Singularly Perturbed Ordinary Differential Equations

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We show the existence of periodic solutions for a couple of ordinary differential equations depending on a parameter $\varepsilon$ when $\varepsilon$ presents a small parameter for the majorest derivative of the first equation. The key idea is to connect a coincidence degree theory with a method from a previous paper by the author (J. Differential Equations 89 (1991), 203–223) to avoid the use of a hard implicit function theorem of Nash–Moser.

1. INTRODUCTION

A hard implicit function theorem of Nash–Moser has been used by several authors [3, 5, 8–10] for the study of periodic solutions of differential equations. Also several papers occurred which have proved results typical for the use of this theorem without it [1, 4, 6].

The author of this paper developed methods [1] which were applied mainly to ordinary differential equations with a small parameter in the majorest derivative to show periodic solutions of these equations. Hence these methods allowed us to solve problems where a loss of derivatives appeared. The main idea of these methods is a procedure based on a Galerkin approximation method connected with the Brouwer degree theory and the realization of this idea is enabled by auxiliary inequalities for a given differential equation.

The purpose of this paper is to proceed in the direction of [1]. We shall study couples of ordinary differential equations where only the first one has a loss of derivatives. It is well known that for the study of periodic solutions of ordinary differential equations without loss of derivatives it is very useful to apply a coincidence degree theory developed by Mawhin [7]. Hence we study such couples of ordinary differential equations where the first one presents a problem which prompts us to use the method of [1] and the second one the Leray–Schauder degree theory; we try to join these two methods in this paper.
The plan of this paper is the following. In the first part we develop several abstract theorems and in the second part we apply them to the abovementioned special couples of ordinary differential equations to show the existence of their periodic solutions.

2. ABSTRACT RESULTS

Let $H_r, H_q, H_s, H_0$ be Hilbert spaces with the properties

(i) $H_r \subset H_q \subset H_s$ ($r \geq q \geq s > 0$) and $H_s$ is compactly imbedded into $H_0$, i.e., $H_s \bigcap H_0$;

(ii) there exists an orthogonal projection $P_N: H_0 \rightarrow H_r \subset H_0$ for each natural number $N \in \mathcal{N}$ such that the restriction of $P_N: H_0 \rightarrow H_r \subset H_s$ to $H_r$ is also orthogonal and $\dim \overline{H}_N \leq \infty$, $\overline{H}_N = \text{Im} P_N$, and

$$|(I - P_N)\eta|_0 \leq N^{-q} \cdot |\eta|_0, \quad |P_N\eta|_r \leq N^{-q} \cdot |\eta|_q$$

for each $\eta \in \bigcup \overline{H}_N$ and $|\cdot|_r$ is the corresponding norm of $H_r$. Let $X$ be a Banach space and let $Y \subset X$ be a dense linear subspace of $X$. Now we consider the mappings

$A: H_r \times X \rightarrow H_r, \quad T: H_r \times X \rightarrow X$

such that

(i) $(A(u, v), Ju) \geq a(|u|_q); \quad |A(u, v)|_s \leq d(|u|_s)$, for each $|u|_0 \leq 1$, $u \in \bigcup \overline{H}_N$, $v \in \Omega \cap Y$, where $J: H_r \rightarrow H_r$, $H_r$ is a Banach space $H_r \subset H_0$, $J$ is a linear mapping satisfying $J(\overline{H}_N) \subset \overline{H}_N$ and $J/\overline{H}_N \rightarrow \overline{H}_N$ is an isomorphism for each $N \in \mathcal{N}$, $(\cdot, \cdot)_j$ is an inner product in $H_j$ corresponding to $|\cdot|_j$, $a: \mathbb{R}^+ \rightarrow \mathbb{R}$ is a mapping such that $a(K) > 0$ for some $K > 0$, $\varnothing \neq \Omega \subset X$ is a bounded open subset, $d: R_+ \rightarrow R_+$ is nondecreasing, and $\lim_{M \rightarrow \infty} M^{-q}d(M'^{-q}) = 0$.

(ii) $(A(u, v), Ju) \geq b(|u|_0)$ for each $|u|_0 \leq 1$, $u \in \bigcup \overline{H}_N$, $v \in \Omega \cap Y$, where $b: R_+ \rightarrow R$ is a mapping such that $b(1) > 0$.

(iii) $T(W \times \Omega) \subset X$ is compact, $T$ is continuous, where $W = \{u \in H_r, |u|_0 \leq 1, |u|_q \leq K\}$.

(iv) The mapping $(A, T): H_r \times X \rightarrow H_r \times X \subset H_0 \times X$ can be continuously extended on $H_0 \times X$.

(v) $|v - T(u, v)| > \delta > 0$ for some $\delta > 0$ and each $v \in \partial \Omega$, $u \in W$.

**Theorem 1.** If $\deg(\cdot - T(0, \cdot), 0, \Omega) \neq 0$ then the equation

$$A(u, v) = 0, \quad T(u, v) = v$$

has a solution in $H_0 \times X$. 
Proof. By using the standard argument [2] we can find a finite dimensional subspace $X_n \subseteq Y$ and a continuous mapping $T_n : H \times X \to X_n$ for each $n \in \mathbb{N}$ such that

$$|T(u, v) - T_n(u, v)| < 1/n$$

for each $(u, v) \in W \times \Omega$ and $\Omega \cap X_n \neq \emptyset$. We solve

$$P_N A(u, v) = 0, \quad T_N(u, v) = v$$

(2.1)
in $W_N \times \Omega_N$ for $N \gg 1$ where $W_N = \{u \in \bar{H}_N, \ |u|_0 \leq 1, \ |u|_q \leq K\}$, $\Omega_N = \Omega \cap X_n \subset Y$.

Since the projections $P_N$ are orthogonal we have

$$(P_N A(u, v), Ju) \geq a(|u|_q)$$

$$(P_N A(u, v), Ju)_0 \geq b(|u|_0) \quad \text{for} \quad u \in W_N \times \Omega_N.$$ We know

$$\partial(W_N \times \Omega_N) = \partial W_N \times \Omega_N \cup W_N \times \partial \Omega_N,$$

$$\partial W_N = \{u \in \bar{H}_N, \ |u|_0 = 1 \text{ or } |u|_q = K\},$$

$$|v - T_n(u, v)| > \delta/2 \quad \text{for each} \quad v \in \partial \Omega_n.$$ Hence for each $t \in [0, 1]$

$$(t \cdot P_N A(u, v) + (1 - t) \cdot Ju, Ju)_0 > 0 \quad \text{for} \quad |u|_0 = 1$$

$$(t \cdot P_N A(u, v) + (1 - t) \cdot Ju, Ju)_0 > 0 \quad \text{for} \quad |u|_q = K,$$

where we have used $b(1) > 0$ and $a(K) > 0$.

This implies the existence of the Brouwer degree

$$\deg(t \cdot P_N A(u, v) + (1 - t) \cdot Ju, v - T_n(u, v), 0, W_N \times \Omega_N)$$

for each $t \in [0, 1]$ and moreover

$$\deg(P_N A(u, v), v - T_n(u, v), 0, W_N \times \Omega_N)$$

$$= \deg(Ju, v - T_n(u, v), 0, W_N \times \Omega_N).$$

Since $J|\bar{H}_N \to \bar{H}_N$ is an isomorphism and $\deg(v - T(0, v), 0, \Omega) \neq 0$ we have

$$\deg(Ju, v - T_n(u, v), 0, W_N \times \Omega_N) = \pm \deg(v - T_n(0, v), 0, \Omega_N) \neq 0.$$
Hence (2.1) has a solution \((u_N, v_N) \in W_N \times \Omega_N\) for each \(N \gg 1\). Using standard arguments [2] we can assume that \(v_N \to \bar{v}\) in \(X\) as \(N \to \infty\). On the other hand,

\[
|A(u_N, v_N)|_0 = |(I - P_N)A(u_N, v_N) + P_NA(u_N, v_N)|_0 \\
\leq N^{-s} |A(u_N, v_N)|_s \leq N^{-s} d(|u_N|_s) \\
\leq N^{-s} d(N^{-q} \cdot |u_N|_q) \leq N^{-s} d(N^{-q} \cdot K) \to 0 \quad \text{as} \quad N \to \infty.
\]

Hence

\[
\lim_{N \to \infty} |A(u_N, v_N)|_0 = 0, \quad \lim_{N \to \infty} u_N = v_N \quad \text{in} \quad X \\
v_N = T_N(u_N, v_N), \quad \lim_{N \to \infty} |T_N - T| = 0 \quad \text{on} \quad W \times \Omega, \quad |u_N|_q \leq K,
\]

and also by using \(H_q \subset H_s \bigcirc \bigcirc H_0\) and \((A, T): H_0 \times H \to H_0 \times X\) is continuous we obtain

\[
A(\bar{u}, \bar{v}) = 0, \quad T(\bar{u}, \bar{v}) = \bar{v}.
\]

This completes the proof.

**Corollary 2.** Let \(A, T\) satisfy the above assumptions (i)--(iv) with \(\Omega = B_M = \{v \in X, |v| \leq M\}\) and moreover

\[
T(W \times B_M) \subset (1 - \delta) \cdot B_M = \{v \in X, |v| \leq M \cdot (1 - \delta)\}
\]

for some \(0 < \delta < 1\). Then \(A(u, v) = 0, \ T(u, v) = v\) has a solution.

**Proof:** Using the homotopy

\[
H(t, u, v) = (A(u, v), v - t \cdot T(u, v)), \quad t \in [0, 1]
\]

we immediately obtain the proof in a similar way as in [2].

**Corollary 3.** Let \(A(u, v) = Lu + e \cdot F(u, v)\) where \(L: H_r \to H_q\) is a continuous linear mapping, \(F: H_r \to H_s\), and for \(|u|_0 \leq 1, u \in \bigcup \bar{B}_N, v \in \Omega \cap Y\) we have

\[
(Lu, Ju) \geq c \cdot |u|_q^2, \quad (Lu, Ju)_0 \geq c \cdot |u|_0^2, \quad |F(u, v)|_s \leq c \cdot (1 + |u|_r) \\
|(F(u, v), Ju)_0| \leq c, \quad |(F(u, v), Ju)_s| \leq c \cdot (1 + |u|_q^2),
\]

where \(J\) has the same properties as in Theorem 1 (c will denote constants). Let \(T(u, v) = S(v) + e \cdot V(u, v)\), where \(S: X \to X, V: H_r \times X \to X\) are continuous, compact, bounded, \(v - S(v) \neq 0\) on \(\partial \Omega\).
Then $A(u, v) = 0$, $T(u, v) = v$ possesses a solution for $\varepsilon$ small provided that $(A, T)$ satisfies the assumption (iv), $r < q + s$, and $\deg(v - S(v), 0, \Omega) \neq 0$.

Proof. Using the hypotheses for $S, r, q, s, F, T$ we obtain our assertion applying Theorem 1 with

$$a(t) = c_1 + c_2 t^2, \quad c_2 > 0, \quad b(t) = c_3 t^2 - c_4, \quad c_3 > c_4 > 0,$$

$$d(t) = c_5 t + c_6, \quad c_5, c_6 > 0,$$

where $c_i, i = 1, \ldots, 6$ are constants for $\varepsilon$ small.

Corollary 4. Consider the same situation as in Corollary 3 with $\Omega = B_M$ and instead of the assumption $\deg(v - S(v), 0, \Omega) \neq 0$ we suppose $S(B_M) \subset (1 - \delta) \cdot B_M$ for some $0 < \delta < 1$. Then $A(u, v) = 0$, $T(u, v) = v$ has a solution for $\varepsilon$ small.

Of course, following the above procedure we can extend the coincidence degree method developed by Mawhin [7] for our case. Hence we consider a linear mapping $M: \text{dom } M \subset X \to Z$, where $Z$ is a Banach space, $M$ is a Fredholm mapping with index zero and let $P: X \to X$, $Q: Z \to Z$ be continuous projections such that $\text{Im } P = \text{Ker } M \neq \{0\}$, $\text{Im } M = \text{Ker } Q$. Then $M_p = M/\text{dom } M \cap \text{Ker } P$ is one-to-one and onto $\text{Im } M$ and its inverse $M_p^{-1}: \text{Im } M \to X$ will be denoted by $K_p$. Let $K_p$ be compact, continuous. Now we consider the equation

$$A(u, v) = 0, \quad Mv = \lambda \cdot \bar{F}(u, v), \quad \lambda \in [0, 1], \quad \text{if } \Omega \cap \text{Ker } M \neq \emptyset,$$

where $A: H_x \times X \to H_x$, $\bar{F}: H_x \times X \to Z$ is continuous and bounded on $H_x \times X$. Using $P, Q$ we modify the equation $Mv = \lambda \cdot \bar{F}(u, v)$ in the following way

$$(I - P)v = \lambda K_p(I - Q)\bar{F}(u, v), \quad 0 = Q\bar{F}(u, v).$$

We put

$$T_\lambda(u, v) = Pv + \lambda K_p(I - Q)\bar{F}(u, v) - DQ\bar{F}(u, v),$$

where $D: \text{Im } Q \to \text{Ker } M$ is an isomorphism. We see that $T_\lambda: H_x \times X \to X$.

Theorem 5. Let $(A, T_\lambda)$ satisfy the assumptions of Theorem 1 uniformly for $\lambda \in [0, 1]$. Then $A(u, v) = 0$, $Mv = \bar{F}(u, v)$ has a solution provided that $\deg(DQ\bar{F}(0, \cdot), 0, \Omega \cap \text{Ker } M) \neq 0$. 

Proof. We must compute

\[
\deg(v - T_1(0, v), 0, \Omega) = \deg(v - T_0(0, v), 0, \Omega)
\]
\[
= \deg((I - P)v - DQ\bar{F}(0, v), 0, \Omega)
\]
\[
= \deg(DQ\bar{F}(0, v), 0, \Omega \cap \text{Ker } M) \neq 0.
\]

Now we consider the special case \( \bar{F}(u, v) = F_1(v) + \varepsilon F_2(u, v) \), where \( F_1: X \to Z \), \( F_2: H \times X \to Z \) are continuous, bounded on \( X, H \times X \), respectively. Then

\[
T_{\lambda}(u, v) = T_{\lambda, e}(u, v) = Pu + \lambda K_p(I - Q)F_1(v) - DQF_1(v)
\]
\[
+ \varepsilon(\lambda K_p(I - Q)F_2(u, v) - DQF_2(u, v)).
\]

**Theorem 6.** Suppose that the following conditions are satisfied for \( \varepsilon \) small

(i) \( A \) satisfies the hypotheses of Theorem 1.

(ii) For each \( \lambda \in (0, 1] \), \( Mv \neq \lambda F_1(v) \) on \( \partial \Omega \).

(iii) For each \( v \in \text{Ker } M \cap \partial \Omega \), \( F_1(v) \neq \text{Im } M \).

(iv) \( \deg(DQF_1(\cdot), 0, \Omega \cap \text{Ker } M) \neq 0 \).

(v) \((A, T_{\lambda, e})\) satisfies the hypothesis (iv) of Theorem 1.

Then the equation \( A(u, v) = 0, Mv = F_1(v) + \varepsilon F_2(u, v) \) possesses a solution for \( \varepsilon \) small.

**Proof.** We apply Corollary 3 for

\[
S(v) = Pv + K_p(I - Q)F_1(\cdot) - DQF_1(v),
\]
\[
V(u, v) = K_p(I - Q)F_2(u, v) - DQF_2(u, v).
\]

We put

\[
S_{\lambda}(v) = Pv + \lambda K_p(I - Q)F_1(v) - DQF_1(v)
\]

and since \( S_{\lambda}(v) \neq v \) on \( \partial \Omega \) we have

\[
\deg(v - S(v), 0, \Omega) = \deg(v - S_0(v), 0, \Omega)
\]
\[
= \deg((I - P)v + DQF_1(v), 0, \Omega)
\]
\[
= \deg(DQF_1(v), 0, \Omega \cap \text{Ker } M) \neq 0.
\]

**Remark 7.** We must notice that the essence of these theorems is the existence of a solution of \( P_N A(u, v) = 0, T_N(u, v) = v \) for the given mapping.
(A, T): \( H_r \times X \to H_s \times X \) in the set \( W_N \times \Omega_N \). Hence by securing this existence in another way we can obtain additional existence theorems.

Remark 8. The assertion of Theorem 6 also holds for the case \( 0 \in \Omega \), \( \text{Ker} \, M = \{0\} \) and in this case we assume all assumptions of this theorem without (iii) and (iv).

3. Applications

In this section we give theorems which are simple consequences of Theorem 6 and Corollary 3. First we consider the equation

\[
x + x' = \varepsilon (g(t, x, x') \cdot x'' + h(t, x, y))
\]

\[
y' = f(t, y) + \varepsilon k(t, x, y),
\]

where \( g, h, f, k \in C^\infty \) are bounded and \( 2\pi \)-periodic in \( t \), \( \dim x = 1 \), \( \dim y \geq 1 \). We look for \( 2\pi \)-periodic solutions of (3.1).

We put [8] for \( m \in \mathcal{N}, r \in \mathcal{N} \cup \{0\} \)

\[ H^m_r = \left\{ u: R \to R^m, \text{u is} 2\pi\text{-periodic, } |u|^2 = \int_0^{2\pi} \left( |u(t)|^2 + |D'u(t)|^2 \right) dt < \infty \right\} , \]

where \( D = d/dt, H^m_r \) is a Hilbert space with the inner product

\[
(u, v) = \int_0^{2\pi} (u(s) \cdot v(s) + D'u(s) \cdot D'v(s)) \, ds.
\]

Putting

\[
Lu = u' + u, \quad L: H^1_r \to H^1_q, \quad q = r - 1, \quad s = r - 2, \quad X = Z = C^r_{r-1},
\]

\[
\dim y = n, \quad F: H^1_r \times C^r_{r-1} \to H^1_s, \quad F(u, v) = g(\cdot, u, u') \cdot u'' + h(\cdot, u, v),
\]

\[ M: \text{dom} \, M \subset X \to Z, \quad Mv = v', \quad F_1(v) = f(\cdot, v), \quad F_2(u, v) = k(\cdot, u, v), \]

\[ J = L, \quad r \geq 10, \quad C^m_r = \{ v \in C^r(R, R^m), v \text{ is } 2\pi\text{-periodic} \}
\]

with the usual norm \( \| \cdot \| \),

we can rewrite (3.1) in the form

\[
Lu = \varepsilon F(u, v), \quad Mv = F_1(v) + \varepsilon F_2(u, v).
\]

We also set \( Y = \{ v \in C^r_{r-1}, v \text{ is } C^\infty\text{-smooth} \} \) and \( P_N \) are usual Fourier truncation operators [8].
By using standard techniques \([1, 8]\) we easily see that \(L, F\) satisfies the conditions of Corollary 3 provided that

\[ \Omega = \{ v \in C^\alpha_{r-1}, \, v(\cdot) \in G, \, \|v\|_{r-1} < K \}, \quad \text{for} \quad K \gg 1 \text{ large}, \]

where \(G\) is an open bounded convex set containing the origin, \(G \subset \mathbb{R}^n\). As a matter of fact, the inequalities \(|(F(u, v), Ju)_0| \leq c, \, |(F(u, v), Ju)_1| \leq c \cdot (1 + |u|_2^2), \, |F(u, v)|_s \leq c \cdot (1 + |u|_s)\) hold for \(u \in C^\infty, \, v \in C^\infty, \, v \in \Omega, \, \|u\|_3 \leq 1\), where

\[ \|u\|_1 = \sup \{ |u(\cdot)|, |u'(\cdot)|, |u''(\cdot)|, |u'''(\cdot)| \}. \]

Indeed, the inequality \(|F(u, v)|_s \leq c \cdot (1 + |u|_s)\) follows by the composition of functions inequality \([8]\) and the most difficult term in \((F(u, v), Ju)_1\) is

\[ I = \int_0^{2\pi} g(t, u, u') \cdot D^s + 2u \cdot D^{s+1} u \, dt. \]

Integrating by parts we have

\[ I = -\int_0^{2\pi} \frac{d}{dt} (g(t, u, u')) \cdot (D^{s+1} u)^2 \, dt - I. \]

Hence

\[ I = -\frac{1}{2} \int_0^{2\pi} \frac{d}{dt} (g(t, u, u')) \cdot (D^{s+1} u)^2 \, dt \]

and

\[ |I| \leq c \cdot \int_0^{2\pi} (D^{s+1} u)^2 \, dt \leq c \cdot |u|_q^2 \]

for \(\|u\|_3 \leq 1, \, u \in C^\infty\).

Lastly, \((Lu, Ju)_s \geq c \cdot |u|_s^2\) and \((Lu, Ju)_0 \geq c \cdot |u|_0^2\) are clear.

On the other hand, from the proof of Theorem 1 we see that in this case we can take

\[ W = \{ u \in H^1, \, |u|_0 \leq \delta, \, |u|_q \leq 1 \} \]

for \(\delta\) small and by using the Sobolev inequalities we have

\[ W \cap C^\infty \subset \{ u \in C^\infty, \, \|u\|_3 \leq 1 \} \]

for \(\delta\) small fixed.
We must investigate the operator $F_1$. Assuming that for each $x \in \partial G$ we can find a normal $n(x)$ to $\partial G$ such that for every $t \in \mathbb{R}$

$$n(x) \cdot f(t, x) > 0.$$ 

Then it is well known [7] that the assumptions (ii), (iii) of Theorem 6 are satisfied. Indeed, if $v(\cdot) \in \mathcal{G}$ and $v' = Af(\cdot, v)$, $A \in \{0, 1\}$, $v \in C_{\mathbb{R}}^{\mathbb{R}}$ then for $K \gg 1$ large independent of $v$ we have $\|v\|_{r, \mathbb{R}} < K$. Hence $v \in \partial \Omega$ implies $v(t) \in \partial G$ for some $t$ and we proceed as in [7]. Note that $\ker M = \{x : \mathbb{R} \to \mathbb{R}^n, x = \text{constant}\}$ and the mapping $DQF_1(\cdot)$ from the assumption (iv) of Theorem 6 has the form

$$DQF_1(x) = \frac{1}{2\pi} \int_0^{2\pi} f(s, x) \, ds.$$ 

Summing up we obtain

**Theorem 9.** Under the above assumptions and

$$\deg \left( \frac{1}{2\pi} \int_0^{2\pi} f(s, \cdot) \, ds, 0, G \right) \neq 0$$

Eq. (3.1) possesses a classical $2\pi$-periodic solution for each $\varepsilon$ small.

Applying Remark 8 to the equation

$$x + x' = \varepsilon(g(t, x, x') \cdot x'' + h(t, x, y))$$

$$Ay + y' = f(t, y) + \varepsilon k(t, x, y)$$

we obtain a classical $2\pi$-periodic solution of this equation for $\varepsilon$ small, when $f, g, h, k$ have the above properties without

$$\deg \left( \frac{1}{2\pi} \int_0^{2\pi} f(s, \cdot) \, ds, 0, G \right) \neq 0$$

and the matrix $A$ has no eigenvalues on the imaginary axis.

Now we consider the higher-dimensional case of (3.1)

$$Bx + x' = \varepsilon(A(t)x'' + h(t, x, y))$$

$$y' = f(t, y) + \varepsilon k(t, x, y),$$

where $\dim x = m \geq 1$, $B \in \mathcal{L}(\mathbb{R}^m)$ is invertible and symmetric, $A(\cdot) \in \mathcal{L}(\mathbb{R}^m)$ are symmetric, and $t \to A(t)$ is smooth and $2\pi$-periodic, $h, f, k$ have the above properties of Theorem 9.
**Theorem 10.** Under the above assumptions and

\[ \text{deg} \left( \frac{1}{2\pi} \int_0^{2\pi} f(s, \cdot) \, ds, 0, G \right) \neq 0 \]

Eq. (3.2) has a classical $2\pi$-periodic solution for each $\varepsilon$ small.

**Proof.** We apply Theorem 6 and Corollary 3 and the assumptions of these results are verified as in the proof of Theorem 9 based on the techniques in [1, 8]. For this purpose it is essential to assume the hypotheses for $A(\cdot)$. In this case $L = J$, $Lu = Bu + u'$ and we have

\[
(Lu, Ju)_s = (Bu + u', Bu + u')_s = (Bu, Bu)_s + (u', u')_s
\]

\[
= \int_0^{2\pi} (Bu \cdot Bu + BD^s u \cdot BD^s u + u' \cdot u' + D^{s+1} u \cdot D^{s+1} u) \, dt
\]

\[
\geq c \int_0^{2\pi} (|u|^2 + |D^s u|^2 + |Du|^2 + |D^{s+1} u|^2) \geq c \cdot |u|^2,
\]

since $(Bu, u')_s = 0$, $|Bu| \geq c \cdot |u|$, $q = s + 1$.

The assertion $(Bu, u')_s - 0$ follows from the symmetricity of $B$, since

\[
\int_0^{2\pi} Bu \cdot u' \, dt = -\int_0^{2\pi} Bu' \cdot u \, dt = -\int_0^{2\pi} u' \cdot Bu \, dt
\]

and

\[
\int_0^{2\pi} Bu \cdot u' \, dt = 0.
\]

If we consider

\[
Bx + x' = \varepsilon(A(t)x'' + h(t, x, y))
\]

\[
Ay + y' = f(t, y) + \varepsilon k(t, x, y),
\]

(3.3)

where $B$, $A(\cdot)$, $h$, $f$, $k$ have the properties from Theorem 10 without \(\text{deg}((1/2\pi) \int_0^{2\pi} f(t, \cdot) \, dt, 0, G) \neq 0\) and $A$ has no eigenvalues on the imaginary axis. Then (3.3) has a $2\pi$-periodic classical solution for each $\varepsilon$ small.

**References**


