Conditions for Choquet integral representation of the comonotonically additive and monotone functional

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Abstract
If the universal set $X$ is not compact but locally compact, a comonotonically additive and monotone functional (for short c.m.) on the class of continuous functions with compact support is not represented by one Choquet integral, but represented by the difference of two Choquet integrals. The conditions for which a c.m. functional can be represented by one Choquet integral are discussed.

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1. Introduction

The Choquet integral with respect to a nonadditive measure is one of the nonlinear functionals defined on the class $B$ of measurable functions on a measurable space $(X, B)$. It was introduced by Choquet [1] in potential theory with the concept of capacity. Then, in the field of economic theory, it has been used for utility theory [17], and has been used for image processing and recognition [4,5], in the context of fuzzy measure theory [9,20].

Essential properties characterizing this functional are comonotonic additivity and monotonicity. Let $f$ and $g$ be measurable functions. We say that $f$ and $g$ are comonotonic if

$$f(x) < f(x') \implies g(x) \leq g(x')$$

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for \( x, x' \in X \). We denote \( f \sim g \) when \( f \) and \( g \) are comonotonic. Let \( I \) be a real valued functional on \( B \). We say \( I \) is comonotonically additive iff

\[
f \sim g \implies I(f + g) = I(f) + I(g)
\]

for \( f, g \in B \), and \( I \) is monotone iff

\[
f \leq g \implies I(f) \leq I(g)
\]

for \( f, g \in B \). We say the functional \( I \) is a c.m. functional if \( I \) is comonotonically additive and monotone.

Schmeidler [16] proves that a c.m. functional \( I \) on \( B \) can be represented by a Choquet integral. Parker [14] fully discusses the Choquet integral representations of comonotonically additive functionals on \( B \) and related concepts. Skala [19] provides a generalization of the Choquet integral and uses this for an integral representation of comonotonically additive operators.

Concerning the problem of whether or not the c.m. functional \( I \) can be represented by a Choquet integral when the domain of \( I \) is smaller than \( B \), Greco [6] proved it when \( I \) has some continuity. Sugeno et al. [21] proved that a c.m. functional \( I \) can be represented by a Choquet integral with respect to a regular nonadditive measure (in the fuzzy context we say a nonadditive measure is a fuzzy measure) when the domain of \( I \) is the class \( K^+ \) of nonnegative continuous functions with compact support on a locally compact Hausdorff space. In [11], it is proved that a c.m. functional is a rank- and sign-dependent functional, that is, the difference of two Choquet integrals, if the domain of \( I \) is the class \( K \) of continuous functions with compact support on a locally compact Hausdorff space \( X \). This functional is used in utility theory [7] and cumulative prospect theory [22,23]. It is also proved in [11] that a rank- and sign-dependent functional is a c.m. functional if \( X \) is not compact. From this fact, in order to represent \( I \) by one Choquet integral, further conditions are needed.

In this paper, we present the conditions for which a c.m. functional can be represented by one Choquet integral. We call these conditions the conjugate conditions. We define the conjugate conditions and show their basic properties in Section 3. The conjugate conditions are stronger than the boundedness. A c.m. functional \( I \) is represented by one Choquet integral when \( I \) satisfies one of the conjugate conditions. Conversely if a c.m. functional \( I \) is represented by one Choquet integral, \( I \) satisfies the conjugate condition when the universal set \( X \) is separable.

## 2. Preliminaries

In this section, we define the nonadditive measure, the Choquet integral and the rank- and sign-dependent functional, and show their basic properties.

Throughout the paper we assume that \( X \) is a locally compact Hausdorff space, \( B \) is the class of Borel subsets, \( O \) is the class of open subsets, and \( C \) is the class of compact subsets.

**Definition 2.1.** A nonadditive measure \( \mu \) is an extended real valued set function

\[
\mu : B \to \bar{\mathbb{R}}^+
\]

with the following properties:

(1) \( \mu(\emptyset) = 0 \);
(2) \( \mu(A) \leq \mu(B) \) whenever \( A \subseteq B \), \( A, B \in \mathcal{B} \),
where \( \bar{R}^+ = [0, \infty) \) is the set of extended nonnegative real numbers.

When \( \mu(X) < \infty \), we define the conjugate \( \mu^c \) of \( \mu \) by
\[
\mu^c(A) = \mu(X) - \mu(A^c)
\]
for \( A \in \mathcal{B} \).

**Definition 2.2.** Let \( \mu \) be a nonadditive measure on measurable space \((X, \mathcal{B})\). \( \mu \) is said to be outer regular if
\[
\mu(B) = \inf \{ \mu(O) \mid O \in \mathcal{O}, O \supseteq B \}
\]
for all \( B \in \mathcal{B} \).

An outer regular nonadditive measure \( \mu \) is said to be regular, if for all \( O \in \mathcal{O} \),
\[
\mu(O) = \sup \{ \mu(C) \mid C \in \mathcal{C}, C \subseteq O \}.
\]

\( K \) denotes the class of continuous functions with compact support, \( K^+ \) denotes the class of nonnegative continuous functions with compact support, and \( K^+_1 \) denotes the class of nonnegative continuous functions with compact support that satisfy \( 0 \leq f \leq 1 \).

\( \text{supp}(f) \) denotes the support of \( f \in K \).

**Definition 2.3** [1,8]. Let \( \mu \) be a nonadditive measure on \((X, \mathcal{B})\).

(1) The Choquet integral of \( f \in K^+ \) with respect to \( \mu \) is defined by
\[
(C) \int f \, d\mu = \int_0^\infty \mu_f(r) \, dr,
\]
where \( \mu_f(r) = \mu(\{x \mid f(x) \geq r\}) \).

(2) Suppose \( \mu(X) < \infty \). The Choquet integral of \( f \in K \) with respect to \( \mu \) is defined by
\[
(C) \int f \, d\mu = (C) \int f^+ \, d\mu - (C) \int f^- \, d\mu^c,
\]
where \( f^+ = f \vee 0 \) and \( f^- = -(f \wedge 0) \). When the right-hand side is \( \infty - \infty \), the Choquet integral is not defined.

If \( \mu \) is not additive, the Choquet integral with respect to \( \mu \) is nonlinear functional. In general, the Choquet integral is comonotonically additive and monotone (for short c.m.) [2,3,15].

Suppose that \( I \) is a c.m. functional, then we have \( I(af) = aI(f) \) for \( a \geq 0 \) and \( f \in K \),
that is, \( I \) is positively homogeneous.
Definition 2.4. Let $I$ be a real valued functional on $K$. $I$ is said to be a rank- and sign-dependent functional (for short a r.s.d. functional) on $K$, if there exist two nonadditive measures $\mu^+, \mu^-$ such that for every $f \in K$,

$$I(f) = (C) \int f^+ \, d\mu^+ - (C) \int f^- \, d\mu^-,$$

where $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$.

The r.s.d. functional is used in utility theory [7] and cumulative prospect theory [22,23]. When $\mu^+ = \mu^-$, we say that the r.s.d. functional is a Šipoš functional [18]. If the r.s.d. functional is a Šipoš functional, we have $I(-f) = -I(f)$.

If $\mu^+(X) < \infty$ and $\mu^- = (\mu^+)^c$, we say that the r.s.d. functional is a Choquet functional.

Theorem 2.1 [11]. Let $I$ be a c.m. functional on $K$.

1. We put
   $$\mu^+_I(O) = \sup \{I(f) \mid f \in K_I^+, \ supp(f) \subset O\}$$
   and
   $$\mu^+_I(B) = \inf \{\mu^+_I(O) \mid O \in \mathcal{O}, \ O \supset B\}$$
   for $O \in \mathcal{O}$ and $B \in \mathcal{B}$. Then $\mu^+_I$ is a regular nonadditive measure.

2. We put
   $$\mu^-_I(O) = \sup \{-I(-f) \mid f \in K_I^+, \ supp(f) \subset O\}$$
   and
   $$\mu^-_I(B) = \inf \{\mu^-_I(O) \mid O \in \mathcal{O}, \ O \supset B\}$$
   for $O \in \mathcal{O}$ and $B \in \mathcal{B}$. Then $\mu^-_I$ is a regular nonadditive measure.

3. A c.m. functional is a r.s.d. functional, that is, there exist unique regular nonadditive measures $\mu^+_I$ and $\mu^-_I$ such that
   $$I(f) = (C) \int (f \vee 0) \, d\mu^+_I - (C) \int -(f \wedge 0) \, d\mu^-_I$$
   for $f \in K$.

4. If $X$ is compact, then a c.m. functional can be represented by one Choquet integral.

5. If $X$ is locally compact but not compact, then a r.s.d. functional is a c.m. functional.

Let $I$ be a c.m. functional on $K$. We say that $\mu^+_I$ defined in Theorem 2.1 is the regular nonadditive measure induced by the positive part of $I$ and $\mu^-_I$ the regular nonadditive measure induced by the negative part of $I$.

Definition 2.5. Let $I$ be a real valued functional on $K$. 

(1) $I$ is said to be bounded above if there exists $M > 0$ such that $I(f) \leq M \|f\|$ for all $f \in K$.

(2) $I$ is said to be bounded below if there exists $M > 0$ such that $-M \|f\| \leq I(f)$ for all $f \in K$.

(3) $I$ is said to be bounded if $I$ is bounded above and below.

**Proposition 2.1** [13]. Let $I$ be a c.m. functional on $K$ and $\mu^+_I$ and $\mu^-_I$ the regular nonadditive measures induced by $I$.

1. $I$ is bounded above iff $\mu^+_I(X) < \infty$.
2. $I$ is bounded below iff $\mu^-_I(X) < \infty$.

**Proposition 2.2** [12]. Let $X$ be separable and $I$ be a c.m. functional on $K$ that is bounded, and $\mu^+_I$ and $\mu^-_I$ the regular nonadditive measure induced by $I$.

1. If $(C) \int f \, d\mu^+_I = (C) \int f \, d(\mu^-_I)^c$ for all $f \in K^+$, then $\mu^+_I(C) = (\mu^-_I)^c(C)$ for all $C \in \mathcal{C}$.
2. If $(C) \int f \, d\mu^-_I = (C) \int f \, d(\mu^+_I)^c$ for all $f \in K$, then $\mu^-_I(C) = (\mu^+_I)^c(C)$ for all $C \in \mathcal{C}$.

Proposition 2.2 says that if a c.m. functional $I$ is Choquet integral with respect to $\mu^+_I$ then we have $\mu^-_I(C) = (\mu^+_I)^c(C)$ for every $C \in \mathcal{C}$. Since $(\mu^+_I)^c$ is not always regular, it is not always true that $\mu^-_I = (\mu^+_I)^c$. That is, $I$ is not always a Choquet functional. See the example in [10].

3. Conjugate condition for compact sets

We have stated in Theorem 2.1 that a c.m. functional is not always represented by one Choquet integral if the universal space $X$ is not compact. In this section we present the conditions for a c.m. functional being represented by one Choquet integral. In the following, we assume that $X$ is not compact.

**Definition 3.1.** Let $I$ be a c.m. functional and $C \in \mathcal{C}$.

1. We say that $I$ satisfies the positive conjugate condition for $C$ if there exists a positive real number $M$ such that for any $\epsilon > 0$ there exist $f_1, f_2 \in K^+_I$ satisfying the following condition:

   $1_C \leq g_1 \leq f_1$ and $f_2 \leq g_2 \leq 1_{C^c}$ with $\text{supp}(f_2) \subseteq \text{supp}(g_2) \subseteq C^c$

   imply

   $|I(-g_1) - I(g_2) + M| < \epsilon$

   for $g_1, g_2 \in K^+_I$. 
(2) We say that $I$ satisfies the negative conjugate condition for $C$ if there exists a positive real number $M$ such that for any $\epsilon > 0$ there exist $f_1, f_2 \in K_1^+$ satisfying the following condition:

\[ l_C \leq g_1 \leq f_1 \quad \text{and} \quad f_2 \leq g_2 \leq l_C \]

with $\text{supp}(f_2) \subset \text{supp}(g_2) \subset C^c$ imply

\[ \left| -I(g_1) + I(-g_2) + M \right| < \epsilon \]

for $g_1, g_2 \in K_1^+$.

The conjugate conditions are stronger than the boundedness.

**Proposition 3.1.** Let $I$ be a c.m. functional.

(2) If $I$ satisfies the positive conjugate condition for $\emptyset$, then $I$ is bounded above.

(2) If $I$ satisfies the negative conjugate condition for $\emptyset$, then $I$ is bounded below.

**Proof.** Suppose that a c.m. functional $I$ satisfies the positive conjugate condition for $\emptyset$. Let $g_1(x) = 0$ for all $x \in X$. Since $\emptyset \subset \text{supp}(g_1)$ and $I(g_1) = 0$, there exists $M > 0$ and for any $\epsilon > 0$ there exists $f_2 \in K_1^+$ such that $\text{supp}(f_2) \subset \text{supp}(g_2) \subset X$ implies

\[ \left| -I(g_2) + M \right| < \epsilon. \]

Then we have $I(g_2) < M + \epsilon$. For any $h \in K_1^+$ there exists $g_2 \in K_1^+$ such that

\[ h \leq f_2 \vee h \leq g_2. \]

It follows from monotonicity of $I$ that

\[ I(h) \leq I(g_2) < M + \epsilon. \]

This means that $I$ is bounded above. \qed

We need the next two lemmas to show that these conditions are necessary and sufficient conditions for representation by one Choquet integral.

**Lemma 3.1.** Let $A \in B$ and $f \in K^+$. Suppose that $A \subset \{x \mid f \geq 1\}$; then we have

\[ \mu_A^+(A) \leq I(f) \quad \text{and} \quad \mu_A^-(A) \leq -I(-f). \]

**Proof.** Let $O_\varepsilon = \{x \mid f(x) > 1 - \varepsilon\}$, where $0 < \varepsilon < 1$. Then $O_\varepsilon$ is open and $A \subset O_\varepsilon$.

Let $g \in K_1^+$ such that $\text{supp}(g) \subset O_\varepsilon$. Then $x \in \text{supp}(g)$ implies $f(x)/(1 - \varepsilon) \geq 1$. Therefore we have $g \leq f/(1 - \varepsilon)$. It follows from the monotonicity and positive homogeneity of $I$ that $I(g) \leq (1/(1 - \varepsilon))I(f)$. Therefore

\[ \mu(A) \leq \mu(O_\varepsilon) \leq \frac{1}{1 - \varepsilon}I(f). \]

Since $\varepsilon$ is arbitrary, we have $\mu(A) \leq I(f)$.

The part of $\mu_A^-(A)$ is much the same. \qed
Applying Lemma 3.1, we have the next lemma. The details of the proof are in Section 4.

**Lemma 3.2.** Let \( C \in \mathcal{C} \), let \( I \) be a c.m. functional and \( \mu_I^+ \) and \( \mu_I^- \) the regular fuzzy measure induced by \( I \).

1. \( I \) satisfies the positive conjugate condition for every \( C \in \mathcal{C} \) if and only if
   \[
   \mu_I^+(C) = (\mu_I^-)^*(C)
   \]
   for every \( C \in \mathcal{C} \).
2. \( I \) satisfies the negative conjugate condition for \( C \) if and only if
   \[
   \mu_I^+(C) = (\mu_I^-)^*(C)
   \]
   for every \( C \in \mathcal{C} \).

Suppose that a c.m. functional \( I \) satisfies the positive conjugate condition for all \( C \in \mathcal{C} \). It follows from Lemma 3.2 that

\[
\mu_I^-(X) = \sup\{ \mu_I^-(C) \mid C \subset X \} = \sup\{ (\mu_I^-)^*(C) \mid C \subset X \}
\]

\[
= \sup\{ \mu_I^+(X) - \mu_I^+(C^c) \mid C \subset X \} \leq \mu_I^+(X).
\]

Therefore we have the next corollary.

**Corollary 3.1.** If a c.m. functional \( I \) satisfies the positive or negative conjugate condition for all \( C \in \mathcal{C} \), then \( I \) is bounded.

Applying Lemma 3.2, we have the next theorem.

**Theorem 3.1.** Let \( I \) be a c.m. functional.

1. If \( I \) satisfies the positive conjugate condition for all \( C \in \mathcal{C} \), we have
   \[
   I(f) = (C) \int f \, d\mu_I^+
   \]
   for all \( f \in L \).

2. If \( I \) satisfies the negative conjugate condition for all \( C \in \mathcal{C} \), we have
   \[
   I(f) = -(C) \int -f \, d\mu_I^-
   \]
   for all \( f \in K \).

**Proof.** We prove (1). The proof of (2) is much the same. Let \( f \in K \). It follows from Theorem 2.1 that there exist unique regular nonadditive measures \( \mu_I^+ \) and \( \mu_I^- \) such that

\[
I(f) = (C) \int (f \vee 0) \, d\mu_I^+ - (C) \int -(f \wedge 0) \, d\mu_I^-.
\]
Let $r$ be a real number such that $0 < r < 1$. Since $\{x \mid f^-(x) \geq r\}$ is a compact set, we have

$$\mu_I^-(\{x \mid f^-(x) \geq r\}) = (\mu_I^+)^c(\{x \mid f(x) \geq r\}).$$

Then we have

$$(C) \int f^- \, d\mu_I^- = \int_0^\infty \mu_I^-(\{x \mid f(x) \geq r\}) \, dr = \int_0^\infty (\mu_I^+)^c(\{x \mid f^-(x) \geq r\}) \, dr = (C) \int f^- \, d(\mu_I^+)^c.$$

Therefore we have

$$I(f) = (C) \int (f \vee 0) \, d\mu_I^+ - (C) \int -(f \wedge 0) \, d(\mu_I^+)^+ = (C) \int f \, d\mu.$$ 

If $X$ is separable, the converse of Theorem 3.1 is valid.

**Theorem 3.2.** Let $X$ be separable and $I$ be a c.m. functional on $K$ that is bounded, and $\mu_I^+$ and $\mu_I^-$ the regular fuzzy measure induced by $I$.

1. If $I(f) = (C) \int f \, d\mu_I^+$ for all $f \in K$, then $I$ satisfies the positive conjugate condition for all $C \in \mathcal{C}$.
2. If $I(f) = -(C) \int -f \, d\mu_I^-$ for all $f \in K$, then $I$ satisfies the negative conjugate condition for all $C \in \mathcal{C}$.

**Proof.** (1) Let $f \in K^+$. It follows from Theorem 2 that

$$I(-f) = -(C) \int f \, d\mu_I^-.$$

Since $-f \in K$, we have

$$I(-f) = -(C) \int f \, d\mu_I^+)^c$$

from Definition 2.3. Therefore we have

$$(C) \int f \, d\mu_I^- = \int f \, d(\mu_I^+)^c$$

for all $f \in K^+$. Applying Proposition 2.2, we have

$$\mu_I^-(C) = (\mu_I^+)^c(C)$$

for every $C \in \mathcal{C}$. It follows from Lemma 3.2 that $I$ satisfies the positive conjugate condition for every $C \in \mathcal{C}$.

(2) It is much the same as (1). 

$\square$
4. Proof of Lemma 3.2

In this section, the proof of Lemma 3.2(1) is shown. (2) can be proved in the same way.

Let \( \epsilon > 0 \) and \( C \in \mathcal{C} \).

First suppose that a c.m. functional \( I \) satisfies the positive conjugate condition for every compact set \( C \). That is, there exists a positive real number \( M \) such that \( \forall \epsilon > 0, \exists f_1, f_2 \in K_1, 1_C \leq g_1 \leq f_1 \), and \( f_2 \leq g_2 \leq 1_C \) with \( \text{supp}(f_2) \subset \text{supp}(g_2) \subset C^c \) imply

\[
M - I(g_2) - \epsilon < -I(-g_1) < M - I(g_2) + \epsilon
\]  

(1)

for \( g_1, g_2 \in K_1 \).

Since \( \mu^- \) is regular, there exists an open set \( O \) such that \( C \subset O \) and

\[
\mu^- (C) + \epsilon \geq \mu^- (O).
\]  

(2)

Using Uryson’s lemma, there exists \( h_1 \in K_1^+ \) such that \( 1_C \leq h_1 \leq 1_O \). Since \( 1_C \leq f_1 \), we may suppose that \( f_1 \geq h_1 \). It follows from Lemma 3.1 that

\[
\mu^- (C) \leq -I(-h_1).
\]  

(3)

Since \( \text{supp}(h_1) \subset O \), we have

\[
\mu^- (O) \geq -I(-h_1)
\]  

(4)

from the definition of \( \mu^- \). Then it follows from (2) and (4) that

\[
\mu^- (C) + \epsilon \geq -I(-h_1).
\]  

(5)

Since \( C^c \) is an open set, it follows from the definition of the induced regular fuzzy measure \( \mu^+ \) that there exists \( h_2 \in K_1^+ \) such that \( \text{supp}(h_2) \subset C^c \) and

\[
I(h_2) \geq \mu^+ (C^c) - \epsilon.
\]  

(6)

We may suppose that \( f_2 \leq h_2 \leq 1_C \). Then applying (5) and (6), we have

\[
\mu^- (C) + \epsilon \geq M - I(h_2) - \epsilon.
\]  

(7)

Since we have \( I(h_2) \leq \mu^+ (C^c) \) from \( \text{supp}(h_2) \subset C^c \), we have

\[
\mu^- (C) + \epsilon \geq M - \mu^+ (C^c) - \epsilon.
\]  

(8)

Since \( I \) satisfies the conjugate condition for \( \emptyset \), we have \( M = \mu^+(X) \). Therefore we have

\[
2\epsilon \geq (\mu^+)^c (C) - \mu^- (C)
\]  

(9)

from (8).

On the other hand, it follows from (1), (2), and (6) that

\[-I(-h_1) \leq M - I(h_0) + \epsilon \leq M - (\mu^+ (C^c) - \epsilon) + \epsilon \leq (\mu^+)^c (C) + 2\epsilon.
\]

Therefore we have

\[
|\mu^- (C) - (\mu^+)^c (C)| \leq 2\epsilon.
\]

Since \( \epsilon \) is an arbitrary, we have \( \mu^- (C) = (\mu^+)^c (C) \).
Next suppose that $\mu_\triangledown(C) = (\mu_\triangledown)^\triangledown(C)$. Define $M = \mu_\triangledown(X)$. Then it follows from the definition of the conjugate of $\mu_\triangledown$ that

$$\mu_\triangledown(C) = M - \mu_\triangledown(C^c). \quad (10)$$

Since $\mu_\triangledown$ is regular, there exists an open set $O$ such that $O \supset C$ and

$$\mu_\triangledown(C) + \varepsilon \geq \mu_\triangledown(O). \quad (11)$$

Using Uryson’s lemma, there exists $f_1 \in K_1^+$ such that $1_C \leq f_1 \leq 1_O$. Then for every $g_1 \in K_1^+$ such that $1_C \leq g_1 \leq f_1$, we have

$$\mu_\triangledown(O) \geq -I(-g_1) \geq \mu_\triangledown(C) \quad (12)$$

from Lemma 3.1. It follows from the definition of the induced regular nonadditive measure $\mu_\triangledown$ that there exists $f_2 \in K_1^+$ such that $\text{supp}(f_2) \subset C^c$ and

$$\mu_\triangledown(C^c) - \varepsilon \leq I(f_2). \quad (13)$$

Therefore for every $g_2 \in K_1^+$ such that $f_2 \leq g_2 \leq C^c$ and $\text{supp}(f_2) \subset \text{supp}(g_2) \subset C^c$, we have

$$\mu_\triangledown(C^c) - \varepsilon \leq I(f_2) \leq I(g_2) \leq \mu_\triangledown(C^c). \quad (14)$$

It follows from (10)–(12) that

$$M - \mu_\triangledown(C^c) + \varepsilon \geq -(g_2). \quad (15)$$

Then we have

$$\varepsilon \geq -M - I(-g_1) + I(g_2) \quad (15)$$

from (14). On the other hand, it follows from (10) and (14) that

$$I(g_2) + \varepsilon \geq M - \mu_\triangledown(C). \quad (16)$$

Then we have

$$\varepsilon \geq M - I(g_2) + I(-g_1) \quad (17)$$

from (12). Therefore we have

$$|I(-g_1) - I(g_2) + M| < \varepsilon \quad (18)$$

from (15) and (17). \hfill \Box

References