Propagation of singularities in Cauchy problems for quasilinear thermoelastic systems in three space variables

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Received 19 June 2003
Submitted by B. Straughan

Abstract

By using microlocal analysis, the propagation of weak singularities in Cauchy problems for quasilinear thermoelastic systems in three space variables are investigated. First, paradifferential operators are employed to decouple the quasilinear thermoelastic systems. Second, by investigating the decoupled hyperbolic–parabolic systems and using the classical bootstrap argument, the property of finite propagation speeds of singularities in Cauchy problems for the quasilinear thermoelastic systems is obtained. Finally, it is shown that the microlocal weak singularities for Cauchy problems of the thermoelastic systems are propagated along the null bicharacteristics of the hyperbolic operators.

Keywords: Microlocal analysis; Paradifferential operators; Hyperbolic–parabolic coupled systems; Cauchy problems

1. Introduction

The equations of thermoelasticity describe the elastic and the thermal behaviors of elastic heat conductive media, in particular the reciprocal actions between the elastic stress and the temperature difference. The equations for the displacement vector $U$ and the temperature difference $\theta$ are hyperbolic–parabolic coupled systems [9,10,12,16]. There are many results on the well-posedness of initial-boundary value problems in the thermoelastic sys-

* This work is partially supported by the NSF and the Educational Ministry of China.
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tems (cf. [9,10,12,16] and references therein). For example, in [12] Racke obtained the well-posedness of the Cauchy problem for the quasilinear thermoelasticity in three space variables. Mukoyama [10] proved the same result of the quasilinear thermoelasticity in two space variables.

As pointed out by Dafermos and Hsiao in [8], and Rivera and Racke in [11], the solutions to the thermoelastic systems will develop singularities in general when time evolves. Then it is an important and interesting question to study how the singularities are propagated in the coupled systems. In order to describe its behaviors accurately, decoupling the systems is necessary. How to decouple it efficiently?

There is a rich literature on propagation of singularities for hyperbolic equations (see [1–5] and references therein). Especially, in [3] Bony introduced paradifferential operators to study the propagation of singularities, which is also employed by the author in this paper. Recently several interesting and important results concerning the propagation of singularities of the linear and semilinear hyperbolic–parabolic coupled systems are obtained by Chen, Racke, Reissig and Wang (see [6,7,13–15]). More precisely, Racke and Wang in [13] studied the propagation of singularities in thermoelasticity in one space variable by using the Fourier analysis, and the corresponding result in three space variables was studied by Reissig and Wang in [14]. In [15] Wang introduced an argument to microlocally decouple the semilinear thermoelastic systems, and obtained results on the propagation of singularities of solutions to the Cauchy problem for the semilinear thermoelastic systems both in one and three space variables by using nonsmooth pseudodifferential operators. In [6] Chen and Wang investigated the propagation of singularities both in Cauchy problems and in interior domains for the semilinear hyperbolic–parabolic coupled systems by using paradifferential operators, and the corresponding problems for the compressible Navier–Stokes equation was studied by Chen and Wang in [7].

Roughly speaking, these results showed that the singularities are propagated along the null bicharacteristics of the hyperbolic operators in coupled systems. The purpose of this paper is to use the idea of decoupling the hyperbolic–parabolic coupled systems by paradifferential operators (see [6,7,15]) to study the propagation of singularities for Cauchy problems of a quasilinear thermoelastic system in three space variables.

Let us consider the following Cauchy problem in three space variables:

$$
\begin{align*}
\frac{\partial U}{\partial t} & = \nabla((2\mu + \lambda)\nabla U) + \nabla \times (\mu \nabla \times U) + \nabla(\gamma \theta) = g(U, \theta, \nabla U), \\
\frac{\partial \theta}{\partial t} & = \left( \sum_{i,j=1}^{3} b_{ij} \partial_{x_i}^2 \partial_{x_j} U + \sum_{i,j=1}^{3} c_{ij} \partial_{x_i}^2 \theta + \sum_{i=1}^{3} e_{i} \partial_{x_i} \theta \right)
\end{align*}
(1.1)
$$

where $U$ and $\theta$ are the displacement and the temperature difference of the elastic media, $\nabla$ and $\nabla'$ represent the gradient and the divergence operators with respect to the spatial variables, and

$$
\begin{align*}
\mu & := \mu(U, \theta), & \lambda & := \lambda(U, \theta), & \gamma & := \gamma(U, \theta), \\
d & := d(U, \theta, \nabla U, \nabla \theta), & c_{ij} & := c_{ij}(\theta, \nabla U, \nabla \theta), \\
g & := g(U, \theta, \nabla U), & b_{ij} & := b_{ij}(\theta, \nabla U, \nabla \theta), & e_{i} & := e_{i}(\theta, \nabla U)
\end{align*}
$$

are all smooth with their arguments.
Furthermore, by $k_0$ representing a positive constant we assume the conditions
\[
\begin{cases}
\mu + \lambda \geq k_0 > 0, \\
\sum_{i, j=1}^3 c_{ij} \xi_i \xi_j \geq k_0 \sum_{i=1}^3 \xi_i^2, \quad \forall (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \setminus \{0\},
\end{cases}
\]
hold in this paper. The system in (1.1) is derived from the quasilinear thermoelastic systems
given by Jiang and Racke in [9,12] for the special case of the coefficients $\mu, \lambda, \gamma, b_{ij}, c_{ij},$
and $e_i$ in order to decouple this quasilinear system later.

As mentioned above, it was shown by Racke in [12] that under the assumptions
\[
U_0 \in H^{s+1}(\mathbb{R}^3), \quad (U_1, \theta_0) \in H^s(\mathbb{R}^3), \tag{1.2a}
\]
for a fixed $s > 9/2$, there is $T > 0$, such that the Cauchy problem of (1.1) admits a unique solution $(U, \theta)$ satisfying
\[
\begin{cases}
U \in C([0, T], H^{s+1}(\mathbb{R}^3)) \cap C^1([0, T], H^s(\mathbb{R}^3)), \\
\theta \in C([0, T], H^{s+1}(\mathbb{R}^3)) \cap C^1([0, T], H^{s-2}(\mathbb{R}^3)). \tag{1.2b}
\end{cases}
\]

For convenience, we shall always assume that the condition (1.2a) is satisfied and the local solution $(U, \theta)$ of (1.1) admits the regularity (1.2b) in the following discussion.

The main results of this paper are as follows.

**Theorem 1.1.** Let $\omega \subset \{ t = 0 \}$ be an open subset of $\mathbb{R}^3$, and for any fixed $s > 9/2$, $(U, \theta)$ be the unique local solution of (1.1) given in (1.2b). Let $\Omega \subset [0, T] \times \mathbb{R}^3$ be the determinacy domain of $\omega$ for the operator $\partial_t^2 - (2\mu + \lambda)\Delta$. If the initial data further satisfy
\[
(U_0, U_1) \in C^\infty(\omega), \quad \theta_0 \in C^\infty(\omega),
\]
then the solution of the Cauchy problem (1.1) satisfies
\[
(U, \theta) \in C^\infty(\Omega). \tag{1.3}
\]

Before giving the results on the microlocal regularities of the solution, let us recall a typical notation from [1,2] as follows.

Let $(\tau, \xi)$ be the dual variables of $(t, x) \in \mathbb{R}^{n+1}$. For any $-\infty < s \leq \gamma < \infty, u \in H^s \cap H^\gamma_{ml}(t_0, x_0; \tau_0, \xi_0)$ means that there exists a smooth $\phi(t, x)$, supported near $(t_0, x_0)$ with $\phi(t_0, x_0) = 1$, and a cone $K$ in $\mathbb{R}^{n+1}$ about the direction $(\tau_0, \xi_0)$ such that
\[
\langle \tau, \xi \rangle^s |\hat{\phi}u(\tau, \xi)| \in L^2(\mathbb{R}^{n+1})
\]
and
\[
\langle \tau, \xi \rangle^\gamma |\chi_K(\tau, \xi)\hat{\phi}u(\tau, \xi)| \in L^2(\mathbb{R}^{n+1}),
\]
where $\chi_K$ is the characteristic function of $K$, $\langle \tau, \xi \rangle = (1 + \tau^2 + |\xi|^2)^{1/2}$ and $\hat{\phi}u$ is the Fourier transform of $\phi u$. If $\Gamma$ is a closed conic set in $T^*(\mathbb{R}^{n+1}) \setminus \{0\}$ (that is, conic in the $(\tau, \xi)$ variables), we shall say $u \in H^s \cap H^\gamma_{ml}(\Gamma)$ if $u \in H^s \cap H^\gamma_{ml}(t, x, \tau, \xi)$ for all $(t, x, \tau, \xi) \in \Gamma$.

**Theorem 1.2.** For any fixed $9/2 < s < 5/2$ and a fixed point $(x_0, \xi_0) \in T^*(\mathbb{R}^3) \setminus \{0\},$
suppose that
\[\psi(\theta, \eta)\]
operators introduced by Bony (see \cite{3,5}). Suppose that
pled systems (consequence of Theorems 1.1 and 1.2, we shall prove Theorem 1.3 in Section 5. As a
proved in Sections 3 and 4, respectively, by using the classical bootstrap argument. As a
moreover, let
\[S(\eta)\] is called a paraproduct operator, where \(\hat{u}\) represents the Fourier transformation of \(u\).

\begin{align}
U_0 & \in H^{r+1} \cap H^{s+1}_{\text{ml}}(x_0, \xi_0), \\
(U_1, \theta_0) & \in H^r \cap H^s_{\text{ml}}(x_0, \xi_0).
\end{align}

and let \((U, \theta)\) be the unique solution of (1.1) given in (1.2b), \(\Gamma(t) = \{(t, x(t), \tau(t), \xi(t)) \in T^*(\mathbb{R}^3) \mid 0 \leq t < T\}\) be a null bicharacteristic for the hyperbolic operator \(\partial_t^2 - (2\mu + \lambda)\Delta\) or \(\partial_t^2 - \mu\Delta\) in (1.1) passing through \((x_0, \xi_0)\). Then the unique solution \((U, \theta)\) satisfies

\begin{align}
U & \in C([0, T], H^{r+1} \cap H^{s+1}_{\text{ml}}(x(t), \xi(t))) \\
& \cap C^1([0, T], H^r \cap H^s_{\text{ml}}(x(t), \xi(t)))).
\end{align}

**Theorem 1.3.** Let \(\{t = g_1(x)\}\) and \(\{t = g_2(x)\}\) be two forward light cones of \(\partial_t^2 - (2\mu + \lambda)\Delta\) and \(\partial_t^2 - \mu\Delta\), respectively, issuing from the origin and \(R_+ = \{t > g_1(x), t \neq g_2(x)\}\). For any fixed \(s \geq 9/2\), suppose that the initial data in the problem (1.1) satisfy

\begin{align}
U_0 & \in H^{r+1} \cap C^\infty(\mathbb{R}^3 \setminus 0), \\
(U_1, \theta_0) & \in H^r \cap C^\infty(\mathbb{R}^3 \setminus 0).
\end{align}

Then the local solution of (1.1) satisfies

\begin{align}
U & \in C([0, T], H^{2s-\epsilon-5/2}_{\text{loc}}(R_+)) \cap C^1([0, T], H^{2s-\epsilon-7/2}_{\text{loc}}(R_+)), \\
\theta & \in C([0, T], H^{2s-\epsilon-7/2}_{\text{loc}}(R_+)).
\end{align}

The remainder of this paper is arranged as follows.

In Section 2, we shall first paralinearize and decouple the hyperbolic–parabolic coupled systems (1.1) by using paradifferential operators, and Theorems 1.1 and 1.2 will be proved in Sections 3 and 4, respectively, by using the classical bootstrap argument. As a consequence of Theorems 1.1 and 1.2, we shall prove Theorem 1.3 in Section 5.

2. Paralinearizing and decoupling thermoelastic systems

For convenience, let us first recall the definitions of paraproduct and paradifferential operators introduced by Bony (see \cite{3,5}). Suppose that \(\psi(\theta, \eta) \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus 0))\) is nonnegative, homogeneous of order zero, and there are small \(0 < \epsilon_1 < \epsilon_2\) such that

\[\psi(\theta, \eta) = \begin{cases} 1, & |\theta| \leq \epsilon_1 |\eta|, \\ 0, & |\theta| \geq \epsilon_2 |\eta|, \end{cases}\]

moreover, let \(S(\eta) \in C^\infty(\mathbb{R}^n)\) satisfy

\[S(\eta) = \begin{cases} 0, & |\eta| \leq R, \\ 1, & |\eta| \geq 2R. \end{cases}\]

Then, for any \(a, u \in S'(\mathbb{R}^n)\), and \(\chi(\theta, \eta) = \psi(\theta, \eta)S(\eta)\), the operator \(T_a\) defined by

\[T_a u(x) = F_{\xi \rightarrow x}^{-1} \left( \int \chi(\xi - \eta) \hat{u}(\xi - \eta) \hat{a}(\eta) \, d\eta \right)\]

is called a paraproduct operator, where \(\hat{u}\) represents the Fourier transformation of \(u\).
Suppose that \( l(x, \xi) \) is homogeneous of order \( m \) for \( \xi \in \mathbb{R}^n \), smooth with respect to \( \xi \neq 0 \), and for any \( \alpha \in \mathbb{N}^n \), \( D_x^\alpha l(x, \xi) \) is in \( H^{s}(s > n/2) \) with respect to \( x \), then the following operator \( T_l \in \text{Op}(\sum_{\rho}) \):

\[
T_l u(x) = F_{\xi \rightarrow x}^{-1} \left( \int \chi(\xi - \eta, \eta) \hat{l}(\xi - \eta, \eta) \hat{u}(\eta) \, d\eta \right)
\]

is called a paradifferential operator of order \( m \) associated with symbol \( l(x, \xi) \), where \( \hat{l}(\theta, \xi) \) is the Fourier transformation of \( l(x, \xi) \) with respect to \( x \in \mathbb{R}^n \).

If the symbol \( l(t, x, \xi) \) is \( k \)th order continuously differentiable with respect to a parameter \( t \in I \), we say the corresponding operator \( T_l \) belonging to \( C^{k}(I, \text{Op}(\sum_{\rho})) \).

For paraproduct and paradifferential operators, as in [7], we have the following results, which can be obtained as in [4].

**Lemma 2.1.** (1) For any \( \gamma > n/2, m \in \mathbb{Z} \), the operator

\[
T_l \in C \left( [0, T], \text{Op}\left( \sum_{\rho} \right) \right) : C([0, T], H^{s}(\mathbb{R}^n)) \rightarrow C([0, T], H^{s-m}(\mathbb{R}^n))
\]

is bounded for any \( s \in \mathbb{R} \).

(2) For any paradifferential operators \( T_{l_1} \in C([0, T], \text{Op}(\sum_{\rho_1})) \) and \( T_{l_2} \in C([0, T], \text{Op}(\sum_{\rho_2})) \) with \( m_1, m_2 \in \mathbb{R} \) and \( \rho > 1 \), we have

\[
[T_{l_1}, T_{l_2}] \in C \left( [0, T], \text{Op}\left( \sum_{\rho} \right) \right).
\]

(3) Suppose that the symbol of \( L \in C([0, T], \text{Op}(\sum_{\rho}(\Omega))) \) is \( l = l_m + l_{m-1} + \cdots + l_{m-[\rho]} \) satisfying \( l_{m-k}(t, x, \xi_0) \neq 0 \) for a fixed \( (x_0, \xi_0) \in T^{\ast}(\Omega) \backslash 0 \), where \( l_{m-k}(t, x, \xi) \) is smooth and homogeneous of order \( m - k \) with respect to \( \xi \) and in \( H^{\rho-k} \) with respect to \( x \). Then there are \( H, H' \in C([0, T], \text{Op}(\sum_{\rho}(\Omega))) \) such that

\[
LH = I + R, \quad H'L = I + R',
\]

where \( R \) and \( R' \) are \((\rho - n/2)\)-regular at \((x_0, \xi_0)\) microlocally, which means that \( R \) and \( R' \) are bounded from \( C([0, T], H^{s}(\Omega)) \) to \( C([0, T], H^{s+\rho-n/2}(\Omega)) \).

**Lemma 2.2.** (1) If \( a \in H^{s} \) and \( b \in H^{s} \) with \( s > n/2 \) and \( t \geq n/2 \), then we have

\[
ab = T_a b + T_b a + r,
\]

where \( r \in H^{s+t-n/2} \).

(2) Suppose that \( F(y_1, y_2, \ldots, y_N) \) is smooth and each derivative of \( F \) is bounded on any compact set \( K \subseteq \mathbb{R}^n \). Then for any \( u^i \in H^{s}(\mathbb{R}^n), s > n/2 \), \( i = 1, 2, \ldots, N \), we have the following paralinearization identity:

\[
F(u^1(x), u^2(x), \ldots, u^N(x)) = \sum_{j=1}^{N} T_{a_j} F(a^1(x), a^2(x), \ldots, a^N(x)) u^j(x) + R(x),
\]

where \( R \in H^{2s-n/2}(\mathbb{R}^n) \).
Return to the system (1.1). First let us decompose the displacement $U$ into its curl-free part $U^P$ and divergence-free part $U^S$, namely, $U = U^P + U^S$, $\nabla \times U^P = 0$, $\nabla \cdot U^S = 0$. Then $(U^P, U^S, \theta)$ satisfy the following system:

$$
\begin{align*}
\begin{cases}
U^P_t & = \nabla((2\mu + \lambda)\nabla U^P) + \nabla(\gamma \theta) = g^P(U^P, U^S, \theta, \nabla U^P, \nabla U^S), \\
U^S_t & = \nabla \times (\mu \nabla \times U^S) = g^S(U^P, U^S, \theta, \nabla U^P, \nabla U^S), \\
\theta_t & = \sum_{i,j=1}^3 c_{ij} \bar{\alpha}^2 \theta + \sum_{i,j=1}^3 b_{ij} \bar{\alpha} \theta (U^P + U^S) + \sum_{i=1}^3 e_i \bar{\alpha}^2 (U^P + U^S) + d(U^P, U^S, \theta, \nabla U^P, \nabla U^S, \nabla \theta), \\
U^P|_{t=0} & = U^P_0(x), \quad U^S|_{t=0} = U^S_0(x), \quad \theta|_{t=0} = \theta_0(x),
\end{cases}
\end{align*}
$$

where $\lambda, \mu, \gamma$ smoothly depend on $(U^P, U^S, \theta)$, $b_{ij}, c_{ij}, e_i$ smoothly depend on $(U^P, U^S, \theta, \nabla U^P, \nabla U^S)$, and $g = g^P + g^S$ with $\nabla \times g^P = 0$ and $\nabla g^S = 0$.

The notations $\bar{\alpha}_x, \bar{\alpha}^2_x, \bar{\alpha}^2_{ij}$ are used here and, for simplicity, we shall use them as well in following discussion.

After slightly modifying $g^P$ and $g^S$, (2.1) can be rewritten as follows:

$$
\begin{align*}
\begin{cases}
U^P_t & = (2\mu + \lambda)\Delta U^P + \gamma \nabla \theta = g^P(U^P, U^S, \theta, \nabla U^P, \nabla U^S), \\
U^S_t & = -\mu \Delta U^S = g^S(U^P, U^S, \theta, \nabla U^P, \nabla U^S), \\
\theta_t & = \sum_{i,j=1}^3 c_{ij} \bar{\alpha}^2 \theta + \sum_{i,j=1}^3 b_{ij} \bar{\alpha} \theta (U^P + U^S) + \sum_{i=1}^3 e_i \bar{\alpha}^2 (U^P + U^S) + d(U^P, U^S, \theta, \nabla U^P, \nabla U^S, \nabla \theta), \\
U^P|_{t=0} & = U^P_0(x), \quad U^S|_{t=0} = U^S_0(x), \quad \theta|_{t=0} = \theta_0(x),
\end{cases}
\end{align*}
$$

In the remainder of this section, we will paralinearize and decouple the system (2.2) microlocally and always use $\psi_{1,0}$ to represent the set of classical pseudodifferential operators of order $s$.

Let

$$
\begin{align*}
U^P_\alpha & = (\bar{\alpha}_x \pm i \alpha \Lambda)U^P, \\
U^S_\beta & = (\bar{\alpha}_x \pm i \beta \Lambda)U^S,
\end{align*}
$$

with $\alpha = (2\mu + \lambda)^{1/2}$, $\beta = \mu^{1/2}$, and $\Lambda = (1 - \Delta)^{1/2}$, then we have

$$
\begin{align*}
\partial_t U^P & = \frac{1}{2}(U^P_\alpha + U^P_\beta), \quad \Delta U^P = \frac{1}{2\mu}(U^P_\alpha - U^P_\beta), \\
\partial_t U^S & = \frac{1}{2}(U^S_\alpha + U^S_\beta), \quad \Delta U^S = \frac{1}{2\mu}(U^S_\alpha - U^S_\beta).
\end{align*}
$$

Using (2.3) and (2.4), we can deduce from (2.2) that

$$
\begin{align*}
\begin{cases}
(\partial_t - \pm i \alpha \Lambda)U^P_\alpha + \gamma \nabla \theta = G^P(U^P_\alpha, U^P_\beta, U^S_\alpha, U^S_\beta, \theta), \\
(\partial_t + \pm i \alpha \Lambda)U^P_\beta + \gamma \nabla \theta = G^P(U^P_\alpha, U^P_\beta, U^S_\alpha, U^S_\beta, \theta), \\
(\partial_t - \pm i \beta \Lambda)U^S_\alpha = G^S(U^P_\alpha, U^P_\beta, U^S_\alpha, U^S_\beta, \theta), \\
(\partial_t + \pm i \beta \Lambda)U^S_\beta = G^S(U^P_\alpha, U^P_\beta, U^S_\alpha, U^S_\beta, \theta), \\
\theta_t & = \sum_{i,j=1}^3 c_{ij} \bar{\alpha}^2 \theta + a_1 P_1(D)U^P_\alpha + a_2 P_2(D)U^P_\beta + a_3 P_3(D)U^S_\alpha + a_4 P_4(D)U^S_\beta = \tilde{d},
\end{cases}
\end{align*}
$$

where $a_j$ $(j = 1, \ldots, 4)$ and $\tilde{d}$ depend on $(U^P_\alpha, U^P_\beta, U^S_\alpha, U^S_\beta, \theta, \nabla \theta)$ smoothly, and $P_j(D)$ $\in \psi_{1,0}(\mathbb{R}^3)$ $(j = 1, \ldots, 4)$.
Denoting by $V = (U_+^P, U_+^P, U_+^s, U_+^s, \theta)$, $V_k$ the $k$th component of $V$, and using Lemma 2.2 to (2.5), we obtain

$$
\begin{cases}
(\partial_t - iT\alpha\Lambda)U_+^P + T\gamma\nabla\theta + Tf_1 V = R_1, \\
(\partial_t + iT\alpha\Lambda)U_+^P + T\gamma\nabla\theta + Tf_2 V = R_2, \\
(\partial_t - iT\beta\Lambda)U_+^s + Tf_3 V = R_3, \\
(\partial_t + iT\beta\Lambda)U_+^s + Tf_4 V = R_4, \\
\theta_t - \sum_{i,j=1}^3 T_{cij} \partial^2_{ij}\theta + T_{a.}\nabla\theta + \sum_{i=1}^4 T_{ia} P_i(D)V_k + Tf_5 V = R_5,
\end{cases}
$$

(2.6)

where, for $1 \leq k \leq 4$, $f_k \in C([0,T], H^{3-1}(R^3))$ smoothly depend on $(V, P_1^1(D)V)$, $f_5 \in C([0,T], H^{2-2}(R^3))$ smoothly depend on $(V, P_1^1(D)V, P_2^2(D)\theta)$ with $P_i^1(D) \in \psi_{1,0}(R^3)$, $R_i \in C([0,T], H^{2-5/2}(R^3))$ for $i = 1, \ldots, 4$, $R_5 \in C([0,T], H^{2-9/2}(R^3))$ and

$$
T_{a.}\nabla\theta = \sum_{i,j=1}^3 T_{cij} \partial^2_{ij}\theta + T_{a.}\nabla\theta + \sum_{i=1}^4 T_{ia} P_i(D)V_k + Tf_5 V = R_5,
$$

which implies $a = (a^1, a^2, a^3) \in C([0,T], H^{3-2}(R^3))$.

Denoting by

$$
A_c = \left(1 - \sum_{i,j=1}^3 T_{cij} \partial^2_{ij}\right)^{1/2}
$$

obviously, the system (2.6) can be expressed as

$$
\partial_t V + A_2 V + A_1 V + A_0 V = R,
$$

(2.7)

where

$$
A_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
$$

$$
A_1 = \begin{pmatrix}
-iT\alpha\Lambda & iT\alpha\Lambda & T\gamma\nabla & T\gamma\nabla & T\gamma\nabla \\
-iT\beta\Lambda & iT\beta\Lambda & 0 & 0 & 0 \\
T_{a_1} P_1(D) & T_{a_2} P_2(D) & T_{a_3} P_3(D) & T_{a_4} P_4(D) & T_{a.}\nabla \\
\end{pmatrix},
$$

$$
A_0 = (T_{f_1}, T_{f_2}, T_{f_3}, T_{f_4}, T_{f_5} - 1)^t, \quad R = (R_1, R_2, R_3, R_4, R_5)^t.
$$

In the following we shall decouple the system (2.7).

Denoting by $A_{11}^{(i)}, A_{12}^{(i)}, A_{21}^{(i)}, A_{22}^{(i)}$ the corresponding blocks of $A_i$, which is $12 \times 12$, $12 \times 1, 1 \times 12,$ and $1 \times 1$, respectively. As in [6,7,15], let

$$
K = \begin{pmatrix}
0 & K_{12} \\
K_{21} & 0 \\
\end{pmatrix},
$$
where $K_{12}$ and $K_{21}$ are $12 \times 1$, $1 \times 12$ matrixes, respectively, with their elements being $(-1)$-order paradifferential operators in spatial variables, which will be determined later. Then $W = (I + K)V$ satisfies the following system:

$$
+ ([K, A_1] - \partial_t K + (I + K)A_0)V = (I + K)R. 
$$

(2.8)

Setting $[K, A_2] + A_1 = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, where $B$ is a $12 \times 12$ matrix, $C$ is scalar, by simple computations we have

$$
K_{12}A_{22}^{(2)} + A_{12}^{(2)} = 0, \quad A_{22}^{(2)}K_{21} = A_{21}^{(2)}.
$$

$$
B = A_{12}^{(1)}, \quad C = A_{22}^{(1)}.
$$

Since the operator $A_{22}^{(2)} \in C([0, T], \text{Op}(\sum_{s-1}^{2}(\Omega)))$ is elliptic, by using Lemma 2.1, the following identities hold for two matrices $F, F' \in C([0, T], \text{Op}(\sum_{s-1}^{-2}(\Omega)))$:

$$
A_{22}^{(2)} F = I + \rho_1, \quad F' A_{22}^{(2)} = I + \rho_2,
$$

where $\rho_1, \rho_2$ are $(s - 3/2)$-regular operators.

Now if defining

$$
K = \begin{pmatrix} 0 & -A_{12}^{(1)} F' \\ FA_{21}^{(1)} & 0 \end{pmatrix},
$$

(2.9)

then $K \in C([0, T], \text{Op}(\sum_{s-1}^{-1}(\Omega))) \cap C([0, T], \text{Op}(\sum_{s-1}^{-2}(\Omega)))$, and we have

$$
[K, A_2] + A_1 = \begin{pmatrix} A_{11}^{(1)} & 0 \\ 0 & A_{22}^{(1)} \end{pmatrix} + \begin{pmatrix} 0 & -A_{12}^{(1)} \rho_2 \\ -\rho_1 A_{21}^{(1)} & 0 \end{pmatrix}
$$

and $(I + K) \in C([0, T], \text{Op}(\sum_{s-1}^{0}(\Omega)))$ is elliptic. Its right and left parainverse-matrix $G, G'$ satisfy

$$
G'(I + K) = I + \rho_3, \quad (I + K)G = I + \rho_4,
$$

where $\rho_3$ and $\rho_4$ are $(s - 3/2)$-regular operators.

By directly simple computations, we obtain

$$
(I - K G')(I + K) = I - K \rho_3,
$$

which gives rise to

$$
V = G'W - \rho_3 V, \quad V = W - KG'W + K \rho_3 V.
$$

Thus, the system (2.8) can be rewritten as

$$
+ ([K, A_2]K \rho_3 - [K, A_1] \rho_3 - \partial_t K - (I + K)A_0 \rho_3)V = (I + K)R,
$$

where

$$
\rho_3, \rho_4 \in C([0, T], \text{Op}(\sum_{s-1}^{0}(\Omega))).
$$
which implies
\[ \partial_t W + A_2 W + \tilde{A}_1 W + \tilde{A}_0 W + \tilde{A}_{-1} V = \tilde{R}, \]  
(2.10)
where
\[ \tilde{A}_1 = \begin{pmatrix} A^{(1)}_{11} & 0 \\ 0 & A^{(1)}_{22} \end{pmatrix}, \]
\[ \tilde{A}_0 = ([K, A_1] + (I + K)A_0 - [K, A_2]K)G' + \begin{pmatrix} 0 & -A^{(1)}_{12} \rho_2 \\ -\rho_1 A^{(1)}_{21} & 0 \end{pmatrix}, \]
\[ \tilde{A}_{-1} = [K, A_2]K\rho_3 - [K, A_1]\rho_3 - \partial_t K - (I + K)A_0\rho_3, \]
\[ \tilde{R} = (I + K)R = \begin{pmatrix} R' - A^{(1)}_{12} F' R_0 \\ FA^{(1)}_{21} R'_{u} + R_{\theta} \end{pmatrix}, \]
and
\[ R'_{u} = (R_1, R_2, R_3, R_4)^t, \quad R_0 = R_5. \]

Denoting by \( W = (W_1, W_2)^t \) with \( W_1 \) being a 4 \( \times \) 1 vector, \( W_2 \) being scalar, using Lemma 2.2 once more, we have
\[ \begin{pmatrix} -iT_a A \\ iT_A A \\ -iT_\beta A \\ iT_\beta A \end{pmatrix} W_1 = \begin{pmatrix} -i\alpha A \\ i\alpha A \\ -i\beta A \\ i\beta A \end{pmatrix} W_1 + Tf_6 V + r_1, \]  
(2.11)
where
\[ Tf_6 \in C\left([0, T], Op(\sum_{s=1}^k (R^3))\right), \quad r_1 \in C([0, T], H^{(2s-5/2)}(R^3)), \]
and
\[ \sum_{i,j=1}^3 T_{ij} \partial^2_{ij} W_2 = \sum_{i,j=1}^3 c_{ij} \partial^2_{ij} W_2 - Tf_7 \nabla W_2 + r_2, \]  
(2.12)
where
\[ Tf_7 \in C([0, T], Op(\sum_{s=2}^k (R^3))), \quad r_1 \in C([0, T], H^{(2s-9/2)}(R^3)). \]

In the following discussion, we will always denote by \( Q_i \in C([0, T], Op(\sum_{s=2}^k (\Omega))) \) the \( k \)th order paradifferential operators.

By substituting (2.11) and (2.12) into (2.10) and shifting the lower order terms from the right to the left, we obtain the following decoupled systems up to order one for \( (W_1, W_2) \):
\[ \begin{cases} \partial_t W_1 + E W_1 + Q_0^1 W_1 = F_1(W), \\ \partial_t W_2 - \sum_{i,j=1}^3 c_{ij} \partial^2_{ij} W_2 + Q_2^1 W_2 = F_2(W), \end{cases} \]  
(2.13)
where
\[
E = \begin{pmatrix}
-\alpha \Lambda & i\beta \Lambda \\
\alpha \Lambda & -i\beta \Lambda
\end{pmatrix},
\]
and
\[
F_1(W) = F_1(Q_1^{-1}W_1, Q_0^0W_2), \quad F_2(W) = F_2(Q_2^0W_1, Q_2^0W_2).
\]

3. Proof of Theorem 1.1

As mentioned in Section 1, we shall use the classical bootstrap argument to gradually improve the regularity of the solutions in \(\Omega\). As in Section 1, we know that under the assumptions
\[
U_0 \in H^{s+1}(\mathbb{R}^3), \quad (U_1, \theta_0) \in H^s(\mathbb{R}^3),
\]
there is a unique solution \((U, \theta)\) satisfying
\[
\begin{cases}
U \in C([0, T], H^{s+1}(\mathbb{R}^3)) \cap C^1([0, T], H^s(\mathbb{R}^3)), \\
\theta \in C([0, T], H^s(\mathbb{R}^3)) \cap C^1([0, T], H^{s-2}(\mathbb{R}^3)),
\end{cases}
\]
(3.1)

Thus from (2.13) and the assumptions in Theorem 1.1, we know that \(W_2\) satisfies the following Cauchy problem:
\[
\begin{cases}
\partial_t W_2 - \sum_{i,j=1}^3 c_{ij} \partial_{ij}^2 W_2 + Q_1^2 W_2 = F_2(W), \\
W_2|_{t=0} \in H^s(\mathbb{R}^3) \cap C^\infty(\omega).
\end{cases}
\]
(3.2)

Taking a cut-off function \(\chi \in C_c^\infty(\Omega)\) such that \(\chi \equiv 1\) in a subdomain \(\Omega_1\) of \(\Omega\), then from (3.1) we know that \((\chi W_2)\) satisfies the following problem:
\[
\begin{cases}
(\partial_t - \sum_{i,j=1}^3 c_{ij} \partial_{ij}^2)(\chi W_2) + Q_1^2(\chi W_2) = \tilde{F}_2(W), \\
(\chi W_2)|_{t=0} \in H^\infty(\mathbb{R}^3),
\end{cases}
\]
(3.3)

where \(\tilde{F}_2(W) = \chi F_2(W) - [\chi, \partial_t - \sum_{i,j=1}^3 c_{ij} \partial_{ij}^2]W_2 - [\chi, Q_1^2]W_2\) belongs to \(L^2([0, T], H^s(\mathbb{R}^3))\) by using (3.1).

Since the principal part of (3.2) is parabolic, and the appearance of the lower order paradifferential operators does not influence the application of the classical theory of parabolic equation. Thus from (3.3) we can obtain
\[
(\chi W_2) \in L^2([0, T], H^{s+2}(\mathbb{R}^3)) \cap H^1([0, T], H^s(\mathbb{R}^3)),
\]
which implies
\[
W_2 \in L^2([0, T], H^{s+2}(\omega_t)) \cap H^1([0, T], H^s(\omega_t)),
\]
(3.4)

where \(\omega_t = \Omega \cap \{t = \tau\}\), by using the arbitrariness of \(\chi \in C_c^\infty(\Omega)\).
To improve the regularity of $W_1$, we know from (2.13) that $W_1$ satisfies the following Cauchy problem:

$$\begin{cases}
\partial_t W_1 + EW_1 + Q_1^0 W_1 = F_1(W), \\
W_1|_{t=0} \in H^s (R^3) \cap C^\infty (\omega),
\end{cases} \tag{3.5}$$

where $F_1(W) \in C([0, T], H^{s+1} (R^3))$ by using (3.4) and (3.1).

Applying the basic hyperbolic theory to (3.5), we have

$$W_1 \in C ([0, T], H_{loc}^{s+1} (\omega_t)) \cap C^1 ([0, T], H_{loc}^s (\omega_t)). \tag{3.6}$$

Combining (3.4) with (3.6), we conclude that $(U, \theta)$ is more regular than that given in (3.1) by order one. By applying the similar arguments as above to the problem (3.2) and (3.5) again, we can finally conclude

$$(U, \theta) \in C^\infty (\Omega).$$

4. Proof of Theorem 1.2

This section is devoted to proving Theorem 1.2. At first, we will give some results for Cauchy problems of linear parabolic and hyperbolic equations.

Lemma 4.1 [6]. Consider the following Cauchy problem of a parabolic equation:

$$\begin{cases}
\partial_t W - \sum_{i,j=1}^n c_{ij} \partial_{ij}^2 W = f(t,x), \\
W(0, x) = \omega_0(x),
\end{cases} \tag{4.1}$$

suppose that $\Gamma(t) = \{(x(t), \xi(t)) \in T^*(R^n) \setminus 0, 0 \leq t < T\}$ is a smooth curve, and

$$\begin{cases}
f \in L^2 ([0, T], H^{s-1}(R^n) \cap H_{ml}^{\gamma-1} (x(t), \xi(t))), \\
\omega_0 \in H^s (R^n) \cap H_{ml}^\gamma (x(0), \xi(0)).
\end{cases}$$

Then the solution of (4.1) satisfies

$$W \in L^2 ([0, T], H^{s+1}(R^n) \cap H_{ml}^{\gamma+1} \Gamma(t)) \cap C^1 ([0, T], H^{s-1}(R^n) \cap H_{ml}^{\gamma-1} \Gamma(t)).$$

Lemma 4.2. Let $s > n/2 + 1$, $A_0 \in Op(\sum_{i=1}^n (\Omega))$, $P_1 \in \psi_{1,0}$, its symbol $p_1(t,x,\xi) \in S^1_{1,0}(R^n \times R^n)$ be scaler, real and smooth with respect to $t \in [0, T]$. Consider the following Cauchy problem:

$$\begin{cases}
(D_t - P_1(t,x,D_x)) u + A_0 \partial_t = f(t,x), \\
u(0,x) = u_0(x).
\end{cases} \tag{4.2}$$

Assuming that $\Gamma(t) = \{(t,x(t), p_1(t,x(t),\xi(t))), \xi(t))\}$ is a null bicharacteristic for $L = D_t - P_1(t,x,D_x)$, $f \in L^2 ([0, T], H^{s} \cap H_{ml}^{\gamma} (x(t), \xi(t)))$, $u_0 \in H^{s} \cap H_{ml}^{\gamma} (x(0), \xi(0))$ with $n/2 + 1 < s < 2s - n/2 - 1$, we have

$$u \in C ([0, T], H^{s} \cap H_{ml}^{\gamma} (x(t), \xi(t))). \tag{4.3}$$
Proof. Choose \( B_0 \in \Psi^0_{1,0}(\mathbb{R}^n) \), its symbol \( b_0(t, x, \xi) \in \mathcal{S}^0_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \), \( \text{supp} b_0 \subseteq K \) and \( b_0(t, x, \xi) = 1 \) on \( F \), with \( K \) being a conic neighborhood of \( \Gamma \) and all assumptions holding on \( K \), such that \( [B_0, D_t - P_1] \in \Psi^0_{1,0}(\mathbb{R}^n) \). Then we have

\[
B_0A_0 = [T_{b_0}, A_0] + A_0B_0 + \text{smooth operator},
\]

which implies \( B_0A_0u \in H^{s+1}(\mathbb{R}^n) \) under the assumptions in Lemma 4.2 and noticing \([T_{b_0}, A_0] \in \text{Op}(\sum_{i=1}^{-1}(\Omega))\). The remainder is similar to the proof of Lemma 3.2 in [6] by using the classical bootstrap argument, so we omit it here. \( \square \)

Lemma 4.3. Assume that \( P_1(t, x, D_x) \) and \( A_0 = (a_{ij}) \) are two \( N \times N \) matrices with elements being in \( \psi^1_{1,0}(\mathbb{R}^n) \) and \( a_{ij} \in \text{Op}(\sum_{i=1}^{N}(\Omega)) \), respectively. \( L = D_t - P_1(t, x, D_x) \) is strictly hyperbolic with respect to \( t \), and \( \Gamma(t) = \{(t, x(t), \tau(t), \xi(t)), 0 \leq t < T\} \) is a null bicharacteristic of \( L \). For the Cauchy problem

\[
\begin{cases}
(D_t - P_1(t, x, D_x))u + A_0u = F(t, x), \\
u(0, x) = u_0(x),
\end{cases}
\tag{4.4}
\]

if \( F \in L^2([0, T], H^s \cap H^r_\text{ml}(x(t), \xi(t))) \), and \( u_0 \in H^s \cap H^r_\text{ml}(x(0), \xi(0)) \) with \( n/2 + 1 < s \leq 2s - n/2 - 1 \), we have

\[
u \in C([0, T], H^s \cap H^r_\text{ml}(x(t), \xi(t))).
\tag{4.5}
\]

Proof. Because \( L \) is a strictly hyperbolic, without loss of generality, we can assume that \( p_1 = \text{diag} \left[ \lambda_1(t, x, \xi), \lambda_2(t, x, \xi), \ldots, \lambda_N(t, x, \xi) \right] \).

Meanwhile, we assume that \( \Gamma(t) \) is the null bicharacteristic of the operator \( D_t - \lambda_1(t, x, D_x) \). Denote by

\[
p_\text{II} = \text{diag} \left[ \lambda_2(t, x, \xi), \ldots, \lambda_N(t, x, \xi) \right]
\]

and \( u = (u_1, u_\text{II}) \) with \( u_\text{II} = (u_2, \ldots, u_N)' \) for any vector \( u \).

From the assumptions \( F \in L^2([0, T], H^s(\mathbb{R}^n)) \) and \( u_0 \in H^s(\mathbb{R}^n) \), we immediately have

\[
u \in C([0, T], H^s(\mathbb{R}^n)).
\tag{4.6}
\]

Since \( D_t - p_\text{II}(t, x, D_x) \) is elliptic near \( \Gamma(t) \), we deduce

\[
u_\text{II} \in C([0, T], H^{s+1}_{\text{ml}}(x(t), \xi(t))).
\tag{4.7}
\]

Replacing \( u_\text{II} \) into the problem of \( u_1 \) in (4.4), and using Lemma 4.2, it follows that

\[
u_1 \in C([0, T], H^s \cap H^{s+1}_{\text{ml}}(x(t), \xi(t))).
\tag{4.8}
\]

If \( s \leq s + 1 \), we know from (4.7) and (4.8) that (4.5) holds.

Otherwise, we have \( \gamma > s + 1 \). Then from (4.8) we know \( u_1 \in C([0, T], H^s \cap H^{s+1}_{\text{ml}}(x(t), \xi(t))) \). Using it in the equation of \( u_\text{II} \) and applying the assumption of \( F(t, x) \), we have \( u_\text{II} \in H^{s+2}_{\text{ml}} \), which is more regular than that given in (4.7) by order one. Repeating the above procedure, we can finally obtain the conclusion (4.5). \( \square \)
In the remainder of this section, we shall prove Theorem 1.2.

By using the decoupling method employed in Section 2, we can find $K_1 = K + C([0, T], \text{Op}(\sum_{j=1}^{-2}(\Omega)))$, with $K$ being given in (2.9) such that those zeroth order terms in (2.10) can also be decoupled (see [5,7,14]). If we define $W = (W_1, W_2)^T := (I + K_1)V$, then (2.10) can be further decoupled as follows:

$$
\begin{align*}
\partial_t W_1 + EW_1 + \tilde{A}_0 W_1 &= \tilde{F}_1(W), \\
\partial_t W_2 - \sum_{i,j=1}^3 c_{ij} \partial_{ij}^2 W_2 + \tilde{B}_1 W_2 &= \tilde{F}_2(W),
\end{align*}
$$

(4.9)

where

$$
\begin{align*}
\tilde{F}_1(W) &= \tilde{F}_1(Q_1^{-1}(W_1), Q_2^{-1}(W_2)), \\
\tilde{F}_2(W) &= \tilde{F}_2(Q_2^{-1}(W_1), Q_0^0(W_2)), \\
Q_i^j &\in C([0, T], \text{Op}(\sum_{k=0}^{j-i}(\Omega))) \quad (i = 0, -1), \\
\tilde{A}_0 &\in C([0, T], \text{Op}(\sum_{k=0}^{j-i}(\Omega))), \\
\tilde{B}_1 &\in C([0, T], \text{Op}(\sum_{k=0}^{j-i}(\Omega))).
\end{align*}
$$

By using the assumptions and the classical theories on parabolic and hyperbolic equations, we know the Cauchy problem

$$
\begin{align*}
\partial_t W_1 + EW_1 + \tilde{A}_0 W_1 &= \tilde{F}_1(W), \\
\partial_t W_2 - \sum_{i,j=1}^3 c_{ij} \partial_{ij}^2 W_2 + \tilde{B}_1 W_2 &= \tilde{F}_2(W), \\
(W_1, W_2)|_{t=0} &\in H^s(R^3),
\end{align*}
$$

(4.10)

has a unique local solution satisfying

$$
\begin{align*}
W_1 &\in C([0, T], H^s(R^3)) \cap C^1([0, T], H^{s-1}(R^3)), \\
W_2 &\in L^2([0, T], H^{s+1}(R^3)) \cap H^1([0, T], H^{s-1}(R^3)).
\end{align*}
$$

(4.11)

Using the classical hyperbolic theory, Lemma 4.3 and (4.11), we can obtain from the Cauchy problem

$$
\begin{align*}
\partial_t W_1 + EW_1 + \tilde{A}_0 W_1 &= \tilde{F}_1(W), \\
W_1|_{t=0} &\in H^s \cap H^s_{ml}(x_0, \xi_0),
\end{align*}
$$

(4.12)

that

$$
W_1 \in C([0, T], H^s \cap H^s_{\text{min,y},x+1}(x(t), \xi(t))).
$$

(4.13)

If $9/2 \leq \gamma \leq s + 1$, (4.13) implies

$$
W_1 \in C([0, T], H^s \cap H^s_{\text{ml}}(x(t), \xi(t))).
$$

(4.14)

From (4.14) and (4.11), we deduce $\tilde{F}_2(W) \in L^2([0, T], H^{s+1} \cap H^{s+1}_{\text{ml}}(x(t), \xi(t)))$. Using Lemma 4.1 for the following Cauchy problem of $W_2$:

$$
\begin{align*}
\partial_t W_2 - \sum_{i,j=1}^3 c_{ij} \partial_{ij}^2 W_2 + \tilde{B}_1 W_2 &= \tilde{F}_2(W), \\
W_2|_{t=0} &\in H^s \cap H^s_{\text{ml}}(x(0), \xi(0)),
\end{align*}
$$

(4.15)

we have

$$
W_2 \in C([0, T], H^s \cap H^s_{\text{ml}}(x(t), \xi(t))),
$$

(4.16)

which implies the results given in Theorem 1.2.
Otherwise, if $\gamma > s + 1$, we know from (4.13) that
\[ W_1 \in C([0, T], H^s \cap H_{ml}^{s+1}(x(t), \xi(t))). \] (4.17)
Then, from (4.11) and (4.17), we know
\[ \tilde{F}_2(W) \in L^2([0, T], H^{s+1} \cap H_{ml}^{s+1}(x(t), \xi(t))). \]
Using it and the parabolic theory in (4.15), we have
\[ W_2 \in C([0, T], H^s \cap H_{ml}^{\min(\gamma, s+2)}(x(t), \xi(t))). \] (4.18)
Combining (4.18) with (4.11) and (4.12), we deduce that
\[ W_1 \in C([0, T], H^s \cap H_{ml}^{\min(\gamma, s+3)}(x(t), \xi(t))). \]
Continuing this procedure, we can eventually conclude
\[(W_1, W_2) \in C([0, T], H^s \cap H_{ml}^\gamma(x(t), \xi(t))), \]
which implies (1.5) immediately.

5. Proof of Theorem 1.3

As an application of Theorems 1.1 and 1.2, we will prove Theorem 1.3 below.

For any $(t_0, x(t_0)) \in R_+ \setminus 0$, let $\tau(t_0) \in R$ be such that $p_0 = (t_0, x(t_0), \tau(t_0), \xi(t_0))$ is a characteristic point for the hyperbolic operator $\partial_t^2 - (2\mu + \lambda)\Delta$ or $\partial_t^2 - \mu\Delta$ appeared in (1.1).

Denote by $\Gamma(t) = \{(t, x(t), \tau(t), \xi(t))\} \subset T^*(R^4) \setminus 0$ a null bicharacteristic of hyperbolic operators passing through $p_0$. Obviously, the projection of $\Gamma(t)$ in the $(t, x)$-space must intersect with $\{t = 0\}$ at $x \neq 0$, where $(U, \theta)$ is smooth by using Theorem 1.1.

Thus, by applying Theorem 1.2, we obtain
\[ \left\{ \begin{array}{l}
U \in C([0, T], H_{ml}^{2s-5/2-\epsilon}(x(t), \xi(t))) \cap C^1([0, T], H_{ml}^{2s-7/2-\epsilon}(x(t), \xi(t))), \\
\theta \in C([0, T], H_{ml}^{2s-7/2-\epsilon}(x(t), \xi(t))),
\end{array} \right. \] (5.1)
for any $\epsilon > 0$, which is equivalent to the assertion (1.7) by using the arbitrariness of $(x(t_0), \xi(t_0))$ in $R_+$, because the result (5.1) holds obviously for the case that $\Gamma(t) = \{(t, x(t), \tau(t), \xi(t))\}$ is a bicharacteristic of the hyperbolic operator $\partial_t^2 - (2\mu + \lambda)\Delta$ or $\partial_t^2 - \mu\Delta$ when $(t_0, x(t_0), \tau(t_0), \xi(t_0))$ is not a characteristic point.

References