Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER Computers and Mathematics with Applications 49 (2005) 539–548
An International Journal
computers & mathematics with applications
www.elsevier.com/locate/camwa

Fuzzy Relation Analysis in Fuzzy Time Series Model

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(Received and accepted July 2004)

Abstract—Fuzzy relation is a crucial connector in presenting fuzzy time series model. However, how to obtain a fuzzy relation matrix to represent a time-invariant relation is still a question. Based on the concept of fuzziness in Information Theory, the concept of *entropy* is applied to measure the degrees of fuzziness when a time-invariant relation matrix is derived. Finally, an example is illustrated to show that the proposed method could obtain more accurate and robust results in forecasting.

Keywords—Fuzzy time series model, Entropy, Fuzzy relation, Secular trend.

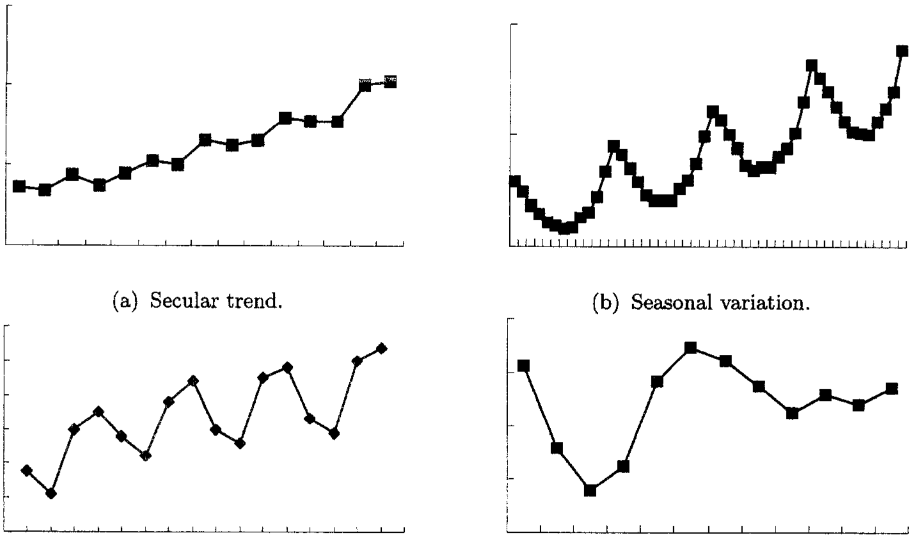
1. INTRODUCTION

Forecasting is important for any organization to control effective resource plan which is often employed by collecting a large number of time series data so that the trend of development could be discovered. In the forecasting models, there are four types of time series patterns that have been studied intensively in the literature. They are secular trend, seasonal variation, cyclical variation, and irregular variation as shown in Figure 1 [1].

In forecasting, statistical methods such as time series model are the commonly used tools. However, if the given datum are in linguistic terms or very little, the statistical methods will fail [2–4]. In order to cope with such a problem, fuzzy time series models [2–7] have been developed and applied in practice. Song and Chissom (S & C in abbreviation) were the pioneers of studying such problems and have proposed fuzzy time series model in 1993. Because of its better performance in some kinds of forecasting problems, fuzzy time series model has drawn much attention to the researchers. Inspired by S & C's approach, Lee [8] has proposed a method to fuzzify the historical data with triangular fuzzy numbers and used S & C's time-variant fuzzy

The authors gratefully acknowledge the financial support from National Science Foundation with project No. NSC 89-2213-E007-110.

0898-1221/05/$ - see front matter Typeset by AMS-TeX
doi:10.1016/j.camwa.2004.07.014



(a) Secular trend.
(b) Seasonal variation.
(c) Cyclical variation.
(d) Irregular variation.

Figure 1. Four types of time series.

time series model to forecast the university enrollments. Besides, Lee also indicated that the universe of discourse, the number of intervals, the size of extended universe, and the operator used in fuzzy time series model could influence the forecasting results. Among these issues, Chen [9] focused on the operator used in the model and simplified the arithmetic calculations to improve the composition operations which showed the proposed method was more efficient and accurate in computation than S & C's method. On the other hand, Hwang [5] applied the variation of historical data to replace the original input of S & C's method and obtained reasonable results.

In fuzzy time series, fuzzy relational equations are employed based on relation matrix to determine the fuzzy relations between time series data [2–4,7] in which most researches focus such fuzzy relation equations on seasonal data [6,10] but few studies are found on secular trends. Therefore, in secular trends, how to obtain a fuzzy relation matrix in order to derive a better forecasting performance is what we are interested in. For this purpose, we shall first consider the issue of how fuzzy relation matrix affects the forecasting performance and propose an arithmetic procedure for deriving fuzzy relation matrix.

In Section 2, the basic concept of fuzzy time series model is introduced. Then, fuzzy relation analysis is performed in Section 3 to derive a revised fuzzy time series model. Besides, the accuracy and the robustness for the proposed method are evaluated and discussed with an illustrated example in Section 4. Finally, summary and conclusions will be drawn in Section 5.

2. BASIC CONCEPT OF FUZZY TIME SERIES

The characteristics of fuzzy time series model have been commonly recognized by four aspects

(1) dynamic process;
(2) fuzzy observations;
(3) a naturally or artificially defined universe of discourse in a fuzzy subset of $\mathbf{R}^1$; and
(4) the conventional time series models are no longer applicable to describe these processes [2–4].

For the fuzzy time series model, if $Y(t)$ $(t = 1, 2, \ldots, n)$ is a subset of $\mathbf{R}^1$ in which the universe of fuzzy sets $f_i(t)$ $(i = 1, 2, \ldots, m)$ are defined and let $F(t)$ be a collection of $f_i(t)$ $(i = 1, 2, \ldots, m)$, then $F(t)$ is called a fuzzy time series on $Y(t)$ $(t = 1, 2, \ldots, n)$. It is noted that $F(t)$ can be regarded as a linguistic variable and $f_i(t)$ $(i = 1, 2, \ldots, m)$ can be viewed as possible linguistic values of $F(t)$. Therefore, the values of $F(t)$ can be different in different time periods because the universe of discourse can be different from different time periods. Besides, the main difference between conventional and fuzzy time series is that the observations of the former are real numbers but the latter can be fuzzy sets. Next, if any two fuzzy sets $f_i(t)$ and $f_j(t-1)$ are considered as matrices $[f_i(t)]_{m\times 1}$ and $[f_j(t-1)]_{1\times m}$, respectively, then the fuzzy relation matrix between two matrices could be the max-min composite form as (1) below

$$R_{ij}(t,t-1) = \max_j \min\{ f_{i1}(t), \quad f_{1j}(t-1)\}, \qquad \forall i,j = 1,2,\dots,m. \tag{1}$$

Finally, by the cylindrical extension, the fuzzy relation matrix $R(t,t-1)$ derived from (1) can be obtained as (2) below

$$R(t,t-1) = \bigcup_{ij} R_{ij}(t,t-1). \tag{2}$$

DEFINITION 1. *(See [3,4].) Suppose that $F(t)$ is caused by $F(t-1)$, then the relation of the first-order model of $F(t)$ can be expressed as $F(t) = F(t-1) \circ R(t,t-1)$ where $R(t,t-1)$ is the relation matrix to describe the fuzzy relationship between $F(t-1)$ and $F(t)$, and '$\circ$' is the max-min composition.*

THEOREM 1. *(See [3,4].) If $F(t)$ is a fuzzy time series, $F(t) = F(t-1)$ for any t and $F(t)$ has only finite elements $f_i(t)$ $(i = 1,2,\dots,m)$, then*

$$R(t,t-1) = f_i(t-1)\times f_j(t) \cup f_i(t-2)\times f_j(t-1) \cup \cdots \cup f_i(t-n)\times f_j(t-n+1), \qquad \text{where } n > 0.$$

DEFINITION 2. *(See [3,4].) Suppose $R(t,t-1)$ is a first-order model of $F(t)$. If for any t, $R(t,t-1)$ is independent of t, i.e., $R(t,t-1) = R(t-1,t-2)$, then $F(t)$ is called a time-invariant fuzzy time series; otherwise it is called a time-variant fuzzy time series.*

Based on the above concepts, Song and Chissom [2] have proposed a procedure for solving fuzzy time series model described as following steps.

STEP 1. Define the universal discourse U for the historical data.

When defining the universe, the minimum data $D_{\min}$ and the maximum data $D_{\max}$ of given historical data are first defined. Based on $D_{\min}$ and $D_{\max}$, we define the universe U as $[D_{\min} - D_1, D_{\max} + D_2]$ where D_1 and D_2 are two proper positive numbers.

STEP 2. Partition universal discourse U into several equal intervals.

If the universal discourse U is partitioned into n equal intervals with its length to be ℓ as defined below

$$\ell = \frac{[(D_{\max} + D_2) - (D_{\min} - D_1)]}{n}, \tag{3}$$

then each interval could be obtained as $u_1 = [D_{\min} - D_1, D_{\min} - D_1 + \ell]$,

$$u_2 = [D_{\min} - D_1 + \ell, D_{\min} - D_1 + 2\ell], \dots, \qquad u_1 = [D_{\min} - D_1 + (n-1), D_{\min} - D_1 + n\ell].$$

STEP 3. Define fuzzy sets on universal discourse U.

In S & C's model, there is no restriction on determining how many linguistic variables to be fuzzy sets. For instance, the 'enrollment' can be described by the fuzzy sets of $\tilde{A}_1 =$ (not many), $\tilde{A}_2 =$ (not too many), $\tilde{A}_3 =$ (many), $\tilde{A}_4 =$ (many many), $\tilde{A}_5 =$ (very many), $\tilde{A}_6 =$ (too many), $\tilde{A}_7 =$ (too many many). In this example, each fuzzy set $\tilde{A}_i$ $(i = 1,2,\dots,7)$ is defined below by $\tilde{A}_i = \{(\mu_{\tilde{A}_i}(u_j)/u_j) \mid \mu_{\tilde{A}_i}(u_j) \in [0,1],\ u_j \in R\ j = 1,\dots,7\}$ with the membership degree $\mu_{\tilde{A}_i}(u_j)$

Table 1. Enrollment data of Alabama University [3].

Year	Historical Data $Y(t)$	u_1	u_2	u_3	u_4	u_5	u_6	u_7
1971	13055	1	0.5	0	0	0	0	0
1972	13563	1	0.8	0.1	0	0	0	0
1973	13867	1	0.9	0.2	0	0	0	0
1974	14696	0.8	1	0.8	0.1	0	0	0
1975	15460	0.2	0.8	1	0.2	0	0	0
1976	15311	0.2	0.8	1	0.2	0	0	0
1977	15603	0	0.6	1	0.6	0.1	0	0
1978	15861	0	0.5	1	0.7	0.2	0	0
1979	16807	0	0.1	0.5	1	0.9	0.2	0
1980	16919	0	0.1	0.5	1	0.9	0.2	0
1981	16388	0	0.2	0.8	1	0.5	0	0
1982	15433	0.2	0.8	1	0.2	0	0	0
1983	15497	0.2	0.8	1	0.2	0	0	0
1984	15145	0.2	0.8	1	0.2	0	0	0
1985	15163	0.2	0.8	1	0.2	0	0	0
1986	15984	0	0.2	1	0.7	0.2	0	0
1987	16859	0	0.1	0.5	1	0.8	0.1	0
1988	18150	0	0	0.1	0.5	0.8	1	0.7
1989	18970	0	0	0	0.25	0.55	1	0.8
1990	19328	0	0	0	0.3	0.5	0.8	1

of u_j

$$\begin{aligned}
\tilde{A}_1 &= \{1/u_1,\ 0.5/u_2,\ 0/u_3,\ 0/u_4,\ 0/u_5,\ 0/u_6,\ 0/u_7\},\\
\tilde{A}_2 &= \{0.5/u_1,\ 1/u_2,\ 0.5/u_3,\ 0/u_4,\ 0/u_5,\ 0/u_6,\ 0/u_7\},\\
\tilde{A}_3 &= \{0/u_1,\ 0.5/u_2,\ 1/u_3,\ 0.5/u_4,\ 0/u_5,\ 0/u_6,\ 0/u_7\},\\
\tilde{A}_4 &= \{0/u_1,\ 0/u_2,\ 0.5/u_3,\ 1/u_4,\ 0.5/u_5,\ 0/u_6,\ 0/u_7\},\\
\tilde{A}_5 &= \{0/u_1,\ 0/u_2,\ 0/u_3,\ 0.5/u_4,\ 1/u_5,\ 0.5/u_6,\ 0/u_7\},\\
\tilde{A}_6 &= \{0/u_1,\ 0/u_2,\ 0/u_3,\ 0/u_4,\ 0.5/u_5,\ 1/u_6,\ 0.5/u_7\},\\
\tilde{A}_7 &= \{0/u_1,\ 0/u_2,\ 0/u_3,\ 0/u_4,\ 0/u_5,\ 0.5/u_6,\ 1/u_7\}.
\end{aligned}$$

STEP 4. Fuzzify the historical data.

This is to find an equivalent fuzzy set for each input data. The commonly used method is to define a cut set for each $\tilde{A}_i$ $(i = 1, \dots, 7)$.

STEP 5. Determine fuzzy relation matrix R.

By the Theorem 1 and fuzzy logical, for example in Table 1, from 1971–1990, we have the relations as $\tilde{A}_1 \to \tilde{A}_1$, $\tilde{A}_1 \to \tilde{A}_2$, $\tilde{A}_2 \to \tilde{A}_3$, $\tilde{A}_3 \to \tilde{A}_3$, $\tilde{A}_3 \to \tilde{A}_4$, $\tilde{A}_4 \to \tilde{A}_4$, $\tilde{A}_4 \to \tilde{A}_3$, $\tilde{A}_3 \to \tilde{A}_3$, $\tilde{A}_3 \to \tilde{A}_4$, $\tilde{A}_4 \to \tilde{A}_6$, $\tilde{A}_6 \to \tilde{A}_6$, $\tilde{A}_6 \to \tilde{A}_7$. Therefore, the fuzzy relation matrix could be obtained by the max-min operator as (4) below.

$$R = \begin{bmatrix} 1 & 1 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 1 & 1 & 0.5 & 0.5 & 0.5 \\ 0 & 0.5 & 1 & 1 & 0.5 & 1 & 0.5 \\ 0 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0.5 \end{bmatrix} \tag{4}$$

STEP 6. Calculate the forecasted outputs.

If the data $Y(t)$ $t \in [1, n]$, is set to the fuzzy set $\tilde{A}_i$, $\forall i = 1, 2, \dots, m$, then the forecast of $F(k+1)$ is obtained by (5) below

$$F(t+1) = \tilde{A}_i \circ R, \qquad t \in [1, n], \quad \forall i = 1, 2, \dots, m. \tag{5}$$

STEP 7. Interpret the forecasted outputs.

If the forecasting output for period $k+1$ is the fuzzy set $\tilde{A}_5 = \{0/u_1, 0/u_2, 0/u_3, 0.5/u_4, 1/u_5, 0.5/u_6, 0/u_7\}$, then by the centroid defuzzy method we can obtain $A_5 = (0.5 \times m_4 + 1 \times m_5 + 0.5 \times m_6)/(0 + 0 + 0 + 0.5 + 1 + 0.5 + 0)$, where m_i is the central value for the interval u_i.

In many researches, although, issues on the input variables and types of operators in fuzzy time series model of $F(t) = F(t-1) \circ R(t, t-1)$ have been discussed, the critical factor of $R(t, t-1)$ has not yet been studied. Therefore, we shall consider the issue of how fuzzy relation matrix R affecting the result of forecasting and then propose a better method for fuzzy relation matrix R in this study. Based on the above structure, once R is derived, the relation between the current and the preceding data could be obtained to revise the fuzzy time series model to describe secular trend time series patterns.

3. A REVISED FUZZY TIME SERIES MODEL

In fuzzy time series model, the universe of discourse U is partitioned into several equal-length intervals in which the input data are defined as fuzzy sets $\tilde{A}_i$, $\forall i = 1, 2, \dots, m$, where m is an positive integer. Therefore, each input datum is only belonged to corresponding fuzzy sets so that the fuzzified input data are defined to have many zeros in $\tilde{A}_i$, $\forall i = 1, 2, \dots, m$, as shown in Columns 3–9 in Table 1.

When a first-order fuzzy time series model is constructed as defined in Definition 2, the redundant computation is complex to contribute the derived fuzzy relation matrix to contain many 1s as below which often leads to nonconvex fuzzy outputs as matrix (4). To avoid the above situations, we shall first investigate the properties of fuzzy relation matrix R and then propose an alternative approach to derive an appropriate matrix R. Since fuzzy relation matrix R is derived by unifying all relations between fuzzy sets $\tilde{A}_i$, $\forall i = 1, 2, \dots, m$, so that if the number of intervals of defining $\tilde{A}_i$ is not large enough, then the derived R also contain many 1s to make both computation cost and estimation accuracy. Therefore, instead of by trial-and-error to trade off between these two factors, we take an analytical approach by relaxing the pre-assumption of time-invariant to derive the possibility of steady relation among secular data in which the concept of fuzziness in Information theory is adopted.

In information theory [11], a steady state implies that the degrees of fuzziness of the system are the same when the system is transient from the current state to next state. Therefore, we make use of the concept of *entropy* in information theory as defined below to measure the degrees of fuzziness of a system and determine the time T of which the data approaches steady state.

DEFINITION 4. *(See [11].) The entropy of a fuzzy set $\tilde{A} = \{(x, \mu_A(x))\}$ is defined as $d(\tilde{A}) = H(\tilde{A}) + H(\tilde{A}^c)$ where $H(\tilde{A}) = -K \sum_{i=1}^{m} \mu_{\tilde{A}}(x_i) \ln(\mu_{\tilde{A}}(x_i))$, m is the number of elements in the support of $\tilde{A}$, $\tilde{A}^c$ is the complement of $\tilde{A}$, and K is a positive constant.*

Let a fuzzy relation matrix R be an $m \times m$ matrix, then the n^{th}-order fuzzy relation matrix R^n is defined as below

$$R^n = R^{n-1} \circ R, \tag{6}$$

where '$\circ$' is the 'max-min' operator. Since at steady state, we have $d(R^{T+1}) = d(R^T)$, $T \in [1, n]$, therefore by letting K=1 we derive model (7) to determine the minimum T as below

$$\begin{aligned} \min \quad & T, \\ \text{s.t.} \quad & d\left(R^{T+1}\right) - d\left(R^{T}\right) = 0, \\ & r_{ij}^{T+1} = r_{ij}^{T}, \qquad \forall\, i,j = 1,2,\ldots,m, \\ & 0 \le r \le 1. \end{aligned} \tag{7}$$

Since $r_{ij}^{T+1} = r_{ij}^{T}$ implies $d(R^{T+1}) - d(R^{T}) = 0$, we rewrite model (7) into (8) as below

$$\begin{aligned} \min \quad & T \\ \text{s.t.} \quad & r_{ij}^{T+1} = r_{ij}^{T}, \quad \forall\, i,j = 1,2,\ldots,m \\ & 0 \le r \le 1. \end{aligned} \tag{8}$$

By solving model (8), we can obtain the objective value T to obtain values of $[n/T]+1$ time-invariant relations matrices per T time-series data, where $[n/T]$ is gauss value. Next, by using fuzzy relation matrix $R^{(k)}\ \forall\, k = 1,2,\ldots,[n/T]+1$, the forecasting output $F(t)$ could be obtained by input fuzzy data $\tilde{A}_i$, $\forall\, i = 1,2,\ldots,m$, as (9) below

$$F(t) = \tilde{A}_i \circ R^{(k)}, \qquad \forall\,(t-1) \le k \cdot T, \quad k = 1,2,\ldots,[n/T]+1, \tag{9}$$

where '$\circ$' is the 'max-min' operator.

After deriving the fuzzy relation matrices $R^{(k)}$, $\forall\, k = 1,2,\ldots,[n/T]+1$, the solving procedure for fuzzy time series model could be obtained as following steps.

STEP 1. Define the universe of discourse U for the historical data.

STEP 2. Partition the universe U into several even-length intervals.

STEP 3. Define fuzzy sets on the universe U.

STEP 4. Fuzzify the historical data.

STEP 5. Determine the minimum value of invariant time index T.

STEP 6. Determine fuzzy relation matrices $R^{(k)}$, $\forall\, k = 1,2,\ldots,[n/T]+1$.

STEP 7. Calculate the forecasted outputs.

STEP 8. Defuzzify the outputs.

4. AN ILLSUTRATED EXAMPLE

In this section, we illustrated the revised fuzzy time-series model by the example used by S & C for comparison.

STEP 1. Define the universe of discourse U within for the historical data.

In Table 1, we have the enrollments of the university from 1971–1992 with $D_{\min} = 13055$ and $D_{\max} = 19337$. For simplicity, we choose $D_1 = 55$ and $D_2 = 663$. Thus, the universe is the interval of $U = [13000, 20000]$.

STEP 2. Partition the universe U into several equal-length intervals.

U is dividend into seven intervals with equal lengths and denote u_1, u_2, u_3, u_4, u_5, u_6, and u_7 for each interval with $u_1 = [13000, 14000]$, $u_2 = [14000, 15000]$, $u_3 = [15000, 16000]$, $u_4 = [16000, 17000]$, $u_5 = [17000, 18000]$, $u_6 = [18000, 19000]$, and $u_7 = [19000, 20000]$.

STEP 3. Define fuzzy sets on the universe U.

The step has the same defined fuzzy sets as in Section 2 proposed by S & C's model.

STEP 4. Fuzzify the historical data.

The process is the same as that of determining the memberships of u_i to $\tilde{A}_i$, $\forall i = 1, 2, \ldots, m$, in Step 3. The equivalent fuzzy sets to each year's enrollment are shown in Table 1 and each fuzzy set has seven elements.

STEP 5. Determine the minimum value of invariant time index T.

By model (8), we obtain the time index T for time invariant relation as value of 5.

STEP 6. Determine $R^{(k)}$, $k = 1, 2, 3, 4$.

Since $T = 5$ from Step 5, we could obtain four fuzzy relation matrices $R^{(k)}$ ($k = 1, 2, 3, 4$) for every five years. For example, from 1971–1975, we have three relations as $A_1 \rightarrow A_1$, $A_1 \rightarrow A_2$, $A_2 \rightarrow A_3$ to derive fuzzy relation matrix $R^{(1)}$, so that we can induct the same fuzzy relation matrices for the rest as follows.

$$R^{(1)} = \begin{bmatrix} 1 & 1 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 1 & 1 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 1 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R^{(4)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 1 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 0.5 & 1 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0.5 \end{bmatrix}.$$

It is obvious that there are less 1s in matrices $R^{(k)}$ ($k = 1, 2, 3, 4$) than original matrix R. This decreases the possibility of defuzzified errors on forecasted outputs.

STEP 7. Calculate the forecasted outputs.

By using $R^{(k)}$ to obtain the forecasting values, for example of 1972, because F (1971) falls in the range of fuzzy relation matrix $R^{(1)}$, therefore, with $F(1971) = \tilde{A}_1 \circ R^{(1)}$, the forecasted output of 1972 is obtained as below.

$$[1, 0.5, 0, 0, 0, 0, 0] \circ \begin{bmatrix} 1 & 1 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0.5 & 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = (1, 1, 0.5, 0.5, 0, 0, 0).$$

Note that because only the first 3×4 submatrix has nonzero values, therefore only 1×3 input vector should be used to derive the 1×4 output.

STEP 8. Defuzzify the outputs.

Because the forecasting outputs are all fuzzy sets, it is necessary to defuzzy the fuzzy output into real values number where S & C's defuzzify method is adopted as follows.

(1) If the membership of an output has only one maximum, then select the midpoint of the interval corresponding to the maximum as the forecasted value.
(2) If the membership of an output has one or more consecutive maximums, then select the midpoint of the corresponding conjunct intervals as the forecasted value.
(3) Otherwise, standardize the fuzzy output and use the midpoint of each interval to calculate the centroid of the fuzzy set as the forecasted value.

Table 2. The forecasted values.

Year	Historical Data	u_1	u_2	u_3	u_4	u_5	u_6	u_7	Forecasted value
1972	13563	1	1	0.5	0.5	0	0	0	14000
1973	13867	1	1	0.8	0.5	0	0	0	14000
1974	14696	1	1	0.9	0.5	0	0	0	14000
1975	15460	0.8	0.8	1	0.5	0	0	0	15500
1976	15311	0.5	0.5	0.8	0.5	0	0	0	15500
1977	15603	0	0.5	1	1	0.5	0	0	16000
1978	15861	0	0.5	1	1	0.5	0	0	16000
1979	16807	0	0.5	1	1	0.5	0	0	16000
1980	16919	0	0.5	0.5	1	0.5	0	0	16500
1981	16388	0	0.5	0.5	1	0.5	0	0	16500
1982	15433	0	0.5	1	0.5	0	0	0	15500
1983	15497	0	0.5	1	0.5	0	0	0	15500
1984	15145	0	0.5	1	0.5	0	0	0	15500
1985	15163	0	0.5	1	0.5	0	0	0	15500
1986	15984	0	0.5	1	0.5	0	0	0	15500
1987	16859	0	0	0.5	1	0.5	0.7	0.5	16500
1988	18150	0	0	0.5	0.5	0.5	1	0.5	18500
1989	18970	0	0	0.5	0.5	0.5	1	1	19000
1990	19328	0	0	0.5	0.5	0.5	1	1	19000
1991	19337	0	0	0.3	0.3	0.5	0.8	0.8	19000

Following the above steps, we have obtained the forecasting values for the enrollments from 1972–1991 as shown in Table 2.

In order to value the estimated error between forecasting value and actual, an estimated methods are listed as below.

$$\text{Forecasting error} = \frac{|(\text{forecasted value–actual value})|}{(\text{actual value})*100\%} \tag{7}$$

$$\text{Average forecasting error} = \frac{(\text{sum of forecasting errors})}{(\text{total number of errors})} \tag{8}$$

The results are listed in Table 3 and shown in Figure 2 where the solid line is the actual enrollment and the dashed line is the forecasted enrollment. From Table 3, the S & C model's

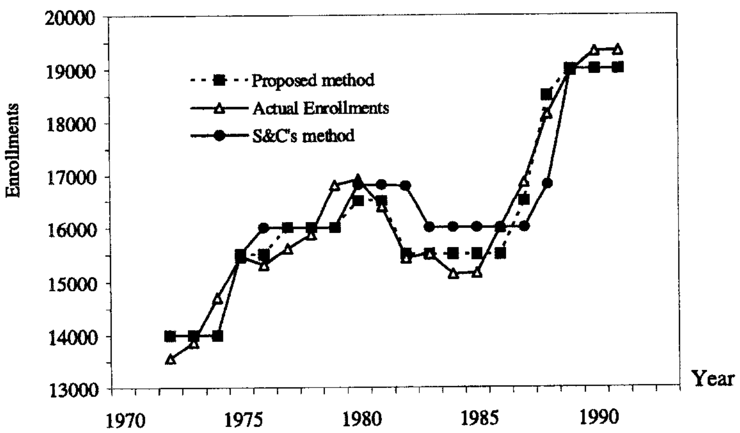


Figure 2. Forecasted enrollments and actual enrollments.

Table 3. Comparative study on forecasting errors.

Year	Given Values	S & C's Results		Proposed Method	
			Error		Error
1972	13563	14000	437	14000	437
1973	13867	14000	133	14000	133
1974	14696	14000	696	14000	696
1975	15460	15500	40	15500	40
1976	15311	16000	689	15500	189
1977	15603	16000	397	16000	397
1978	15861	16000	139	16000	139
1979	16807	16000	807	16000	807
1980	16919	16813	106	16500	419
1981	16388	16813	425	16500	112
1982	15433	16789	1356	15500	67
1983	15497	16000	503	15500	3
1984	15145	16000	855	15500	355
1985	15163	16000	837	15500	337
1986	15984	16000	16	15500	484
1987	16859	16000	859	16500	359
1988	18150	16813	1337	18500	350
1989	18970	19000	30	19000	30
1990	19328	19000	328	19000	328
1991	19337	19000	337	19000	337
Average error		3.18%		1.86%	

forecasting errors range from 0.1% to 8.7% with the average error being 3.18%. The forecasting errors of the modified model are between 0.02% and 4.8% and the average error is 1.86%. Besides, from the Figure 2, the time lag of the proposed method is much shorter than S & C's to have quick response ability.

Furthermore, the robustness of the proposed method is tested by using the same data set of university enrollment but randomly selected the enrollments of 1974, 1978, 1985, and 1990 to increase their values by 5%. With the proposed procedure, we obtained the fuzzy logical relationships as $A_1 \to A_1$, $A_1 \to A_3$, $A_3 \to A_3$, $A_3 \to A_4$, $A_4 \to A_4$, $A_4 \to A_3$, $A_4 \to A_6$, $A_6 \to A_6$, and $A_6 \to A_7$. Also, by model (8), we solve $T = 4$. The forecasted enrollments from 1972–1991 and the actual enrollments are shown in Figure 3 for comparison. The forecasting errors range from 0.16% to 9.36%, and the average error is 3.49% which is also smaller than 3.9%

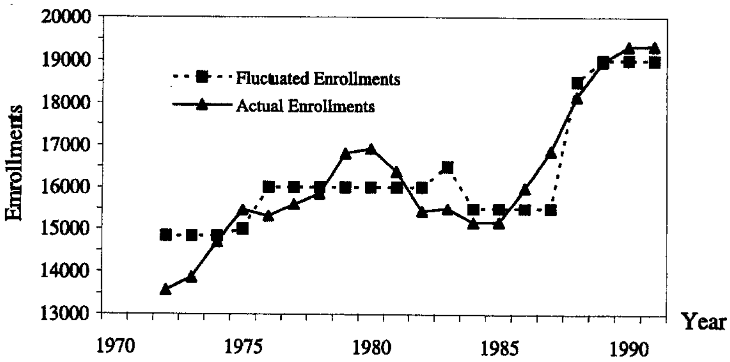


Figure 3. Robustness analysis of the proposed method.

Table 4. Comparison of forecasting errors with three types of methods.

Method	S & C's Method	Proposed Method	Quadratic Regression
Average Forecasting Errors	3.18%	1.86%	5.0%

in S & C's error on robust test. Beside, from Figure 3, it also can be noted that when time T increases, the forecasting error decreases. This indicates that because of quick-response ability of our method, even if the historical data are fluctuated, satisfactory forecasting results still can be expected.

Finally, in order to show the forecasting performance, the best fitted regression equations are also applied to compare with the proposed method. From the third column of Table 4, it is obviously that the forecasting error of the best fitted regression equation is also larger than the proposed method.

5. SUMMARY AND CONCLUSION

Accurate prediction plays an important role in the era of e-commerce business. In order to obtain realistic results from historical data, a good forecasting method is necessary. Because the existing statistical time series methods could not effectively analyze time series with small amount of data, fuzzy time series methods were developed. Apart from this necessity, fuzzy time series methods also provide a tool to deal with the problems when historical data are linguistic values. Although S & C and the following research have considered the fuzzy time series extensively, the effect of the fuzzy relation matrix has never been investigated. Therefore, this study proposed an analytical approach to find the steady state of fuzzy relation matrix to revise the logic forecasting process. From the illustrated examples, the proposed model has been shown to be more accurate and robust.

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