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The Non-equivalent Circulant D-Optimal Designs for $n \equiv 2 \bmod 4$, $n \leqslant 54$, $n = 66$

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All non-equivalent circulant D-optimal designs for $n \equiv 2 \bmod 4$, $n \leqslant 54$ and $n = 66$ are given and were found by an exhaustive search. There is a unique non-equivalent circulant design for each value of $n \leqslant 18$, 3 for $n = 26$ and $n = 30$, 8 for $n = 38$, 31 for $n = 42$, 17 for $n = 46$, 39 for $n = 50$, 48 for $n = 54$, and 1025 for $n = 66$. These are presented in tables in the form of the corresponding non-equivalent supplementary difference sets. Most of the given designs are new.

1. INTRODUCTION

If $n \equiv 2 \bmod 4$ and A, B are $n/2 \times n/2$ commuting matrices, with elements ± 1, such that

$$AA^T + BB^T = (n-2)\, I_{n/2} + 2J_{n/2}, \tag{1}$$

where $J_{n/2}$ is an $n/2 \times n/2$ matrix of 1's, then the $n \times n$ matrix

$$R = \begin{pmatrix} A & B \\ -B^T & A^T \end{pmatrix}$$

has the maximum determinant (see [4, 6]) among all $n \times n$, ± 1 matrices. Such matrices are called D-optimal designs of order n.

Now form the two sets $P = \{p_1, p_2, ..., p_r\}$ and $Q = \{q_1, q_2, ..., q_s\}$ where p_i, q_j denote the positions of -1's in the first row of A, B respectively.

If the matrices A, B are circulant, then they satisfy (1) if and only if (see [2]) they are supplementary difference sets $2 - \{n/2; r, s; \lambda\}$, where $\lambda = r + s - (n-2)/4$ and $s \geqslant r \geqslant 0$ are found from

$$(n/2 - 2r)^2 + (n/2 - 2s)^2 = 2n - 2. \tag{2}$$

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Hence the construction of the two circulant matrices A, B satisfying (1) is equivalent to the construction of the corresponding supplementary difference sets. For $n = 22$, 34, 58, D-optimal designs satisfying (1) do not exist because $n-1$ is not the sum of two squares (see [2]).

In this paper we construct all non-equivalent circulant D-optimal designs for $n \leqslant 54$ and $n = 66$. D-optimal designs for these values of n have been given before in the literature (see [2–15]), and Yang [13] published all non-equivalent circulant designs for $n \leqslant 38$. Bridges *et al.* [1] and Trung [9] have constructed a D-optimal design for $n = 82$ which is not of circulant type. Here most of the D-optimal designs we give for $n \geqslant 42$ are new.

In Section 2 we define and give some results on non-equivalent circulant designs, in Section 3 we describe briefly the algorithm, and in Section 4 we present the tables of the non-equivalent circulant designs for $n \leqslant 54$. For $n = 66$, there are 1025 such designs and since the space is limited, we give only 30 here, the remaining are available on request.

2. Non-equivalent Designs

From now on we assume that A, B are circulant and let a_i and b_i, $i = 0, 1, \ldots, m-1$, where $m = n/2$, be the element of their first row.

Define the non-periodic autocorrelation function

$$N_A(t) = \sum_{i=0}^{m-t-1} a_i a_{i+t} \qquad t = 0, 1, \ldots, m-1, \tag{3}$$

then (1) is equivalent to

$$\begin{aligned} N_A(0) + N_B(0) &= 2m && \text{if} \quad t = 0 \\ N_A(t) + N_B(t) + N_A(m-t) + N_B(m-t) &= 2 && \text{if} \quad 1 \leqslant t \leqslant m-1 \end{aligned} \tag{4}$$

Let

$$A(z) = a_0 + a_1 z + \cdots + a_{m-1} z^{m-1}$$
$$B(z) = b_0 + b_1 z + \cdots + b_{m-1} z^{m-1}$$

be polynomials associated with A and B, then

$$A(z)A(z^{-1}) = N_A(0) + \sum_{t=1}^{m-1} N_A(t)(z^t + z^{-t}) \qquad z \neq 0$$

and

$$A(z)\,A(z^{-1})+B(z)\,B(z^{-1})=N_A(0)+N_B(0)+\sum_{t=1}^{m-1}(N_A(t)+N_B(t)\\ +z^{-m}(N_A(m-t)+N_B(m-t)))z^t$$

If A and B satisfy (1) which is equivalent to (4) and $z^m=1$, then

$$A(z)\,A(z^{-1})+B(z)\,B(z^{-1})=\begin{cases}4m-2 & \text{if } z=1\\ 2m-2 & \text{if } z^m=1,\quad z\neq 1\end{cases} \tag{5}$$

Therefore (1), (4), and (5) are equivalent.

Now if $(A(z), B(z))$ is a pair of $(m-1)$th-degree polynomials, with coefficients ± 1, satisfying (5), then (5) is also satisfied by the following pairs:

(i) $(-A(z), B(z))$
(ii) $(A(z), -B(z))$
(iii) $(B(z), A(z))$
(iv) $(z^{m-1}A(z^{-1}), B(z))$
(v) $(A(z), z^{m-1}B(z^{-1}))$
(vi) $(z^uA(z), z^vB(z))$
(vii) $(A(z^d), B(z^d))$ $\quad (d, m)=1$.

All powers of z are taken mod m.

This is because (i) and (iv) leave $A(z)\,A(z^{-1})$ invariant, (ii) and (v) leave $B(z)\,B(z^{-1})$ invariant, (iii), (vi), and (vii) leave $A(z)\,A(z^{-1})+B(z)\,B(z^{-1})$ invariant.

The designs produced from A, B by applying operations (i)–(vii) are called equivalent. In the tables in Section 4, we give one design from every equivalent class.

3. The Algorithm

Here we describe briefly the algorithm.

Let

$$x=A(1)=\sum_{i=0}^{m-1}a_i,\qquad y=B(1)=\sum_{i=0}^{m-1}b_i,$$

then x, y are odd, found from (5), i.e.,

$$x^2 + y^2 = 4m - 2,$$

and we can always take $x \geqslant y > 0$ by applying (i), (ii), and (iii) of Section 2. Let

$$x_{ik} = \sum_{j \equiv i \bmod k}^{m-1} a_j, \qquad y_{ik} = \sum_{j \equiv i \bmod k}^{m-1} b_j \qquad i = 0, ..., k-1,$$

then

$$\sum_{i=0}^{k-1} x_{ik} = x, \qquad \sum_{i=0}^{k-1} y_{ik} = y$$

$$|x_{ik}|, |y_{ik}| \leqslant [(m-1-i)/k] + 1 \tag{6}$$

$$x_{ik}, y_{ik} \equiv ([m-1-i)/k] + 1) \bmod 2.$$

We applied two different algorithms depending on m being prime or not.

3.1. *m Prime*

(i) Take $k = 2$, find $x_{02}, x_{12}, y_{02}, y_{12}$ satisfying (6), i.e.,

$$x_{02} + x_{12} = x, \qquad |x_{02} - x_{12}| = 1,$$

$$y_{02} + y_{12} = y, \qquad |y_{02} - y_{12}| = 1;$$

this is always possible by applying (vi).

(ii) Find $x_{i4}, y_{i4}, i = 0, 1, 2, 3$, from

$$x_{04} + x_{24} = x_{02}, \qquad x_{14} + x_{34} = x_{12},$$

$$y_{04} + y_{24} = y_{02}, \qquad y_{14} + y_{34} = y_{12}.$$

By applying (iv), (v) we can take

$$x_{34} \leqslant x_{14}, \quad y_{34} \leqslant y_{14} \qquad \text{when} \quad m \equiv 1 \bmod 4$$

$$x_{04} \leqslant x_{24}, \quad y_{04} \leqslant y_{24} \qquad \text{when} \quad m \equiv 3 \bmod 4.$$

(iii) Set $k = 8$, find $x_{i8}, y_{i8}, i = 0, ..., 7$, from x_{i4}, y_{i4} and continue until $k \geqslant m$.

(iv) Examine the sequences

$$x_{ik},\ y_{ik} \qquad i=0, ..., m-1$$

and keep the sequences satisfying (4).

(v) Divide these sequences into non-equivalence classes and take one from every equivalence class.

As m increases, the number of generated sequences increases and for $m \leqslant 23$ we examined all of them. For $m \geqslant 31$, m prime, the situation becomes more difficult to handle.

3.2. *m Not a Prime*

Let $m=pq$, $1<q<m$, q prime, then

(i) Take $k=q$ and find x_{ik}, y_{ik}, $i=0, ..., k-1$, satisfying (6). By applying (vi) take

$$x_{0k}=\max(x_{ik}), \qquad y_{0k}=\max(y_{ik})$$

and by applying (vii) take x_{1k} to be the second largest among x_{ik}. Also by applying (v) and (vi) take

$$y_{(k-1)/2,\,k} \geqslant y_{(k+1)/2,\,k}.$$

The number of sequences we examine can be reduced further if we take

$$x_{(k-1)/2,\,k} \geqslant x_{(k+3)/2,\,k} \qquad \text{whenever} \quad x_{0k}=x_{1k}$$

$$y_{(k-3)/2,\,k} \geqslant y_{(k+3)/2,\,k} \qquad \text{whenever} \quad y_{(k-1)/2,\,k}=y_{(k+1)/2,\,k}.$$

This can be done by applying (v) and (vi) of Section 2.

(ii) Set $z^k=1$, then $z^m=1$ and

$$A(z)=\sum_{i=0}^{k-1} x_{ik} z^i, \qquad B(z)=\sum_{i=0}^{k-1} y_{ik} z^i$$

satisfy (5).

Knowing x_{ik}, y_{ik}, compute from (3), $N_X(t)$, $N_Y(t)$, $t=0, ..., k-1$.

(iii) Examine if

$$N_X(0)+N_Y(0)=2m+2(m/k-1)$$

$$N_X(t)+N_Y(t)+N_X(k-t)+N_Y(k-t)=2(m/k) \quad \text{if} \quad 1 \leqslant t \leqslant (k-1)/2$$

If the answer is yes, continue.

(iv) Find $x_{i,2k}$, $y_{i,2k}$ from (6) and go to (iii).

(v) Stop when $k = m$ and keep the sequences x_{im}, y_{im} satisfying (6).

(iv) Divide the sequences into non-equivalence classes and keep one from every equivalence class.

This algorithm was applied for $m = 25, 27, 33$.

4. Tables

In this section we give the tables. The numbers inside the parentheses denote the position of $-1'$ in the first row of A and B, respectively. These sequences give also the non-equivalent supplementary difference sets

$$2 - \{n/2; r, s; \lambda\}, \qquad \text{where} \quad \lambda = r + s - (n-2)/4.$$

For $n \leqslant 54$, we give representatives of all the non-equivalence classes of D-optimal circulant designs.

For $n = 66$, we give only 30 representatives, because there are 1025 such non-equivalence classes of D-optimal designs and the space here is limited; the remaining are available on request.

All non-equivalence classes of circulant designs for $n \leqslant 38$ have appeared before in [13].

For $n = 42$, a representative of the following equivalence classes of designs, as listed in our tables, has appeared before: 1, 2, 4, 6, 8, 10, 17, 20, and 21 appeared in [15] and class 29 appeared in [14]; the remaining 21 classes of our list are new.

For $n = 46$, class 11 of our list appeared in [14]; the remaining 16 classes of our list are new.

For $n = 50$, class 36 of our list appeared in [14]; the remaining 38 classes of our list are new.

For $n = 54$, classes 25, 40 of our list appeared in [14]; the remaining 46 clases of our list are new.

For $n = 66$, $2 - (33; 12, 13; 9)$, there are 509 non-equivalence classes, class 425 of our list appeared in [15], and class 431 of our list appeared in [4]; the remaining 507 classes of our list are new.

For $n = 66$, $2 - (33; 11, 15; 10)$, there are 516 non-equivalence classes; all of them are new.

TABLE I

All Non-equivalent Circulant D-Optimal Designs for $n \equiv 2 \bmod 4$, $n \leqslant 54$, $n = 66$

$n = 6;\quad 2-(3;\,0,1;\,0)$

$A_1 = \{\ \}$ $B_1 = \{1\}$

$n = 10;\quad 2-(5;\,1,1;\,0)$

$A_1 = \{4\}$ $B_1 = \{4\}$

$n = 14;\quad 2-(7;\,1,3;\,1)$

$A_1 = \{6\}$ $B_1 = \{3,5,6\}$

$n = 18;\quad 2-(9;\,2,3;\,1)$

$A_1 = \{7,8\}$ $B_1 = \{3,6,8\}$

$n = 26;\quad 2-(13;\,3,6;\,3)$

$A_1 = \{8,11,12\}$ $B_1 = \{3,4,5,7,10,12\}$
$A_2 = \{8,11,12,\}$ $B_2 = \{4,5,7,9,11,12\}$

$n = 26;\quad 2-(13;\,4,4;\,2)$

$A_1 = \{5,7,8,12\}$ $B_1 = \{5,7,8,12\}$

$n = 30;\quad 2-(15;\,4,6;\,3)$

$A_1 = \{6,9,12,14\}$ $B_1 = \{3,7,8,9,10,14\}$
$A_2 = \{8,9,12,14\}$ $B_2 = \{4,5,7,8,12,14\}$
$A_3 = \{8,9,12,14\}$ $B_3 = \{5,6,9,11,13,14\}$

$n = 38;\quad 2-(19;\,6,7;\,4)$

$A_1 = \{6,7,8,10,13,18\}$ $B_1 = \{4,5,7,8,12,14,18\}$
$A_2 = \{6,7,8,10,13,18\}$ $B_2 = \{5,6,10,12,14,15,18\}$
$A_3 = \{7,8,9,11,13,18\}$ $B_3 = \{4,5,9,12,15,16,18\}$
$A_4 = \{7,10,13,14,16,18\}$ $B_4 = \{6,7,8,11,13,17,18\}$
$A_5 = \{7,8,9,12,15,18\}$ $B_5 = \{6,7,11,12,14,16,18\}$
$A_6 = \{8,10,11,15,16,18\}$ $B_6 = \{4,5,7,8,12,14,18\}$
$A_7 = \{8,10,11,15,16,18\}$ $B_7 = \{5,6,10,12,14,15,18\}$
$A_8 = \{9,10,13,15,17,18\}$ $B_8 = \{4,5,7,11,13,14,18\}$

$n = 42;\quad 2-(21;\,6,10;\,6)$

$A_1 = \{5,6,8,12,15,20\}$ $B_1 = \{4,8,9,12,14,16,17,18,19,20\}$
$A_2 = \{6,9,12,13,18,20\}$ $B_2 = \{4,8,9,12,14,16,17,18,19,20\}$
$A_3 = \{6,11,13,16,19,20\}$ $B_3 = \{6,7,8,11,12,14,16,17,18,20\}$
$A_4 = \{7,13,17,18,19,20\}$ $B_4 = \{3,4,7,10,11,13,15,16,18,20\}$
$A_5 = \{7,9,10,14,16,20\}$ $B_5 = \{4,5,8,12,13,14,15,17,18,20\}$
$A_6 = \{7,9,10,14,16,20\}$ $B_6 = \{4,5,6,7,8,11,12,15,17,20\}$

TABLE I—*Continued*

$A_7 = \{7,9,11,14,19,20\}$	$B_7 = \{5,7,8,9,11,12,14,15,19,20\}$
$A_8 = \{8,13,16,18,19,20\}$	$B_8 = \{3,4,6,10,11,12,14,15,18,20\}$
$A_9 = \{8,10,11,16,18,20\}$	$B_9 = \{4,7,8,11,12,13,14,17,18,20\}$
$A_{10} = \{8,9,14,16,17,20\}$	$B_{10} = \{4,5,6,8,10,12,13,15,16,20\}$
$A_{11} = \{8,12,14,15,18,20\}$	$B_{11} = \{4,5,6,10,13,14,15,17,18,20\}$
$A_{12} = \{8,10,11,14,18,20\}$	$B_{12} = \{5,6,10,11,13,15,16,18,19,20\}$
$A_{13} = \{9,10,11,15,18,20\}$	$B_{13} = \{3,6,7,10,12,13,14,15,18,20\}$
$A_{14} = \{9,13,16,18,19,20\}$	$B_{14} = \{3,4,5,9,12,13,15,17,18,20\}$
$A_{15} = \{9,12,13,17,18,20\}$	$B_{15} = \{3,4,6,10,12,13,14,15,18,20\}$
$A_{16} = \{9,11,12,15,16,20\}$	$B_{16} = \{4,5,7,9,10,12,16,18,19,20\}$
$A_{17} = \{9,12,15,16,18,20\}$	$B_{17} = \{4,6,8,9,12,13,17,18,19,20\}$
$A_{18} = \{9,11,12,15,16,20\}$	$B_{18} = \{4,5,6,8,10,13,16,18,19,20\}$
$A_{19} = \{9,10,14,17,19,20\}$	$B_{19} = \{4,6,8,10,11,12,13,16,19,20\}$
$A_{20} = \{9,12,13,16,18,20\}$	$B_{20} = \{4,5,6,7,11,12,14,16,17,20\}$
$A_{21} = \{9,10,14,17,19,20\}$	$B_{21} = \{5,6,8,10,12,13,14,17,18,20\}$
$A_{22} = \{10,12,13,15,19,20\}$	$B_{22} = \{3,5,6,9,11,15,16,18,19,20\}$
$A_{23} = \{10,11,13,14,19,20\}$	$B_{23} = \{3,6,7,8,11,13,14,16,18,20\}$
$A_{24} = \{10,13,14,18,19,20\}$	$B_{24} = \{3,5,6,8,12,13,15,16,18,20\}$
$A_{25} = \{10,11,13,14,19,20\}$	$B_{25} = \{3,4,5,7,8,11,13,15,18,20\}$
$A_{26} = \{10,11,13,17,18,20\}$	$B_{26} = \{3,7,8,10,12,13,16,18,19,20\}$
$A_{27} = \{10,11,14,15,18,20\}$	$B_{27} = \{4,5,7,9,10,12,16,18,19,20\}$
$A_{28} = \{10,11,13,15,19,20\}$	$B_{28} = \{4,6,7,10,13,14,15,17,19,20\}$
$A_{29} = \{10,11,13,15,19,20\}$	$B_{29} = \{4,6,7,8,11,12,14,17,18,20\}$
$A_{30} = \{10,11,13,14,19,20\}$	$B_{30} = \{4,6,8,9,12,13,15,17,19,20\}$
$A_{31} = \{10,13,14,18,19,20\}$	$B_{31} = \{4,5,7,8,11,13,15,17,18,20\}$

$n = 46;\quad 2-(23;\ 7,\ 10;\ 6)$

$A_1 = \{8,11,15,16,17,19,22\}$	$B_1 = \{4,5,6,8,9,14,16,18,21,22\}$
$A_2 = \{8,12,16,17,19,20,22\}$	$B_2 = \{4,5,7,9,14,15,18,20,21,22\}$
$A_3 = \{9,13,15,18,20,21,22\}$	$B_3 = \{4,6,7,10,11,12,16,19,20,22\}$
$A_4 = \{9,11,12,13,17,19,22\}$	$B_4 = \{4,7,8,11,13,14,18,19,20,22\}$
$A_5 = \{9,10,12,14,16,19,22\}$	$B_5 = \{4,5,6,10,13,17,18,19,21,22\}$
$A_6 = \{9,12,14,16,18,21,22\}$	$B_6 = \{4,5,7,8,9,14,15,16,19,22\}$
$A_7 = \{9,12,13,14,19,21,22\}$	$B_7 = \{4,7,10,11,12,14,16,18,21,22\}$
$A_8 = \{10,12,14,17,20,21,22\}$	$B_8 = \{3,6,7,8,9,13,14,17,20,22\}$
$A_9 = \{10,11,12,14,17,21,22\}$	$B_9 = \{3,6,7,8,11,13,14,16,20,22\}$
$A_{10} = \{10,13,14,16,20,21,22\}$	$B_{10} = \{3,6,7,8,12,14,16,17,19,22\}$
$A_{11} = \{10,11,12,15,19,20,22\}$	$B_{11} = \{4,5,7,9,10,13,14,16,20,22\}$
$A_{12} = \{10,11,13,15,19,21,22\}$	$B_{12} = \{4,7,9,13,14,16,17,20,21,22\}$
$A_{13} = \{10,12,13,14,16,21,22\}$	$B_{13} = \{4,5,7,10,12,14,17,18,21,22\}$
$A_{14} = \{10,11,12,16,19,20,22\}$	$B_{14} = \{4,7,8,10,13,14,15,18,20,22\}$
$A_{15} = \{11,12,13,15,19,21,22\}$	$B_{15} = \{3,5,8,10,13,16,17,20,21,22\}$
$A_{16} = \{11,12,14,17,20,21,22\}$	$B_{16} = \{3,4,7,9,11,14,18,20,21,22\}$
$A_{17} = \{11,12,13,14,18,21,22\}$	$B_{17} = \{3,4,7,9,10,13,15,18,20,22\}$

$n = 50;\quad 2-(25;\ 9,\ 9;\ 6)$

$A_1 = \{6,8,10,13,14,15,16,20,24\}$	$B_1 = \{6,9,10,11,15,18,21,23,24\}$

TABLE I—*Continued*

$A_2 = \{6,9,11,12,13,14,18,22,24\}$	$B_2 = \{6,9,10,15,16,18,20,23,24\}$
$A_3 = \{7,8,15,16,18,20,21,22,24\}$	$B_3 = \{4,8,9,11,14,17,18,22,24\}$
$A_4 = \{7,8,9,10,12,13,17,20,24\}$	$B_4 = \{4,6,10,12,15,16,21,22,24\}$
$A_5 = \{7,8,11,12,13,14,16,23,24\}$	$B_5 = \{4,6,10,13,16,18,20,23,24\}$
$A_6 = \{7,11,14,15,19,20,21,22,24\}$	$B_6 = \{5,6,10,12,16,19,21,22,24\}$
$A_7 = \{7,9,13,16,17,18,19,21,24\}$	$B_7 = \{5,7,8,12,14,19,20,23,24\}$
$A_8 = \{7,8,12,14,15,16,18,21,24\}$	$B_8 = \{6,8,10,13,18,19,20,23,24\}$
$A_9 = \{7,9,10,11,13,16,17,23,24\}$	$B_9 = \{6,8,10,11,15,16,19,21,24\}$
$A_{10} = \{7,9,12,13,15,17,18,22,24\}$	$B_{10} = \{6,10,11,13,14,17,22,23,24\}$
$A_{11} = \{7,8,9,12,13,15,17,22,24\}$	$B_{11} = \{6,7,10,13,16,18,19,20,24\}$
$A_{12} = \{7,8,9,11,12,16,18,23,24\}$	$B_{12} = \{7,10,12,14,17,18,20,23,24\}$
$A_{13} = \{7,9,13,14,16,18,21,22,24\}$	$B_{13} = \{7,10,12,13,14,19,20,23,24\}$
$A_{14} = \{7,8,11,12,14,16,18,21,24\}$	$B_{14} = \{8,9,11,13,16,17,22,23,24\}$
$A_{15} = \{8,10,11,12,14,16,17,23,24\}$	$B_{15} = \{4,6,7,10,14,15,19,21,24\}$
$A_{16} = \{8,12,13,14,16,17,21,23,24\}$	$B_{16} = \{5,7,9,12,17,18,20,23,24\}$
$A_{17} = \{8,10,14,17,19,20,22,23,24\}$	$B_{17} = \{5,6,10,13,14,16,18,23,24\}$
$A_{18} = \{8,11,14,16,17,18,22,23,24\}$	$B_{18} = \{5,8,9,10,13,17,20,22,24\}$
$A_{19} = \{8,10,14,17,19,20,22,23,24\}$	$B_{19} = \{5,7,12,13,15,19,20,23,24\}$
$A_{20} = \{8,12,13,16,19,21,22,23,24\}$	$B_{20} = \{5,6,11,14,16,18,20,23,24\}$
$A_{21} = \{8,9,12,14,16,18,21,23,24\}$	$B_{21} = \{5,6,11,16,19,20,22,23,24\}$
$A_{22} = \{8,9,12,13,14,17,19,21,24\}$	$B_{22} = \{5,6,7,13,16,19,20,22,24\}$
$A_{23} = \{8,9,11,16,17,19,20,23,24\}$	$B_{23} = \{6,8,11,12,13,16,18,22,24\}$
$A_{24} = \{8,9,11,12,13,16,18,22,24\}$	$B_{24} = \{6,7,9,11,14,17,18,23,24\}$
$A_{25} = \{8,9,14,15,17,18,20,22,24\}$	$B_{25} = \{6,7,8,11,14,18,19,22,24\}$
$A_{26} = \{8,9,11,12,14,17,19,23,24\}$	$B_{26} = \{6,7,10,11,13,15,17,23,24\}$
$A_{27} = \{8,10,14,15,17,20,22,23,24\}$	$B_{27} = \{7,9,12,13,15,19,20,23,24\}$
$A_{28} = \{8,9,12,13,16,19,21,22,24\}$	$B_{28} = \{7,9,11,13,16,17,18,23,24\}$
$A_{29} = \{9,10,12,17,18,19,20,23,24\}$	$B_{29} = \{3,7,9,11,12,16,18,21,24\}$
$A_{30} = \{9,10,13,16,18,20,21,22,24\}$	$B_{30} = \{4,6,11,12,15,16,21,22,24\}$
$A_{31} = \{9,11,12,13,16,17,21,23,24\}$	$B_{31} = \{5,7,9,12,14,15,18,23,24\}$
$A_{32} = \{9,12,13,16,18,21,22,23,24\}$	$B_{32} = \{5,6,10,14,16,18,21,23,24\}$
$A_{33} = \{9,10,11,13,15,18,20,21,24\}$	$B_{33} = \{5,6,7,11,12,15,19,22,24\}$
$A_{34} = \{9,11,12,14,16,18,22,23,24\}$	$B_{34} = \{5,6,10,13,14,15,18,21,24\}$
$A_{35} = \{9,12,13,14,17,19,21,23,24\}$	$B_{35} = \{5,6,10,12,13,18,21,22,24\}$
$A_{36} = \{9,10,13,14,16,18,22,23,24\}$	$B_{36} = \{6,7,8,11,13,15,18,21,24\}$
$A_{37} = \{9,10,12,13,17,18,20,22,24\}$	$B_{37} = \{6,8,11,14,17,18,22,23,24\}$
$A_{38} = \{9,10,12,13,15,17,19,23,24\}$	$B_{38} = \{6,7,8,11,14,17,19,23,24\}$
$A_{39} = \{9,10,13,15,17,20,22,23,24\}$	$B_{39} = \{7,8,11,13,16,17,19,23,24\}$

$n = 54; \quad 2-(27;\ 9,\ 11;\ 7)$

$A_1 = \{5,6,10,14,16,18,19,21,26\}$	$B_1 = \{5,6,7,8,10,12,13,16,22,25,26\}$
$A_2 = \{5,10,11,15,18,19,22,24,26\}$	$B_2 = \{6,8,11,12,13,14,16,17,23,24,26\}$
$A_3 = \{6,8,11,16,17,19,20,21,26\}$	$B_3 = \{5,9,12,13,15,16,18,20,24,25,26\}$
$A_4 = \{6,8,9,11,16,17,22,23,26\}$	$B_4 = \{5,10,11,13,14,15,17,18,22,24,26\}$
$A_5 = \{6,8,10,12,13,18,22,23,26\}$	$B_5 = \{6,7,12,13,14,15,17,18,21,24,26\}$
$A_6 = \{6,10,14,15,16,18,21,24,26\}$	$B_6 = \{6,9,10,12,17,18,19,22,23,24,26\}$
$A_7 = \{6,9,11,12,13,15,18,23,26\}$	$B_7 = \{6,8,12,13,14,17,18,22,24,25,26\}$

TABLE I—*Continued*

$A_8 = \{6,10,12,13,18,19,21,22,26\}$	$B_8 = \{6,8,9,10,11,14,16,19,20,24,26\}$
$A_9 = \{6,9,10,12,14,18,19,24,26\}$	$B_9 = \{7,8,9,10,12,14,15,18,21,25,26\}$
$A_{10} = \{6,7,11,15,17,20,23,24,26\}$	$B_{10} = \{7,11,13,14,18,19,21,23,24,25,26\}$
$A_{11} = \{7,10,11,13,14,15,20,23,26\}$	$B_{11} = \{4,6,11,12,13,17,19,21,22,23,26\}$
$A_{12} = \{7,9,16,18,19,21,22,25,26\}$	$B_{12} = \{4,7,8,10,12,14,15,20,21,22,26\}$
$A_{13} = \{7,10,15,17,20,21,22,24,26\}$	$B_{13} = \{5,6,9,10,12,14,15,18,24,25,26\}$
$A_{14} = \{7,8,12,17,19,20,23,25,26\}$	$B_{14} = \{5,7,9,14,15,16,18,19,21,22,26\}$
$A_{15} = \{7,10,12,14,18,19,23,24,26\}$	$B_{15} = \{5,6,7,8,10,12,13,16,22,25,26\}$
$A_{16} = \{7,9,14,17,18,20,23,24,26\}$	$B_{16} = \{6,8,10,11,12,13,17,18,22,25,26\}$
$A_{17} = \{7,8,11,13,18,19,20,22,26\}$	$B_{17} = \{7,9,10,13,14,16,18,19,23,24,26\}$
$A_{18} = \{7,8,9,14,16,19,22,23,26\}$	$B_{18} = \{7,9,11,12,16,17,18,20,22,23,26\}$
$A_{19} = \{7,9,12,15,16,18,21,22,26\}$	$B_{19} = \{7,9,11,14,15,16,21,22,23,25,26\}$
$A_{20} = \{8,13,14,15,17,19,23,25,26\}$	$B_{20} = \{4,9,11,12,14,18,19,22,23,25,26\}$
$A_{21} = \{8,9,14,17,19,22,23,24,26\}$	$B_{21} = \{5,9,12,13,15,16,18,20,24,25,26\}$
$A_{22} = \{8,9,14,16,17,19,22,23,26\}$	$B_{22} = \{5,6,7,9,10,14,16,18,20,21,26\}$
$A_{23} = \{8,10,11,16,17,21,23,24,26\}$	$B_{23} = \{5,7,9,10,14,15,16,18,19,22,26\}$
$A_{24} = \{8,10,12,16,17,20,23,24,26\}$	$B_{24} = \{5,6,7,8,11,12,13,16,21,23,26\}$
$A_{25} = \{8,9,13,17,18,20,23,24,26\}$	$B_{25} = \{5,7,9,10,12,13,17,19,20,21,26\}$
$A_{26} = \{8,13,14,15,16,19,22,25,26\}$	$B_{26} = \{6,7,9,11,14,15,19,20,22,24,26\}$
$A_{27} = \{8,12,13,18,19,21,23,25,26\}$	$B_{27} = \{6,8,11,14,15,17,21,22,23,25,26\}$
$A_{28} = \{8,9,11,12,15,16,21,24,26\}$	$B_{28} = \{6,8,12,14,17,18,19,22,24,25,26\}$
$A_{29} = \{8,12,14,17,19,22,23,25,26\}$	$B_{29} = \{6,7,11,13,17,18,21,23,24,25,26\}$
$A_{30} = \{8,13,14,15,17,20,22,25,26\}$	$B_{30} = \{6,7,8,10,12,15,16,20,22,23,26\}$
$A_{31} = \{9,10,11,13,14,17,20,22,26\}$	$B_{31} = \{4,5,6,7,11,13,16,20,21,24,26\}$
$A_{32} = \{9,10,11,15,16,18,19,23,26\}$	$B_{32} = \{4,5,6,8,10,12,17,20,22,23,26\}$
$A_{33} = \{9,11,13,14,16,17,20,25,26\}$	$B_{33} = \{4,6,8,9,10,15,16,18,22,23,26\}$
$A_{34} = \{9,12,13,17,19,20,22,24,26\}$	$B_{34} = \{4,5,10,12,13,16,21,22,24,25,26\}$
$A_{35} = \{9,11,12,14,18,20,21,25,26\}$	$B_{35} = \{4,8,9,11,12,14,18,19,20,22,26\}$
$A_{36} = \{9,10,11,14,17,18,20,24,26\}$	$B_{36} = \{4,6,7,9,11,12,13,18,22,23,26\}$
$A_{37} = \{9,10,11,14,16,18,20,23,26\}$	$B_{37} = \{5,7,8,12,15,16,20,21,22,23,26\}$
$A_{38} = \{9,11,13,16,17,20,23,25,26\}$	$B_{38} = \{5,6,11,13,16,20,21,22,24,25,26\}$
$A_{39} = \{9,11,13,14,18,19,20,23,26\}$	$B_{39} = \{5,8,9,10,12,16,18,21,24,25,26\}$
$A_{40} = \{9,11,14,16,17,20,24,25,26\}$	$B_{40} = \{5,7,8,9,11,13,14,18,21,25,26\}$
$A_{41} = \{9,11,12,17,18,21,22,24,26\}$	$B_{41} = \{6,8,9,14,15,17,19,21,22,25,26\}$
$A_{42} = \{9,11,14,17,18,20,21,25,26\}$	$B_{42} = \{6,7,8,10,14,16,19,20,21,24,26\}$
$A_{43} = \{9,12,14,15,18,22,23,24,26\}$	$B_{43} = \{6,7,10,12,14,17,18,19,23,24,26\}$
$A_{44} = \{9,10,11,15,16,19,23,24,26\}$	$B_{44} = \{7,9,10,13,15,16,18,20,22,25,26\}$
$A_{45} = \{10,11,13,14,16,19,21,25,26\}$	$B_{45} = \{4,5,8,12,15,16,17,18,22,24,26\}$
$A_{46} = \{10,12,13,15,16,19,21,25,26\}$	$B_{46} = \{5,6,7,11,14,15,19,21,22,24,26\}$
$A_{47} = \{11,12,13,15,18,20,22,25,26\}$	$B_{47} = \{4,5,7,8,11,15,16,17,21,23,26\}$
$A_{48} = \{11,12,14,17,18,21,23,25,26\}$	$B_{48} = \{4,5,6,8,10,15,16,18,22,23,26\}$

$n = 66$; $2-(33;\ 12,\ 13;\ 9)$

$A_1 = \{5,8,9,10,11,16,18,20,23,27,28,32\}$
$B_1 = \{5,7,11,15,18,23,24,25,26,28,29,31,32\}$

$A_2 = \{5,11,12,13,16,17,18,20,21,27,30,32\}$
$B_2 = \{5,7,8,9,11,14,15,19,22,27,29,31,32\}$

TABLE I—*Continued*

$A_3 = \{5,7,10,11,15,17,18,21,26,30,31,32\}$
$B_3 = \{6,11,12,14,15,16,18,19,21,23,29,30,32\}$

$A_4 = \{5,6,8,9,10,11,12,18,23,27,31,32\}$
$B_4 = \{6,8,10,13,15,16,19,21,24,25,29,31,32\}$

$A_5 = \{5,10,12,17,18,22,24,25,26,27,28,32\}$
$B_5 = \{6,7,8,10,11,15,16,19,22,25,28,30,32\}$

$A_6 = \{5,9,11,16,19,20,21,23,27,28,29,32\}$
$B_6 = \{6,12,13,14,15,17,19,22,25,27,28,31,32\}$

$A_7 = \{5,10,14,18,19,20,21,24,26,28,31,32\}$
$B_7 = \{7,10,11,12,14,15,20,21,23,27,29,30,32\}$

$A_8 = \{5,7,13,14,15,17,18,19,24,28,31,32\}$
$B_8 = \{7,11,12,14,18,19,21,23,24,27,29,30,32\}$

$A_9 = \{5,6,9,10,11,12,18,19,21,23,27,32\}$
$B_9 = \{7,8,11,15,17,21,22,24,25,27,29,30,32\}$

$A_{10} = \{5,6,8,9,11,13,17,18,21,23,28,32\}$
$B_{10} = \{8,11,12,13,14,19,20,21,24,27,28,30,32\}$

$A_{11} = \{5,6,7,10,11,15,17,23,24,25,27,32\}$
$B_{11} = \{8,10,12,13,17,18,20,21,23,24,27,30,32\}$

$A_{12} = \{5,6,9,10,11,16,19,23,27,29,31,32\}$
$B_{12} = \{8,11,13,15,16,17,18,23,24,26,27,30,32\}$

$A_{13} = \{5,6,10,14,19,21,24,26,29,30,31,32\}$
$B_{13} = \{9,11,12,13,15,19,20,23,24,26,29,30,32\}$

$A_{14} = \{6,12,13,17,18,19,21,22,23,29,31,32\}$
$B_{14} = \{3,7,10,12,15,17,21,24,27,28,29,30,32\}$

$A_{15} = \{6,7,9,13,14,15,17,19,26,29,31,32\}$
$B_{15} = \{4,6,10,15,18,19,20,21,24,25,27,28,32\}$

$n = 66; \quad 2 - (33;\ 11,\ 15;\ 10)$

$A_1 = \{5,11,13,14,15,16,20,23,27,31,32\}$
$B_1 = \{4,7,9,10,11,12,13,17,18,20,22,23,26,30,32\}$

$A_2 = \{5,7,8,11,15,17,22,27,30,31,32\}$
$B_2 = \{5,7,10,11,13,14,18,20,21,22,23,25,26,27,32\}$

$A_3 = \{5,8,13,14,15,18,20,25,28,29,32\}$
$B_3 = \{5,7,8,9,10,13,14,15,16,17,21,25,27,29,32\}$

TABLE I—*Continued*

$A_4 = \{5,7,11,12,14,15,18,23,26,28,32\}$
$B_4 = \{6,7,8,12,13,17,19,20,21,22,23,24,27,30,32\}$

$A_5 = \{5,7,10,11,14,17,21,26,30,31,32\}$
$B_5 = \{6,8,9,10,11,12,14,16,17,21,22,24,25,30,32\}$

$A_6 = \{5,11,15,16,17,20,23,24,25,27,32\}$
$B_6 = \{6,8,10,11,12,14,17,21,22,24,25,27,30,31,32\}$

$A_7 = \{5,6,7,10,14,17,20,21,26,30,32\}$
$B_7 = \{8,9,11,13,17,18,19,22,24,25,26,27,29,30,32\}$

$A_8 = \{6,9,11,12,13,18,22,24,29,30,32\}$
$B_8 = \{4,5,9,12,13,16,17,18,19,20,21,23,27,29,32\}$

$A_9 = \{6,13,14,16,17,21,22,23,26,28,32\}$
$B_9 = \{4,7,8,9,10,13,16,20,22,24,27,28,29,30,32\}$

$A_{10} = \{6,11,15,17,21,23,24,25,27,28,32\}$
$B_{10} = \{4,6,9,11,12,13,14,19,22,23,24,25,28,31,32\}$

$A_{11} = \{6,10,13,17,22,23,25,26,27,28,32\}$
$B_{11} = \{4,5,9,11,13,17,18,19,20,21,24,27,29,30,32\}$

$A_{12} = \{6,10,11,12,16,18,21,27,30,31,32\}$
$B_{12} = \{4,5,7,12,15,19,21,22,24,25,26,28,29,30,32\}$

$A_{13} = \{6,11,14,16,17,23,24,25,26,28,32\}$
$B_{13} = \{4,5,8,10,12,14,17,18,21,22,24,25,26,27,32\}$

$A_{14} = \{6,8,9,10,14,15,19,26,27,29,32\}$
$B_{14} = \{5,7,9,12,15,16,17,18,19,21,23,24,27,31,32\}$

$A_{15} = \{6,8,11,12,16,20,26,27,29,31,32\}$
$B_{15} = \{5,7,10,14,15,16,17,18,19,21,24,25,27,31,32\}$

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