the preceding inequality, (3.18) and Proposition A.1 in the appendix that $D^{1+\varepsilon} \widetilde{u} \in L^2_t(L^2_x)$ (recall that $q = 4/6$) and

$$|D^{1+\varepsilon} \widetilde{u}|_{L^2_t(L^2_x)} \leq C |\Phi|_{L^2}^{2^\frac{q}{2}}. \quad (3.19)$$

Also we have

$$|\widetilde{u}|_{L^2_t(L^2_x)} \leq C |\Phi|_{L^2}^{2^\frac{q}{2}}. \quad (3.20)$$

Indeed we can write

$$|\widetilde{u}|_{L^2_t(L^2_x)} = \int_R \mathbb{E}\left(\left(\int_0^T \left|\int_0^\tau S(t - \tau) \Phi \, dW(\tau)\right|^2 \, dt\right)^{\frac{q}{2}}\right) \, dx$$

$$\leq C \int_R \mathbb{E}\left(\left(\int_0^T \left|\int_0^\tau S(t - \tau) \Phi \, dW(\tau)\right|^2 \, dt\right)^{\frac{q}{2}}\right) \, dx$$

$$\leq C \int_R \mathbb{E}\left(\sum_{\tau} \left(\int_0^T \left|\int_0^\tau \Phi_{e_i} \, dW(\tau)\right|^2 \, dt\right)^{\frac{q}{2}}\right) \, dx$$

$$\leq C(T) \int_R \left(\int_0^T \left|\int_0^\tau \Phi_{e_i} \, dW(\tau)\right|^2 \, dt\right)^{\frac{q}{2}} \, dx$$
and by Minkowski inequality,

\[ |\bar{u}|^2_{L^q_t(L^{\infty}_x)} \leq C \sum_i \left( \int_{\mathbb{R}} \left( \int_0^T |S(\tau) \phi_i|^2 \, d\tau \right)^{q/2} \, dx \right)^{2/q} \]

\[ \leq C \sum_i |S(\tau) \phi_i|^2_{L^{\infty}_x} \]

\[ \leq C |\Phi|^2_{L^{\infty}_x}. \]

We have used the embedding \( H^\alpha \subset L^q_x \), \( \alpha \geq 1/2 \) and the fact that \( S(\tau) \) defines an isometry in \( H^\alpha_x \). We deduce from (3.19) and (3.20) that

\[ |\bar{u}|^2_{L^q_t(L^{\infty}_x)} \leq C |\Phi|^2_{L^{\infty}_x}. \]

Since \( qe/2 = 2 \), we have

\[ W^{\gamma;2-n}(L^2_x) \subset L^{\infty}_x(L^2_x), \]

it follows that \( D^{1+\delta-\epsilon} u \) is in \( L^{\infty}_x(L^2_x(\mathbb{R})) \) and

\[ |D^{1+\delta-\epsilon} u|^2_{L^{\frac{2^\gamma}{\alpha}}(L^{\infty}_x)} \leq C |\Phi|^2_{L^{\infty}_x}. \]

To end the proof, we write

\[
D^{\gamma-\gamma/2} \bar{u} = \int_0^T D^{\gamma-\gamma/2} \bar{u} \, S(t-\tau) \, \Phi \, dW(\tau)
\]

\[
= \int_0^T D^{1+\delta-\epsilon} S(t-\tau) \, \mathcal{H} \Phi \, dW(\tau)
\]

where \( \mathcal{H} \) is the Hilbert transform. We deduce

\[ |D^{\gamma-\gamma/2} \bar{u}|_{L^{\frac{2^\gamma}{\alpha}}(L^{\infty}_x)} \leq C |\mathcal{H} \Phi|^2_{L^{\infty}_x} \leq C |\Phi|^2_{L^{\infty}_x}. \]

**Proposition 3.4.**

\( \bar{u} \in L^2(\Omega; L^4_x(\mathbb{R})) \)

and

\[ E \left( \left( \sup_{x \in \mathbb{R}} \int_0^T |\partial_x u|^4 \, dt \right)^{1/2} \right) \leq |\Phi|^2_{L^{\infty}_x}. \]
Proof. Let $\varepsilon = \alpha - 3/4$ and $q = 4(1 + 1/\varepsilon)$. We will use interpolation arguments and Sobolev embeddings again. We start with $|D^{1+\alpha\varepsilon}|_{L^2_{T_x}(L^2_{x,\varepsilon}(L^2_{z,\varepsilon})))}$, we have

$$|D^{1+\alpha\varepsilon}|_{L^2_{T_x}(L^2_{x,\varepsilon}(L^2_{z,\varepsilon})))} \leq C \left( \int_0^T \sup_{x \in \mathbb{R}} \left( \sum_{i} \int_0^T |D^{1+\alpha\varepsilon}S(t-\tau)\Phi_{e_i}|^2 d\tau \right)^{2} d\tau \right)^{\frac{1}{2}} $$

By Theorem 2.4 in [17] (with $\nu = 2$, $\theta = 1$, $\beta = 1/2$) we know that

$$\int_0^T \sup_{x \in \mathbb{R}} |D^{1+\alpha\varepsilon}S(t)\Phi_{e_i}|^4 dt \leq C |D\Phi_{e_i}|^{4}_{L^2}$$

it follows

$$|D^{1+\alpha\varepsilon}|_{L^2_{T_x}(L^2_{x,\varepsilon}(L^2_{z,\varepsilon})))} \leq C |\Phi|_{L^2}^{\alpha + \varepsilon}. \quad (3.21)$$

The following estimate is easy to derive:

$$|\phi|_{L^2_{T_x}(L^2_{x,\varepsilon}(L^2_{z,\varepsilon})))} \leq C |\Phi|_{L^2}^{\alpha + \varepsilon} \leq C |\Phi|_{L^2}^{\alpha + \varepsilon}.$$

By Proposition A.1, we obtain for a.e. $t \in [0, T]$

$$|D^{1+\alpha\varepsilon}\phi|_{L^2_{T_x}(L^2_{x,\varepsilon}(L^2_{z,\varepsilon})))} \leq C |\phi|_{L^2_{T_x}(L^2_{x,\varepsilon}(L^2_{z,\varepsilon})))}^{1-\varepsilon}$$

and, since $q = 4(1 + 1/\varepsilon) \geq 4$,

$$|D^{1+\alpha\varepsilon}\phi|_{L^2_{T_x}(L^2_{x,\varepsilon}(L^2_{z,\varepsilon})))} \leq |D^{1+\alpha\varepsilon}\phi|_{L^2_{T_x}(L^2_{x,\varepsilon}(L^2_{z,\varepsilon})))}$$

Using Sobolev embedding and Fubini Theorem, we also have:

$$|\phi|_{L^2_{T_x}(L^2_{x,\varepsilon}(L^2_{z,\varepsilon})))} \leq C |\phi|_{L^2_{T_x}(L^2_{x,\varepsilon}(L^2_{z,\varepsilon})))}.$$

By Proposition A.1, we obtain for a.e. $t \in [0, T]$

$$|D^{1+\alpha\varepsilon}\phi|_{L^2_{T_x}(L^2_{x,\varepsilon}(L^2_{z,\varepsilon})))} \leq C |\phi|_{L^2_{T_x}(L^2_{x,\varepsilon}(L^2_{z,\varepsilon})))}^{1-\varepsilon}$$

and, since $q = 4(1 + 1/\varepsilon) \geq 4$,
Thus
\[
|\tilde{u}|_{L^p_t(L^q_x)} \leq C |\Phi|_{L^p_t}
\]
and, since \(\varphi < 2 > 1\),
\[
|\tilde{u} + \tilde{u}|_{L^p_t(L^q_x)} \leq C |\tilde{u}|_{L^p_t(L^q_x)}
\leq C |\Phi|_{L^p_t}
\]

The proof of Theorem 3.2 follows from Propositions 3.1 to 3.4.

3.2. The Nonlinear Problem

We now want to prove Theorem 3.1, i.e. to solve (3.4). We work pathwise and use a fixed point argument in \(X_\sigma(T)\) for some \(T > 0\) and for \(3/4 < \sigma < 1\); then using an a priori estimate, we prove that the solution is global in \(H^1\).

Let \(T_0 > 0\) be fixed. From the previous section, we know that \(\tilde{u}\) is in \(X_\sigma(T_0)\) for almost all \(\omega \in \Omega\), and we fix such an \(\omega\). We use the following results taken from [17].

**Proposition 3.5.** For any \(\sigma > 3/4\) and any \(T > 0\), there exists \(C(T, \sigma)\), nondecreasing with respect to \(T\) such that

(i) \[
\left| \int_0^T S(t-\tau)(u_0, v) \, d\tau \right|_{X_\sigma(T)} \leq C(T, \sigma) T^{1/2} |u|_{X_\sigma(T)} \, |v|_{X_\sigma(T)}
\]

for any \(u, v \in X_\sigma(T)\).

(ii) \[
|S(t) u_0|_{X_\sigma(T)} \leq C(T, \sigma) |u_0|_{H^\sigma(\mathbb{R})}
\]

for any \(u_0 \in H^\sigma(\mathbb{R})\).

**Proof of Theorem 3.1.** We introduce the mapping \(\mathcal{F}\) defined by

\[
\mathcal{F}u(t) = S(t) u_0 + \int_0^t S(t-\tau)(u_0, u) \, d\tau + \tilde{u}(t).
\]

Let \(3/4 < \sigma < 1\); since (3.3) holds with \(\sigma = 1\), we know by Theorem 3.2 and Proposition 3.5 that if \(u_0\) is in \(H^\sigma(\mathbb{R})\), \(\mathcal{F}\) maps \(X_\sigma(T)\) into itself. Moreover, let \(R_0\) be such that

\[
R_0 \geq C(T_0, \sigma) |u_0|_{H^\sigma(\mathbb{R})} + |\tilde{u}|_{X_\sigma(T)}
\]

and choose \(T\) satisfying

\[
8C(T, \sigma) T^{1/2} R_0 \leq 1.
\]
Then, it is easily checked that $\mathcal{F}$ maps the ball of center 0 and radius $2R_0$ in $X_\sigma(T)$ into itself and that it is strictly contracting:

$$|\mathcal{F}u - \mathcal{F}v|_{X_\sigma(T)} \leq \frac{1}{2} |u - v|_{X_\sigma(T)}$$  \hspace{1cm} (3.24)

for any $u, v \in X_\sigma(T)$ with norm less than $2R_0$. Thus $\mathcal{F}$ has a unique fixed point, denoted by $u$, in this ball. It is obviously the unique solution of (3.4) in $X_\sigma(T)$. Of course, $T$ depends on $\omega$ through the choice of $R_0$ satisfying (3.22). We prove below that the solution can be extended to the whole interval $[0, T_0]$ when $u_0$ satisfies the assumptions of Theorem 3.1. Let $(\Phi_n)_{n \in \mathbb{N}}$ be a sequence in $L^2_\omega$ such that

$$\Phi_n \to \Phi \quad \text{in} \quad L^2_\omega$$  \hspace{1cm} (3.25)

and let $(u_{0,n})_{n \in \mathbb{N}}$ be a sequence in $H^1(\mathbb{R})$ such that

$$u_{0,n} \to u_0 \quad \text{in} \quad L^4(\Omega; L^2(\mathbb{R})) \cap L^2(\Omega; H^1(\mathbb{R})) \quad \text{and in} \quad H^1(\mathbb{R}) \text{ a.s.}$$  \hspace{1cm} (3.26)

Thanks to Theorem 3.2, we know that for any $3/4 < \sigma < 1$,

$$\bar{u}_n = \int_0^t S(t - \tau) \Phi_n dW(\tau) \rightarrow \bar{u} \quad \text{in} \quad L^2(\Omega; X_\sigma(T_0)).$$

It follows that there exists a subsequence, still denoted by $(\bar{u}_n)$, such that

$$\bar{u}_n \to \bar{u} \quad \text{in} \quad X_\sigma(T_0) \text{ a.s.}$$  \hspace{1cm} (3.27)

**Lemma 3.2.** For any $n$, there exists a unique solution $u_n$ $P$-almost surely in $L^\infty(0, T_0; H^4(\mathbb{R}))$, of

$$
\begin{cases}
    du_n + \left( \frac{\partial^3 u_n}{\partial x^3} + u_n \frac{\partial u_n}{\partial x} \right) dt = \Phi_n dW, \\
    u_n(0) = u_{0,n}.
\end{cases}
$$

**Proof.** We know by Proposition 3.1 that $\bar{u}_n$ is in $L^\infty(0, T; H^4(\mathbb{R}))$ almost surely. We set

$$v_n = u_n - \bar{u}_n,$$

and look for a solution almost surely of

$$
\begin{cases}
    \frac{\partial v_n}{\partial t} + \frac{\partial^3 v_n}{\partial x^3} + (v_n + \bar{u}_n) \frac{\partial}{\partial x} (v_n + \bar{u}_n) = 0, \\
    v_n(0) = u_{0,n}.
\end{cases}
$$

Thanks to the property (3.24), it is easily seen that $\mathcal{F}$ maps $X_\sigma(T)$ into itself and that it is strictly contracting:

$$|\mathcal{F}u - \mathcal{F}v|_{X_\sigma(T)} \leq \frac{1}{2} |u - v|_{X_\sigma(T)}$$  \hspace{1cm} (3.24)

for any $u, v \in X_\sigma(T)$ with norm less than $2R_0$. Thus $\mathcal{F}$ has a unique fixed point, denoted by $u$, in this ball. It is obviously the unique solution of (3.4) in $X_\sigma(T)$. Of course, $T$ depends on $\omega$ through the choice of $R_0$ satisfying (3.22). We prove below that the solution can be extended to the whole interval $[0, T_0]$ when $u_0$ satisfies the assumptions of Theorem 3.1. Let $(\Phi_n)_{n \in \mathbb{N}}$ be a sequence in $L^2_\omega$ such that

$$\Phi_n \to \Phi \quad \text{in} \quad L^2_\omega$$  \hspace{1cm} (3.25)

and let $(u_{0,n})_{n \in \mathbb{N}}$ be a sequence in $H^1(\mathbb{R})$ such that

$$u_{0,n} \to u_0 \quad \text{in} \quad L^4(\Omega; L^2(\mathbb{R})) \cap L^2(\Omega; H^1(\mathbb{R})) \quad \text{and in} \quad H^1(\mathbb{R}) \text{ a.s.}$$  \hspace{1cm} (3.26)

Thanks to Theorem 3.2, we know that for any $3/4 < \sigma < 1$,

$$\bar{u}_n = \int_0^t S(t - \tau) \Phi_n dW(\tau) \rightarrow \bar{u} \quad \text{in} \quad L^2(\Omega; X_\sigma(T_0)).$$

It follows that there exists a subsequence, still denoted by $(\bar{u}_n)$, such that

$$\bar{u}_n \to \bar{u} \quad \text{in} \quad X_\sigma(T_0) \text{ a.s.}$$  \hspace{1cm} (3.27)

**Lemma 3.2.** For any $n$, there exists a unique solution $u_n$ $P$-almost surely in $L^\infty(0, T_0; H^4(\mathbb{R}))$, of

$$
\begin{cases}
    du_n + \left( \frac{\partial^3 u_n}{\partial x^3} + u_n \frac{\partial u_n}{\partial x} \right) dt = \Phi_n dW, \\
    u_n(0) = u_{0,n}.
\end{cases}
$$

**Proof.** We know by Proposition 3.1 that $\bar{u}_n$ is in $L^\infty(0, T; H^4(\mathbb{R}))$ almost surely. We set

$$v_n = u_n - \bar{u}_n,$$

and look for a solution almost surely of

$$
\begin{cases}
    \frac{\partial v_n}{\partial t} + \frac{\partial^3 v_n}{\partial x^3} + (v_n + \bar{u}_n) \frac{\partial}{\partial x} (v_n + \bar{u}_n) = 0, \\
    v_n(0) = u_{0,n}.
\end{cases}
$$
Using a parabolic regularization as in [27] or a fixed point argument as in [14], it can be proved that there exists a unique local solution. Then, by using the invariant quantities of the deterministic KdV equation (see for example [12]), we can check that this solution is global and is in $L^\infty(0, T_0; H^1(\mathbb{R}))$ almost surely.

**Lemma 3.3.** The sequence $u_n$ is bounded in $L^4(\Omega, L^\infty(0, T_0; H^1(\mathbb{R}))).$

**Proof.** We apply Itô formula to

$$|u_n|^4 = \mathcal{E}_0(u_n),$$

we obtain

$$|u_n(t)|^4 = |u_{0,n}|^4 + 4 \int_0^t |u_n|^2 \langle u_n, \Phi_n dW \rangle + \frac{1}{4} \int_0^t \text{Tr}(\mathcal{E}_0^*(u_n) \Phi_n \Phi_n^*) \, ds$$

with

$$\mathcal{E}_0^*(u_n) v = 8(\langle u_n, v \rangle u_n + 4 |u_n|^2 v).$$

Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R})$, we have:

$$\text{Tr}(\mathcal{E}_0^*(u_n) \Phi_n \Phi_n^*) = \sum_{i \in \mathbb{N}} 8\langle u_n, \Phi_n(e_i) \rangle^2 + 4 |u_n|^2 |\Phi_n(e_i)|^2$$

$$\leq 12 |u_n|^2 |\Phi_n|_{L^2}^2$$

$$\leq \frac{1}{4} |u_n|^4 + 72 |\Phi_n|_{L^2}^4.$$

Also, using a martingale inequality (see e.g. [10], Theorem 3.14),

$$\mathbb{E} \left( \sup_{t \in [0, T_0]} |u_n|^4 \langle u_n, \Phi_n dW \rangle \right) \leq 3\mathbb{E} \left( \int_0^T |u_n|^4 |\Phi_n^* u_n|^2 \, ds \right)^{1/2}$$

$$\leq \frac{1}{16} \mathbb{E} \left( \sup_{t \in [0, T_0]} |u_n|^4 \right) + C |\Phi_n|_{L^2}^4.$$

We deduce

$$\mathbb{E} \left( \sup_{t \in [0, T_0]} |u_n(t)|^4 \right) \leq 2\mathbb{E} |u_{0,n}|^4 + C |\Phi_n|_{L^2}^4. \quad (3.28)$$

In the same way, we have

$$\mathbb{E} \left( \sup_{t \in [0, T_0]} |u_n(t)|^2 \right) \leq 2\mathbb{E} |u_{0,n}|^2 + C |\Phi_n|_{L^2}^2. \quad (3.29)$$
We now apply Itô formula to
\[ E_1(u_n) = \int_R \left[ \frac{1}{2} \left( \frac{\partial^2 u_n}{\partial x^2} \right)^2 - \frac{1}{6} u_n^3 \right] dx \]
and obtain
\[ E_1(u_n(t)) = E_1(u_n,0) - \int_0^t \left( \frac{\partial^2 u_n}{\partial x^2} + \frac{1}{2} u_n^2 \Phi_n dW \right) \]
\[ + \frac{1}{2} \int_0^t \text{Tr}(E^*_1(u_n) \Phi_n \Phi_n^*) \, dt, \]  
(3.30)
with
\[ E^*_1(u_n) \, v = \frac{\partial^2 v}{\partial x^2} - u_n v. \]
We have
\[ \text{Tr}(E^*_1(u_n) \Phi_n \Phi_n^*) = - \sum_{i \in N} \int_R \left[ \frac{\partial^2}{\partial x^2} \left( \Phi_n \phi_i \right) \Phi_n \phi_i + u_n(\Phi_n \phi_i)^2 \right] dx \]
\[ \leq \sum_{i \in N} \left( \left| \frac{\partial}{\partial x} (\phi_i \phi_i) \right| + |u_n| |\phi_i| L^2(R) \right) \]
\[ \leq C |\phi_n| L^2 \left( |u_n| + 1 \right) \]  
(3.31)
where the embedding \( H^1(R) \subset L^4(R) \) has been used. Also, by the same martingale inequality as above,
\[ \mathbb{E} \left( \sup_{t \in [0,T_0]} \left| \int_0^t \left( \frac{\partial^2 u_n}{\partial x^2} + \frac{1}{2} u_n^2 \Phi_n dW \right) \right|^2 \right) \]
\[ \leq 3 \mathbb{E} \left( \left( \int_0^{T_0} \left| \Phi_n \left( \frac{\partial^2 u_n}{\partial x^2} + \frac{1}{2} u_n^2 \right) \right|^2 ds \right)^{1/2} \right). \]
Using
\[ \left| \Phi_n \left( \frac{\partial^2 u_n}{\partial x^2} + \frac{1}{2} u_n^2 \right) \right|^2 = \sum_{i \in N} \left[ \left( \frac{\partial^2 u_n}{\partial x^2}, \Phi_n \phi_i \right) + \frac{1}{2} (\phi_i^2, \Phi_n \phi_i)^2 \right] \]
\[ \leq C \sum_{i \in N} \left( |u_n| H^1(R) \right) |\phi_i| L^2(R) + |u_n| L^4(R) |\phi_i| L^2(R) \]
\[ \leq C |u_n| H^1(R) + |u_n| L^4(R) |\phi_n| L^2 \]
by the embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$; it can be derived

$$
\mathbb{E} \left( \sup_{t \in [0, T_\infty)} \left( \frac{\partial^2 u_n}{\partial x^2} + \frac{1}{2} u_n^2, \Phi_n \right) dW \right)
\leq \frac{1}{8} \mathbb{E} \left( \sup_{t \in [0, T_\infty)} |u_n|_{H^1(\mathbb{R})}^2 \right) + C \mathbb{E} \left( \sup_{t \in [0, T_\infty)} |u_n|^4 \right) + C |\Phi_n|_{L^2}^2.
$$

Thus, by (3.30) and (3.31),

$$
\mathbb{E} \left( \sup_{t \in [0, T_\infty)} \mathcal{E}(u_n(t)) \right) \leq \mathbb{E} \left( \mathcal{E}(u_{n,0}) + \frac{1}{8} \mathbb{E} \left( \sup_{t \in [0, T_\infty)} |u_n(t)|_{H^1(\mathbb{R})}^2 \right) \right)
+ C \mathbb{E} \left( \sup_{t \in [0, T_\infty)} |u_n(t)|^4 \right) + C \mathbb{E} \left( \sup_{t \in [0, T_\infty)} |u_n(t)|^2 \right)
+ C |\Phi_n|_{L^2}^2 (1 + |\Phi_n|_{L^2}^2).
$$

The result follows by using (3.28) and (3.29) since it can be checked, using Sobolev embeddings and interpolation of Sobolev spaces, that

$$
\frac{1}{8} \left| \frac{\partial u_n}{\partial x} \right|^2 - C |u_n|^{10/3} \leq \mathcal{E}(u_n) \leq C |u_n|_{H^1(\mathbb{R})}^2 + |u_n|^{10/3}. \tag{1}
$$

We deduce from this lemma that we can find a function $\tilde{u}$ in $L^2(\Omega; L^p(0, T; H^1(\mathbb{R})))$ such that, after extraction of a subsequence,

$$
u_n \rightarrow \tilde{u} \quad \text{in} \quad L^2(\Omega; L^p(0, T; H^1(\mathbb{R}))) \text{ weak star}.
$$

Also, we know that $\tilde{u} \in L^p(0, T; H^1(\mathbb{R}))$ almost surely. We now choose in (3.22)

$$
R_\infty(\omega) = 2C(T_\infty, \sigma) |\tilde{u}|_{L^p(0, T; H^1(\mathbb{R}))) + 2 |\tilde{u}|_{X_d(T)}
$$

where $\sigma$ is fixed in (3.4, 1), and take $T(\omega)$ satisfying (3.23). It is not difficult to see that $u_n$ is in $X_d(T)$ for any $T \geq 0$ and is the unique fixed point of $\mathcal{F}_n$ defined by

$$
\mathcal{F}_n v = S(t) u_{0,n} + \int_0^t S(t - \tau) v \frac{\partial v}{\partial x} dt + u_n
$$

which is also a strict contraction:

$$
|\mathcal{F}_n v_1 - \mathcal{F}_n v_2|_{X_d(T)} \leq \frac{1}{2} |v_1 - v_2|_{X_d(T)}.
$$
By (3.26) and (3.27), we prove easily that for almost all \( \omega \),
\[
u_n \rightarrow u \quad \text{in} \quad X_{\xi}(T(\omega))
\]
where we recall that \( u \) is the unique fixed point of \( \mathcal{F} \), and it is not difficult to check that necessarily, \( u = \bar{u} \) on \([0, T(\omega)]\) (note that \( u \) continued by 0 outside \([0, T(\omega)]\) is in \( L^2(\Omega, X_{\xi}(T)) \)). Thus,
\[
|u(T(\omega))|_{H^1(\mathbb{R})} \leq |\bar{u}|_{L^2(0, T_0, H^1(\mathbb{R}))}
\]
so that we can construct a solution on \([T(\omega), 2T(\omega)]\) starting from \( u(T(\omega)) \). We can reiterate this to get a solution on \([0, T_0]\). Since \( T_0 \) is arbitrary, this ends the proof of Theorem 3.1.

4. MULTIPLICATIVE NOISE

We now consider the case of a multiplicative noise and an initial datum in \( L^2(\mathbb{R}) \). The equation writes in Itô form:
\[
du + \left( \frac{\partial^4 u}{\partial x^4} + u \frac{\partial u}{\partial x} \right) dt = \Phi(u) \, dW \tag{4.1}
\]
for \( x \in \mathbb{R}, t \geq 0 \). It is supplemented with an initial condition:
\[
u(x, 0) = \nu_0(x). \tag{4.2}
\]
Here \( \nu_0 \) is a deterministic function in \( L^2(\mathbb{R}) \) and \( \Phi \) is a continuous mapping from \( L^2(\mathbb{R}) \) to \( L^2_0(L^2(\mathbb{R})) \), the space of Hilbert–Schmidt operators from \( L^2(\mathbb{R}) \) to itself. We assume that \( \Phi \) is such that there exists a constant \( \kappa_1 \) satisfying
\[
|\Phi(u)|_{L^2_0(L^2(\mathbb{R}))} \leq \kappa_1 (|u|_{L^2(\mathbb{R})} + 1), \quad \text{for any} \quad u \in L^2(\mathbb{R}), \tag{4.3}
\]
and that \( \Phi \) is local in the sense that for any \( a, b \) in \( L^2(\mathbb{R}) \) with compact support, the mapping
\[
u \mapsto (\Phi(u) \, a, \Phi(u) \, b)_{L^2(\mathbb{R})}
\]
\[
L^2(\mathbb{R}) \rightarrow \mathbb{R}
\]
is continuous for the \( L^2_0(\mathbb{R}) \) topology.

We will prove the existence of a martingale solution to (4.1), (4.2), that means that the stochastic basis \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]}, (W(t))_{t \in [0, T]} \) is also an unknown of the problem. The main result of this section is the following theorem.
**Theorem 4.1.** For any $T > 0$ and $u_0 \in L^2(\mathbb{R})$, there exists a martingale solution to (4.1), (4.2). That is, there exist a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0, T]}, (W(t))_{t \in [0, T]})$ and an adapted process on this basis $u$ solution to (4.1), (4.2). Moreover $u$ has paths in $L^\infty(0, T; L^2(\mathbb{R})) \cap L^2(0, T; H^1_{\text{loc}}(\mathbb{R})) \cap \mathcal{G}(0, T; H^s_{\text{loc}}(\mathbb{R}))$ for any $s < 0$.

**Proof.** To prove this theorem we consider the following parabolic regularization of equation (4.1)

$$
du' = \left( \frac{\partial^4 u'}{\partial x^4} + \frac{\partial^3 u'}{\partial x^3} + \frac{\partial u'}{\partial x} \right) dt + \Phi(u') dW \tag{4.4}
$$

The following proposition will be proved at the end of this section.

**Proposition 4.1.** For any $\epsilon > 0$, $T > 0$ and $u_0 \in L^2(\mathbb{R})$, there exists a martingale solution to (4.4), (4.2). That is, there exist a stochastic basis $(\Omega', \mathcal{F}', \mathbb{P}', (\mathcal{F}'_t)_{t \in [0, T]}, (W'(t))_{t \in [0, T]})$ and an adapted process on this basis $u'$ solution to (4.4), (4.2). Moreover $u'$ has paths in $L^\infty(0, T; L^2(\mathbb{R})) \cap L^2(0, T; H^1_{\text{loc}}(\mathbb{R})) \cap \mathcal{G}(0, T; H^s_{\text{loc}}(\mathbb{R}))$ for any $s < 0$.

We now will show that the laws of $u'$ are tight in a conveniently chosen space. This will follow from a priori bounds and an argument similar to [11]. All the constants appearing below are independent of $\epsilon$.

**Lemma 4.1.** For any $p \in \mathbb{N}$, there exists a constant $K_p$ such that

$$
E(\sup_{t \in [0, T]} |u'(t)|^{2p}) \leq K_p(|u_0|^{2p} + 1)
$$

for any $\epsilon > 0$.

The proof of this lemma is not difficult. It suffices to apply Itô formula on the functional $F_p(u') = |u'|^{2p}$ and to use a martingale inequality, as in the proof of Lemma 3.3.

We now prove the following generalization of T. Kato’s smoothing effect ([15]).

**Lemma 4.2.** There exists $\epsilon_0$ such that we can find a constant $c_1$ and, for any $k > 0$, a constant $C_1(k)$ such that for any $\epsilon_0 > \epsilon > 0$:

$$
\epsilon E(\|u'|_{L^2(0, T; H^2(\mathbb{R}))}) \leq C_1
$$

and

$$
E(\|u'|_{L^2(0, T; H^1(\mathbb{R}))}) \leq C_1(k).
$$
Proof. We apply Ito’s formula to $F(u') = \int_\mathbb{R} p(x)(u'(x))^2 \, dx$ where $p$ is a positive and increasing smooth function whose derivatives are all bounded and $p(x) > \delta > 0$, $x \in \mathbb{R}$. The first and second derivatives of $F$ are given by

$$F'(u') \, v = 2 \int_\mathbb{R} p u' \, v \, dx$$

and

$$F''(u') \, v = 2 p v$$

for $v$ in $L^2(\mathbb{R})$. As in the deterministic theory, it can be shown that there exist $c_2, c_3, c_4$ such that

$$\left( \frac{\partial^4 u}{\partial x^4}, pu' \right) = \int_{\mathbb{R}} p \left( \frac{\partial^2 u}{\partial x^2} \right)^2 + 2 \int_{\mathbb{R}} p' \frac{\partial^3 u}{\partial x^3} \, u' \, dx + \int_{\mathbb{R}} p'' \frac{\partial^4 u}{\partial x^4} \, u' \, dx$$

$$\geq \frac{1}{2} \int_{\mathbb{R}} p \left( \frac{\partial^2 u}{\partial x^2} \right)^2 \, dx - c_2 |u'|^2 - c_3 \int_{\mathbb{R}} p' \left( \frac{\partial u}{\partial x} \right)^2 \, dx,$$

$$\left( \frac{\partial^3 u'}{\partial x^3}, pu' \right) = \frac{3}{2} \int_{\mathbb{R}} p' \left( \frac{\partial u}{\partial x} \right)^2 \, dx - \frac{1}{2} \int_{\mathbb{R}} p''(u')^2 \, dx,$$

$$\left( u' \frac{\partial u'}{\partial x}, pu' \right) = -\frac{1}{3} \int_{\mathbb{R}} p'(u')^3 \, dx$$

$$\geq -c_4(1 + |u'|^6) - \frac{1}{2} \int_{\mathbb{R}} p' \left( \frac{\partial u}{\partial x} \right)^2 \, dx.$$
by (4.3). Now, taking the expectation of the Itô formula on \( F(u') \), we obtain
\[
\mathbb{E}(F(u'(t))) + c_7 \int_0^t p'(t) \left( \frac{\partial^2 u'}{\partial x^2} \right)^2 \, dx \, ds + \mathbb{E} \left[ \int_0^t \left( 1 + |u'| \right)^{8} \, ds \right]
\leq F(u_0) + c_7 \int_0^t (1 + |u'|) \, ds
\]
using Lemma 4.1 in the last step. To end the proof of the lemma, we notice that \( p' \) is bounded from below by a positive number on any compact set.

**Lemma 4.3.** The family of laws \( (\mathcal{L}(u')) \) is tight in \( L^2(0, T; L^2_{\text{loc}}) \cap \mathcal{F}(0, T; H^1(-k, k)) \).

**Proof.** We choose a \( k \in \mathbb{N} \) and write
\[
u'(t) = u_0 - \int_0^t \left( \frac{\partial^4 u'}{\partial x^4} - \frac{\partial^2 u'}{\partial x^2} - u' \frac{\partial u'}{\partial x} \right) \, ds + \int_0^t \Phi(u') \, dW'(s).
\]
Since, by an interpolation inequality and a Sobolev embedding, we have
\[
\left| \frac{\partial u'}{\partial x} \right|_{H^{-\gamma}} \leq \frac{1}{2} \| (u')^2 \|_{L^2(-k, k)} \leq C_5(k) \| u' \|^2 \| u' \|_{H^1(-k, k)}^{1/2}
\]
and \( \mathbb{E} \left[ \int_0^t \left( \frac{\partial^4 u'}{\partial x^4} - \frac{\partial^2 u'}{\partial x^2} - u' \frac{\partial u'}{\partial x} \right) \, ds \right] \leq C_3(k). \) (4.5)
Moreover, we infer from Lemma 2.1 in [11] that for any \( \alpha < 1/2 \) and \( p \in \mathbb{N} \)
\[
\mathbb{E} \left[ \left| \int_0^t \Phi(u') \, dW'(s) \right|_0^T \right] \leq c_9 \mathbb{E} \left[ \int_0^T \left| \Phi(u') \right|_0^T \, ds \right] \leq c_9 \kappa \int_0^T \left( |u'| + 1 \right)^{2p} \, ds \leq c_{10}. \) (4.6)
by Lemma 4.1. We first take \( p = 1 \) in (4.6); since both spaces \( W^{1, 2}(0, T; H^{-\gamma}(-k, k)) \) and \( W^{\infty, 2}(0, T; L^2(\mathbb{R})) \) are included in \( W^{\infty, 1}(0, T; H^{-\gamma}(-k, k)) \), we have
\[
\mathbb{E} \left[ |u'|^2 \right]_{W^{\infty, 1}(0, T; H^{-\gamma}(-k, k))} \leq C_4(k). \) (4.7)
Then, taking $0 < \beta < \frac{1}{2}$ and $\alpha, p$ such that $\beta < \alpha - 1/2p$ and using the embedding of $W^{1,2}(0, T; H^{-1}(k, k))$ and $W^{\infty,2}(0, T; L^2(\mathbb{R}))$ in $C(0, T; H^{-1}(k, k))$, we have

$$\mathbb{E}(|u'|^2_{L^2(0, T; H^{-1}(k, k))}) \leq C_\delta(k). \quad (4.8)$$

Let $\eta > 0$, we set

$$\alpha_k = 2^k(C_1(k) + C_4(k) + C_5(k)) \eta^{-1}$$

then we deduce from Chebyshev inequality and Lemma 4.2, (4.7) and (4.8)

$$\mathbb{P}(u' \in A(\{\alpha_k\})) \geq 1 - \eta.$$  

The result follows from Lemma 2.1.

By Prohorov theorem, there exists a converging subsequence, of $(\mathcal{L}(u'))$, (which we still denote by $(\mathcal{L}(u'))$, and by Skohorod theorem (see e.g [13]), which applies since a Fréchet space is metrizable, there exist a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables $\tilde{u}', \tilde{u}$ with values in $L^2(0, T; L^2_{loc}(\mathbb{R}))$ such that

$$\tilde{u}' \to \tilde{u} \quad \text{in} \quad L^2(0, T; L^2_{loc}) \quad \text{and} \quad \mathcal{C}(0, T; H^{-1}_{loc}(\mathbb{R})) \mathcal{P} \ \text{a.s.}$$

and

$$\mathcal{L}(\tilde{u}') = \mathcal{L}(u'), \quad \text{for} \quad \varepsilon > 0. \quad (4.9)$$

Moreover, we also have

$$\mathbb{E}(\sup_{t \in [0, T]} |\tilde{u}'(t)|^{2p}) \leq K_p(1 + |u_0|^{2p})$$

for any $p \in \mathbb{N}$ and

$$\mathbb{E}(|\tilde{u}'|^{2p}_{L^2(0, T; H^{-1}(k, k))}) \leq C_1(k)$$

for any $k \in \mathbb{N}$. It is not difficult to see that we can deduce that for any $k \in \mathbb{N}$

$$\tilde{u}' \to \tilde{u} \quad \text{in} \quad L^2(\tilde{\Omega}; L^2(0, T; H^1(0, k)))$$

weakly.

Let us define

$$M'(t) = u'(t) - u_0 + \int_0^t \left( \varepsilon \frac{\partial^2 u'}{\partial X^2} + \frac{\partial^3 u'}{\partial X^3} + u' \frac{\partial u'}{\partial X} \right) ds$$
\[ \tilde{M}(t) = \tilde{u}(t) - u_0 + \int_0^t \left( \frac{\partial^4 \tilde{u}}{\partial x^4} + \frac{\partial^3 \tilde{u}}{\partial x^3} + \frac{\partial \tilde{u}}{\partial x} \right) ds. \]

We know that \((M'(t))_{t \in [0, T]}\) is a \(L^2(\mathbb{R})\) valued square integrable martingale for the filtration \(\sigma\{\tilde{u}'(s), 0 \leq s \leq t\}\) with quadratic variation \(\int_0^t \Phi(\tilde{u}') \Phi^*(\tilde{u}') \, ds\). Moreover, using martingale inequalities, it can be checked that for any \(p \in \mathbb{N}\) there exists \(K'_p\) such that

\[ \mathbb{E} \left( \sup_{t \in [0, T]} |M'(t)|^{2p} \right) \leq K'_p. \]

Let 0 \(\leq s \leq t \leq T\) and \(\phi\) be a bounded continuous function on \(L^2([0, s]; L^2_{\text{loc}}(\mathbb{R}))\) or \(C([0, s]; H^k_{\text{loc}}(\mathbb{R}))\) and \(a, b \in H^k_{\text{loc}}(\mathbb{R})\) for some \(k \in \mathbb{N}\); since \(M'\) is a martingale, we have

\[ \mathbb{E}((M'(t) - M'(s), a, \phi(u'))) = 0, \]

and

\[ \mathbb{E} \left( \left( (M'(t), a)(M'(t), b) - (M'(s), a)(M'(s), b) \right) \right) = 0. \]

By (4.9) and the definition of \(M'\), \(\tilde{M}\) we also have

\[ \mathbb{E}((\tilde{M}'(t) - \tilde{M}'(s), a, \phi(\tilde{u}')) = 0, \]

\[ \mathbb{E} \left( \left( (\tilde{M}'(t), a)(\tilde{M}'(t), b) - (\tilde{M}'(s), a)(\tilde{M}'(s), b) \right) \right) = 0. \]

Thus \(\tilde{M}'\) is a square integrable martingale for the filtration \(\sigma\{\tilde{u}'(s), 0 \leq s \leq t\}\) with quadratic variation \(\int_0^t \Phi(\tilde{u}') \Phi^*(\tilde{u}') \, ds\).

Let us now define

\[ \tilde{M}(t) = \tilde{u} - u_0 + \int_0^t \frac{\partial^4 \tilde{u}}{\partial x^4} + \frac{\partial \tilde{u}}{\partial x} \, ds. \]
We have
\[ \tilde{M}(t) \to M(t), \quad \tilde{M}(s) \to \tilde{M}(s) \]
\[ \mathbb{P} \text{ a.s. in } H_{-}^{2}(\mathbb{R}) \text{ and} \]
\[ \phi(\tilde{u}) \to \phi(\tilde{u}) \]
\[ \mathbb{P} \text{ a.s.} \] Thus using the continuity of \( \Phi \) for the \( L_{-}^{2}(\mathbb{R}) \) topology, the boundedness of \( \phi \), and \( (4.10) \), it can be checked that
\[ \mathbb{E}(\tilde{M}(t) - \tilde{M}(t), a \phi(\tilde{u}^r)) \to \mathbb{E}(\tilde{M}(t) - \tilde{M}(s), a \phi(\tilde{u}^r)), \]
and
\[ \mathbb{E} \left( \left( (\tilde{M}(t), a)\tilde{M}(t), b) - (\tilde{M}(s), a)\tilde{M}(s), b) \right. \]
\[ \left. - \int_{s}^{t} (\Phi(\tilde{u}^r)^* a, \Phi(\tilde{u}^r)^* b) \ ds \right) \phi(\tilde{u}^r) \]
\[ \to \mathbb{E} \left( \left( (\tilde{M}(t), a)\tilde{M}(t), b) - (\tilde{M}(s), a)\tilde{M}(s), b) \right. \]
\[ \left. - \int_{s}^{t} (\Phi(\tilde{u}^r)^* a, \Phi(\tilde{u}^r)^* b) \ ds \right) \phi(\tilde{u}^r) \].

We deduce that \( \tilde{M} \) is a square integrable martingale on \([0, T] \) for the filtration \( \sigma\{\tilde{u}(s), 0 \leq s \leq t\} \) with quadratic variation \( \int_{u}^{t} \Phi(\tilde{u}) \Phi^{*}(\tilde{u}) \ ds \). To end the proof of Theorem 4.1, it suffices to apply the theorem on representation of martingales in \([10]\). (Theorem 8.2 p. 220.)

It remains to prove Proposition 4.1. The proof is only sketched since it is easier than the proof of Theorem 4.1. We consider a Galerkin approximation of \((4.4), (4.2)\). Let \((e_{i})_{i \in \mathbb{N}}\) be an orthonormal basis of \( L_{-}^{2}(\mathbb{R}) \) and, for \( m \in \mathbb{N}, P_{m} \) the orthogonal projector on \( Sp(e_{0}, ..., e_{m}) \). We consider the following finite dimensional system of stochastic differential equation in \( P_{m}L_{-}^{2}(\mathbb{R}) \)
\[ dh_{m} + P_{m} \left( \frac{\partial^{4} u_{m}^{2}}{\partial x^{4}} + \frac{\partial^{3} u_{m}^{2}}{\partial x^{3}} + \theta \left( \frac{|u_{m}^{2}|}{m} \right) u_{m}^{2} \frac{\partial u_{m}^{2}}{\partial x} \right) = P_{m} \Phi(u_{m}) \ dW_{m}, \]
with initial datum
\[ u_{m}^{2}(0) = P_{m} u_{0}. \]
Here \( \theta \) is a \( C^\infty \) real valued function on \( \mathbb{R} \) such that

\[
\begin{align*}
\theta(x) &= 1, & x &\in [0, 1], \\
0 &\leq \theta(x) \leq 1, & x &\in [1, 2], \\
\theta(x) &= 0, & x &\geq 2.
\end{align*}
\]

All the coefficients of this system have at most linear growth, thus we know that there exists a martingale solution \( u^m \) on a stochastic basis \((\Omega^m, \mathcal{F}^m, \mathbb{P}^m, (\mathcal{F}^m_t)_{t \in [0, T]}, (W^m(t))_{t \in [0, T]} )\). Using Itô’s formula, it is not difficult to prove the following a priori estimates

\[
\mathbb{E}(\sup_{t \in [0, T]} \|u^m\|_{2^p}^2) \leq C_p,
\]

and

\[
\mathbb{E}(\|u^m\|^2_{L^2(0, T; H^1_{\text{loc}}(\mathbb{R}))}) \leq C_1.
\]

(Here the constants might depend on \( \varepsilon \) but not on \( m \).) Then using similar arguments as in the proof of Theorem 4.1, we deduce that the family \((\mathcal{L}(u^m))_{m \in \mathbb{N}}\) is tight in \( L^2(0, T; L^2_{\text{loc}}(\mathbb{R})) \cap \mathcal{C}(0, T; H^1_{\text{loc}}(\mathbb{R})) \) and end the proof.

**APPENDIX**

In this appendix, we prove the following interpolation result, which was used in the proofs of Propositions 3.3 and 3.4.

**Proposition A.1.** Let \( A = L^q(\mathcal{L}^2) \) or \( A = L^q(\Omega) \), with \( 1 < q < +\infty \), and let \( u \) be an \( A \)-valued function of \( x \in \mathbb{R} \). Assume that for some \( p \), with \( 1 < p < +\infty \) and for some \( \sigma > 0 \),

\[
u \in L^p(A) \quad \text{and} \quad D^\sigma u \in L^\sigma(A);
\]

then for any \( \alpha \in [0, \sigma] \), \( D^\alpha u \in L^{\alpha}(A) \), with \( p_\alpha \) defined by \( 1/p_\alpha = 1/p(1-\alpha/\sigma) \). Furthermore, there is a constant \( C \) such that

\[
|D^\alpha u|_{L^\alpha(A)} \leq C |u|_{L^p(A)}^{1-\alpha/\sigma} |D^\sigma u|_{L^\sigma(A)}^{\alpha/\sigma}.
\]

**Proof.** It is adapted from [3], where the case of \( \alpha \) and \( \sigma \) integers, and \( D^\sigma u \in L^\sigma(A) \) with \( \sigma < +\infty \) instead of \( L^\sigma(A) \) is treated. We set

\[
\overline{W}^\sigma_{p, \infty} = \{ v \in \mathcal{F}(L^\sigma(A)) , |\tilde{v}|^\sigma v \in \mathcal{F}(L^\sigma(A)) \}
\]
where we recall that $\mathcal{F}$ is the Fourier transform in $x$; we will prove that
$$T_\zeta: v \mapsto |\zeta|^n v$$
is continuous from $W^\sigma_{p, \infty}$ into $[\mathcal{F}(L^p_0(A)), \mathcal{F}(\text{BMO}(A))]_{\zeta^n}$, for any $\sigma$ with $0 < \sigma < \sigma_0$. The result then follows, since from Lemma 2.1 in [3] and Corollary 1 in [5],
$$[\mathcal{F}(L^p_0(A)), \mathcal{F}(\text{BMO}(A))]_{\zeta^n} = \mathcal{F}([L^p_0(A), \text{BMO}(A)]_{\zeta^n}) = \mathcal{F}(L^p_0(A)).$$

As in [3], it suffices to prove that for a fixed $\hat{u}$ in $W^\sigma_{p, \infty}$, $U: z \mapsto T_\zeta \hat{u}$
is continuous from $\{0 < \text{Re} z < 1\}$ into $[\mathcal{F}(L^p_0(A)) + \mathcal{F}(\text{BMO}(A))]$, analytic from $\{0 < \text{Re} z < 1\}$ into $[\mathcal{F}(L^p_0(A)) + \mathcal{F}(\text{BMO}(A))]$, and satisfies the estimates
$$\sup_{\eta \in \mathbb{H}} |U(i\eta)|_{\mathcal{F}(L^p_0(A))} \leq C_0 |\hat{u}|_{\mathcal{F}(L^p_0(A))},$$  
release{A.1}
$$\sup_{\eta \in \mathbb{H}} |U(1 + i\eta)|_{\mathcal{F}(\text{BMO}(A))} \leq C_1 |\zeta|^n |\hat{u}|_{\mathcal{F}(L^p_0(A))},$$  
release{A.2}
it will then follow from complex interpolation theory (see for example [4]) that $U(x/\sigma) \in [\mathcal{F}(L^p_0(A)), \mathcal{F}(\text{BMO}(A))]_{\zeta^n/\sigma}$ and that
$$\left| \mathcal{F}\left( \frac{\zeta^n}{\sigma} \right) \right|_{[\mathcal{F}(L^p_0(A)), \mathcal{F}(\text{BMO}(A))]_{\zeta^n/\sigma}} \leq C \left| \hat{u} \right|_{\mathcal{F}(L^p_0(A))}^{1 - \frac{n\sigma}{\zeta^n}} \left| \zeta^n \right| |\hat{u}|_{\mathcal{F}(L^p_0(A))}. \qquad \text{(A.3)}$$

Setting $S(z) = D^\sigma u$, i.e. $\mathcal{F}(z) u = U(z) \hat{u}$, (A.1) and (A.2) are equivalent to
$$\sup_{\eta \in \mathbb{H}} |S(i\eta)| u|_{L^p_0(A)} \leq C_0 |\hat{u}|_{L^p_0(A)},$$  
release{A.4}
and
$$\sup_{\eta \in \mathbb{H}} |S(i\eta)| u|_{\text{BMO}_{1/4}} \leq C_1 |\hat{u}|_{L^p_0(A)},$$  
release{A.5}
(A.4) and (A.5) are consequences of Theorem 1.3 in [25] and of the fact that
$$\sup_{\eta \in \mathbb{H}} |S(i\eta)| \frac{\zeta^n}{\sigma} u|_{L^p_0(A)} \leq C |\hat{u}|_{L^p_0(A)}.$$
thanks to the Parseval equality $|u|_{L^2(A)} = |\tilde{u}|_{L^2(A)}$, which can easily be seen to be true in the space $A$ that we consider. To prove that $U$ is analytic from $\{z, 0 < \Re z < 1\}$ into $\tilde{X} = \mathcal{F}(L^p(A)) + \mathcal{F}(\text{BMO}_x(A))$, and is continuous and bounded from $\{0 \leq \Re z \leq 1\}$ into $\tilde{X}$, we use an adaptation of Hörmander–Mihlin theorem to operators with values in $L^p(A)$. For the sake of completeness, we give the statement of the theorem at the end of the appendix (see Lemma A.1). Applying Lemma A.1 with $m(\xi) = |\xi|^{\sigma_0} \ln |\xi|$ for $0 < \Re \sigma_0 < 1$, we obtain $|\xi|^{\sigma_0} \ln |\xi| \hat{u} \in \tilde{X}$. Since $z \mapsto |\xi|^{\sigma_0}$ is a complex-valued analytic function on the strip $\{z \in \mathbb{C}, 0 < \Re z < 1\}$, we have for any circle $\Gamma \subset \{0 < \Re z < 1\}$ containing $z_0$ and any $h \in \mathbb{C}$ sufficiently small,

$$\frac{1}{h} \left( |\xi|^{\sigma_0+h} - |\xi|^{\sigma_0} - \sigma |\xi|^{\sigma_0} \ln |\xi| \right)$$

$$= \frac{1}{2\pi i} \oint_{\Gamma} \left( \frac{1}{z-z_0+h} - \frac{1}{z-z_0} \right) - \frac{1}{(z-z_0)^2} |\xi|^{\sigma_0} \ln |\xi| \, dz.$$ 

Hence, we easily get for $\hat{u} \in W_0^{\infty, \sigma}$,

$$\left| \frac{1}{h} \left( |\xi|^{\sigma_0+h} \hat{u} - |\xi|^{\sigma_0} \hat{u} - \sigma |\xi|^{\sigma_0} \ln |\xi| \right) \hat{u} \right|_{\tilde{X}}$$

$$\leq \frac{1}{2\pi} \int_{\mathbb{R}} \left( \frac{1}{z-z_0+h} - \frac{1}{z-z_0} \right) - \frac{1}{(z-z_0)^2} \left| \left| |\xi|^{\sigma_0} \hat{u} \right|_{\tilde{X}} \right| \, dz$$

$$\leq \frac{1}{2\pi} \sup_{z \in \Gamma} \left| |\xi|^{\sigma_0} \hat{u} \right|_{\tilde{X}} \frac{h}{\int_{\mathbb{R}} \left( \frac{1}{z-z_0+h} - \frac{1}{z-z_0} \right) - \frac{1}{(z-z_0)^2} \, dz}.$$ 

The analyticity of $U$ on $\{z \in \mathbb{C}, 0 < \Re z < 1\}$ follows by applying Lemma A.1 with $m(\xi) = |\xi|^{\sigma_0}/(1 + |\xi|^{\sigma_0})$. The continuity and the boundedness of $U$ on $\{z \in \mathbb{C}, 0 \leq \Re z \leq 1\}$ are proved in the same way.

In the proof of Proposition A.1, we have used the following adaptation of Hörmander–Mihlin multipliers theorem for vector-valued operators. The proof of Lemma A.1 follows from Theorem 1.3 in [25] by using a Littlewood–Paley decomposition, as in the scalar case.

**Lemma A.1.** (Hörmander–Mihlin theorem for $A$-valued operators). Let $A = L^p_x(\mathbb{R})$ or $A = L^p(\Omega)$ as in Proposition A.1. Let $m \in L^{\infty}(\mathbb{R}) \cap C(\mathbb{R}^*)$ satisfying

$$\left| \frac{d}{d\zeta} m(\zeta) \right| \leq C_k |\xi|^{-k} \quad \text{for} \quad k = 0, 1, \xi \neq 0.$$
and let $T_m$ be the operator defined on $L^2(\mathbb{R}; A)$ by

$$T_m u(\xi) = m(\xi) \tilde{u}(\xi).$$

Then $T_m$ is continuous from $L^p(A)$ into itself, and from $L^\infty(A)$ into $BMO(A)$.

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