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On the error estimation for a mixed type of interpolation

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Abstract: We analyse the error induced by the approximation of a function by an interpolation function which is a combination of algebraic and first-order trigonometric polynomials. We prove that, under certain conditions, this error can be expressed in a similar form as in the purely polynomial case. As an application we establish in closed form the local truncation error for a class of extended linear multistep methods of the Adams' type.

Keywords: Mixed interpolation, error estimation, ODEs, linear multistep method, local truncation error.

1. Introduction

Recently we considered the problem of approximating a function $f(x)$ by a combination of algebraic and first-order trigonometric polynomials [1], i.e.,

$$f(x) \approx f_n(x) := a \cos kx + b \sin kx + \sum_{i=0}^{n-2} c_i x^i, \quad n \geqslant 1. \tag{1.1}$$

Herein k denotes a positive, real parameter. Let $X := \{x_i \mid i = 0, 1, \ldots, n\}$ be a set of $n+1$ points satisfying $x_0 < x_1 < \cdots < x_n$. It can be shown that with each X, there corresponds a subset $I_X \subset \mathbb{R}^+$ such that for each $f(x) \in C^0([x_0, x_n])$ and each $k \in I_X$, there exist unique coefficients $a, b, c_0, \ldots, c_{n-2}$ such that

$$f_n(x_j) = f(x_j), \quad j = 0, 1, \ldots, n. \tag{1.2}$$

It can also be shown that the set of functions $\{\cos kx, \sin kx, 1, x, \ldots, x^{n-2}\}$ forms an extended complete Tchebycheff system (ECT-system) [3] over the interval $]x_0, x_n[$ if $0 < k(x_n - x_0) < \pi$. This implies the following inclusion:

$$I_X \supset \left]0, \frac{\pi}{x_n - x_0}\right[. \tag{1.3}$$

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For the particular case of equidistant interpolation nodes, namely $x_i = x_0 + ih$ $(i = 0, 1, \dots, n)$, we have proven in [1] that a unique interpolation function $f_n(x)$ exists if and only if

$$k \notin \left\{ \frac{l\pi}{h} \mid l \in \mathbb{Z} \right\}. \tag{1.4}$$

In this case the nth-order interpolation function $f_n(x)$ can be expressed as follows:

$$f_n(x) = \sum_{j=0}^{n} (-1)^j \binom{-s}{j} \nabla^j f(x_n) - k^2 \phi_n(x - x_0) \nabla^{n-1} f(x_n) - k^2 \phi_{n+1}(x - x_0) \nabla^n f(x_n), \tag{1.5}$$

whereby $s = (x - x_n)/h$ and ∇ denotes the backward difference operator defined by $\nabla f(x) := f(x) - f(x - h)$. Also in [1], useful properties of the function $\phi_n(x)$ have been established. Among those, we list:

- $$\phi_n(x) = \frac{1}{k^2} \binom{x/h - 1}{n - 1} - \frac{1}{4 \sin^2(\frac{1}{2}kh)} \phi_{n-2}(x - h). \tag{1.6}$$

  This relation enables the recursive generation of the functions $\phi_n(x)$ $(n \geqslant 2)$ when starting from:

  $$\phi_0(x) = \frac{2}{k^2} \tan \tfrac{1}{2}kh \sin kx, \qquad \phi_1(x) = \frac{1}{k^2}\left(1 - \frac{\cos k(x - \frac{1}{2}h)}{\cos \frac{1}{2}kh}\right).$$

- $$\Delta\phi_n(x) := \phi_n(x + h) - \phi_n(x) = \phi_{n-1}(x), \quad n \geqslant 1, \tag{1.7}$$

- $$\phi_n(\tfrac{1}{2}nh - x) = (-1)^{n+1} \phi_n(\tfrac{1}{2}nh + x). \tag{1.8}$$

- Defining the differential operator

  $$L_n := (\mathrm{D}^2 + k^2)\mathrm{D}^{n-1}, \quad \mathrm{D} := \frac{\mathrm{d}}{\mathrm{d}x}, \tag{1.9}$$

  the function $\phi_n(x) \in C^\infty(\mathbb{R})$ is the unique solution of the inhomogeneous linear differential equation

  $$(L_n \phi_n)(x) = h^{1-n}, \tag{1.10}$$

  which satisfies the $n + 1$ boundary conditions

  $$\phi_n(jh) = 0, \quad j = 0, \dots, n. \tag{1.11}$$

In general,

$$E_n(x) = E_n(f;\ X;\ x) := f(x) - f_n(x) \tag{1.12}$$

represents the error due to the approximation of the function $f(x)$ by the interpolation function $f_n(x)$ of the mixed type defined by (1.1) and (1.2). In the next section, we will set up sufficient conditions for casting $E_n(x)$ into a form completely analogous to the well-known form of the error term for Lagrange interpolation.

2. Estimation of the error $E_n(f; X; x)$

With the notations introduced so far, the following holds.

Theorem 1. *For any* $k \in]0, \pi/(x_n - x_0)[$ *there exists a smooth function* $\psi_n : \mathbb{R} \to \mathbb{R}$ *(depending only on* X *and* k*), such that if* $f \in C^{n+1}([x_0, x_n])$*, then we can find for each* $x \in [x_0, x_n]$ *some* $\xi \in]x_0, x_n[$ *such that*

$$E_n(x) = \psi_n(x)(L_n f)(\xi). \tag{2.1}$$

Our proof of this theorem will be based on the following lemma.

Lemma 2. *Let* $y_0 < y_1 < \cdots < y_n < y_{n+1}$ *and* $k \in]0, \pi/(y_{n+1} - y_0)[$. *Let* $g \in C^{n+1}([y_0, y_{n+1}])$ *be such that*

$$g(y_j) = 0, \quad j = 0, 1, \ldots, n+1. \tag{2.2}$$

Then there exists some $\xi \in]y_0, y_{n+1}[$ *such that*

$$(L_n g)(\xi) = 0. \tag{2.3}$$

Proof of Theorem 1. Fix some $f \in C^{n+1}([x_0, x_n])$, and let f_n be its nth-order interpolation function, and $E_n(x)$ the corresponding error function. We have $(L_n f_n)(x) = 0$, and therefore E_n satisfies:

$$\begin{cases} (L_n E_n)(x) = (L_n f)(x), \\ E_n(x_j) = 0, \quad j = 0, 1, \ldots, n. \end{cases} \tag{2.4}$$

Since $k \in]0, 0/(x_n - x_0)[$, it follows from (1.3) that also $k \in I_X$ and therefore (2.4) has a unique solution.

Next, let $\psi_n(x) \in C^\infty(\mathbb{R})$ be the unique solution of the problem

$$\begin{cases} (L_n \psi)(x) = 1 \quad \forall x \in \mathbb{R}, \\ \psi(x_j) = 0, \quad j = 0, 1, \ldots, n. \end{cases} \tag{2.5}$$

It is then clear that (2.1) holds for $x \in X$: we can take any $\xi \in]x_0, x_n[$. To prove (2.1) for all other $x \in [x_0, x_n]$ we fix some $\tilde{x} \in [x_0, x_n]$, with $\tilde{x} \notin X$, and define $g_{\tilde{x}} \in C^{n+1}([x_0, x_n])$ by

$$g_{\tilde{x}} := \psi_n(x)E_n(\tilde{x}) - \psi_n(\tilde{x})E_n(x). \tag{2.6}$$

It follows immediately from the properties of $\psi_n(x)$ and $E_n(x)$ that

$$g_{\tilde{x}}(x) = 0 \quad \forall x \in X \quad \text{and} \quad g_{\tilde{x}}(\tilde{x}) = 0.$$

Hence $g_{\tilde{x}}(x)$ vanishes in $n+2$ different points, and since also $k \in]0, \pi/(x_n - x_0)[$, we can apply Lemma 2 to conclude that there exists some $\xi \in]x_0, x_n[$ such that

$$(L_n g_{\tilde{x}})(\xi) = 0.$$

Using (2.4) and (2.5) this gives

$$E_n(\tilde{x}) - \psi_n(\tilde{x})(L_n f)(\xi) = 0,$$

i.e., we have (2.1) for $x = \tilde{x}$. Since the point $\tilde{x} \in [x_0, x_n]$ can be chosen arbitrarily this proves Theorem 1.

It remains to prove Lemma 2. Suppose the hypotheses of Lemma 2 are satisfied. Since $g(y_j) = g(y_{j+1}) = 0$ $(j = 0, 1, \ldots, n)$, there exists for each $j = 0, 1, \ldots, n$ a point $z_j \in]y_j, y_{j+1}[$ such that $(\mathrm{D}g)(z_j) = 0$. Repeating this argument $n-1$ times we arrive at three different points $a, b, c \in]y_0, y_{n+1}[$, with $a < b < c$, and such that

$$(\mathrm{D}^{n-1}g)(a) = (\mathrm{D}^{n-1}g)(b) = (\mathrm{D}^{n-1}g)(c) = 0. \tag{2.7}$$

We will show that this implies that

$$(\mathrm{D}^2 + k^2)(\mathrm{D}^{n-1}g)(\xi) = 0 \tag{2.8}$$

for some $\xi \in]a, c[$. Since $L_n = (\mathrm{D}^2 + k^2)\mathrm{D}^{n-1}$, this proves Lemma 2. □ (Lemma 2)

To prove (2.8) we define $u \in C^2([0, \pi])$ and $\tilde{u} \in C^0([0, \pi])$ by

$$u(x) := (\mathrm{D}^{n-1}g)\left(a + x\left(\frac{c-a}{\pi}\right)\right), \quad x \in [0, \pi], \tag{2.9}$$

and

$$\begin{aligned} \tilde{u}(x) &:= (\mathrm{D}^2 + \tilde{k}^2)u(x), \quad \tilde{k} = \frac{c-a}{\pi}k, \\ &= \left(\frac{c-a}{\pi}\right)^2 (\mathrm{D}^2 + k^2)(\mathrm{D}^{n-1}g)\left(a + x\left(\frac{c-a}{\pi}\right)\right). \end{aligned} \tag{2.10}$$

Since $0 < k < \pi/(y_{n+1} - y_0) < \pi/(c-a)$, it follows that $0 < \tilde{k} < 1$, and proving (2.8) is equivalent to showing that

$$\tilde{u}(\tilde{\xi}) = 0 \tag{2.11}$$

for some $\tilde{\xi} \in]0, \pi[$. Moreover (2.7) implies that

$$u(0) = u(\hat{x}) = u(\pi) = 0, \quad \text{with } \hat{x} := \left(\frac{b-a}{c-a}\right)\pi \in]0, \pi[.$$

It follows from the first equality in (2.10) and from $u(0) = 0$, that

$$u(x) = \frac{A}{\tilde{k}} \sin \tilde{k}x + \frac{1}{\tilde{k}} \int_0^x \sin \tilde{k}(x-y)\, \tilde{u}(y)\, \mathrm{d}y, \tag{2.12}$$

for some constant A. The condition $u(\pi) = 0$ gives then

$$A = -\frac{1}{\sin \tilde{k}\pi} \int_0^\pi \sin \tilde{k}(\pi - y)\, \tilde{u}(y)\, \mathrm{d}y,$$

and hence

$$\begin{aligned} u(x) &= -\frac{1}{\tilde{k} \sin \tilde{k}\pi} \int_0^\pi \sin \tilde{k}(\pi - y) \sin \tilde{k}x\, \tilde{u}(y)\, \mathrm{d}y \\ &\quad + \frac{1}{\tilde{k}} \int_0^x \sin \tilde{k}(x-y)\, \tilde{u}(y)\, \mathrm{d}y \\ &= \int_0^\pi G(x, y) \tilde{u}(y)\, \mathrm{d}y, \end{aligned} \tag{2.13}$$

with

$$G(x, y) = \frac{-1}{\tilde{k} \sin \tilde{k}\pi} \sin \tilde{k}(\pi - x) \sin \tilde{k}y \quad \text{if } 0 \leqslant y \leqslant x \leqslant \pi,$$
$$= \frac{-1}{\tilde{k} \sin \tilde{k}\pi} \sin \tilde{k}x \sin \tilde{k}(\pi - y) \quad \text{if } 0 \leqslant x \leqslant y \leqslant \pi. \tag{2.14}$$

It follows that $G \in C^0([0, \pi] \times [0, \pi])$ and that

$$G(x, y) < 0 \quad \forall (x, y) \in]0, \pi[\times]0, \pi[. \tag{2.15}$$

Now suppose that (2.11) does not hold, i.e., we have either $\tilde{u}(x) > 0$ for all $x \in]0, \pi[$ or $\tilde{u}(x) < 0$ for all $x \in]0, \pi[$. In both cases it then follows from (2.13) and (2.15) that $u(\hat{x}) \neq 0$, which contradicts our earlier condition $u(\hat{x}) = 0$. Hence $\tilde{u}(x)$ must vanish at some point $\tilde{\xi} \in]0, \pi[$, and the proof is complete. □

Remarks. (i) It should be mentioned that within the framework of the theory of ECT-systems [3], an alternative but less elementary proof of Theorem 1 could be given. Moreover, it is readily seen from our proof that Theorem 1 can be extended in the following way:

Let a, b be two points such that $-\infty < a \leqslant x_0 < \cdots < x_n \leqslant b < +\infty$. For any $k \in]0, \pi/(b - a)[$ there exists a smooth function $\psi_n : \mathbb{R} \to \mathbb{R}$ (depending only on X and k), such that if $f \in C^{n+1}([a, b])$, then we can find for each $x \in [a, b]$ some $\xi \in]a, b[$ such that (2.1) *holds.*

(ii) The function $\psi_n(x)$ does not change sign in each of the intervals $]x_i, x_{i+1}[$ $(i = 0, 1, \ldots, n - 1)$, i.e., we have $\psi_n(x) \neq 0$ for all $x \in [x_0, x_n] \setminus X$. To see this, suppose the contrary; then Lemma 2 would imply that $(L_n\psi_n)(\xi) = 0$ for some $\xi \in]x_0, x_n[$, which contradicts $(L_n\psi_n)(x) \equiv 1$. A similar argument shows that also $(\mathrm{D}\psi_n)(x) \neq 0$ for $x \in X$. Indeed, suppose that $(\mathrm{D}\psi_n)(x_j) = 0$ for some $j \in \{0, 1, \ldots, n\}$. Since also $(\mathrm{D}\psi_n)(z_i) = 0$ for some $z_i \in]x_i, x_{i+1}[$ and for all $i = 0, 1, \ldots, n - 1$, we conclude that $(\mathrm{D}\psi_n)(x)$ vanishes in $n + 1$ points. An application of Lemma 2 then implies that $(L_{n-1}\mathrm{D}\psi_n)(x) = (L_n\psi_n)(x)$ vanishes at some intermediate point ξ, which again contradicts $(L_n\psi_n)(x) \equiv 1$.

It follows from these properties of $\psi_n(x)$ that the function $(\tilde{L}_n f)(x)$ defined by

$$(\tilde{L}_n f)(x) = \begin{cases} \dfrac{E_n(f; X; x)}{\psi_n(x)} & \text{for } x \in [x_0, x_n] \setminus X, \\[2ex] \dfrac{(\mathrm{D}E_n)(f; X; x)}{(\mathrm{D}\psi_n)(x)} & \text{for } x \in X, \end{cases} \tag{2.16}$$

is a continuous function. From Theorem 1 we can then conclude that for each $x \in [x_0, x_n]$ there exists some $\eta \in [x_0, x_n]$ such that

$$(\tilde{L}_n f)(x) = (L_n f)(\eta). \tag{2.17}$$

Under the same assumptions as in (i) we can replace the interval $[x_0, x_n]$ by the interval $[a, b]$.

(iii) The function $\psi_n(x)$ appearing in (2.1) can easily be determined for given X. Indeed, if we define the particular function $\tilde{f}(x) := x^{n-1}/k^2(n-1)!$, then $(L_n\tilde{f})(x) = 1$ and hence (using (2.4))

$$\psi_n(x) = E_n(\tilde{f}; X; x).$$

For the particular case of equidistant nodes $x_j = x_0 + jh$ $(j = 0, 1, \ldots, n)$ the function $\psi_n(x)$ turns out to be proportional to the function $\phi_n(x - x_0)$ introduced in (1.5). This can be verified by comparing (2.5) with (1.10) and (1.11); this shows that the proportionality factor is h^{n-1}. Hence we have in this case that

$$E_n(x) = h^{n-1}\phi_n(x - x_0)(L_n f)(\xi), \quad x_0 < \xi < x_n, \quad \text{if } 0 < nhk < \pi. \tag{2.18}$$

Again, under the assumptions of (i) we also have

$$E_n(x) = h^{n-1}\phi_n(x - x_0)(L_n f)(\xi), \quad a < \xi < b, \quad \text{if } 0 < (b - a)k < \pi. \tag{2.19}$$

3. Application

An interesting application of polynomial interpolation is the construction of multistep methods for ODEs. Using our mixed type of interpolation and the corresponding error, new multistep methods and their local truncation errors can be derived. We illustrate this by considering methods of the Adams' type for first order ODEs. It can be verified that the new methods are superior when one exploits the particular form of the appearing error terms.

We want to calculate numerically the value in the equidistant nodes $x_j = x_0 + jh$ $(j = 0, 1, \ldots)$ of the solution $y(x)$ of the following first-order initial-value problem:

$$y' = f(x, y), \quad y(x_0) = \alpha. \tag{3.1}$$

The first step consists in integrating (3.1) over the interval $[x_p, x_{p+1}]$:

$$y(x_{p+1}) - y(x_p) = \int_{x_p}^{x_{p+1}} f(x, y(x))\, \mathrm{d}x. \tag{3.2}$$

The classical Adams–Bashforth formulae [2] are obtained by replacing under the integral sign the function $f(x, y(x))$ by the interpolation polynomial $p_n(x)$ passing through the points $(x_p, f(x_p, y(x_p))), (x_{p-1}, f(x_{p-1}, y(x_{p-1}))), \ldots, (x_{p-n}, f(x_{p-n}, y(x_{p-n})))$. For $n = 3$, e.g., this results in the following explicit formula:

$$y_{p+1} - y_p = h\left[f(x_p, y_p) + \tfrac{1}{2}\nabla f(x_p, y_p) + \tfrac{5}{12}\nabla^2 f(x_p, y_p) + \tfrac{3}{8}\nabla^3 f(x_p, y_p)\right], \tag{3.3}$$

where y_p is the calculated approximation of $y(x_p)$. The local truncation error for this method is known [2] to be given by

$$\tfrac{251}{720} h^4 f^{\text{iv}}(\eta), \quad x_{p-3} < \eta < x_{p+1}. \tag{3.4}$$

An analogous formula, which we call "extended" Adams–Bashforth formula, is obtained by replacing $f(x, y(x))$ in the right-hand side of (3.2) by the mixed interpolation function $f_n(x)$ through the $n + 1$ points $(x_p, f(x_p, y(x_p))), (x_{p-1}, f(x_{p-1}, y(x_{p-1}))), \ldots, (x_{p-n}, f(x_{p-n}, y(x_{p-n})))$. Using expression (1.5) and definition (2.16) in the extended form whereby $a = x_{p-n}$ and $b = x_{p+1}$, (3.2) can be rewritten as:

$$\begin{aligned} y(x_{p+1}) - y(x_p) = & \sum_{j=0}^{n} (-1)^j h \nabla^j f(x_p, y_p) \int_0^1 \binom{-t}{j} \mathrm{d}t \\ & - k^2 \nabla^{n-1} f(x_p, y_p) \int_{x_p}^{x_{p+1}} \phi_n(x - x_{p-n}) \,\mathrm{d}x \\ & - k^2 \nabla^n f(x_p, y_p) \int_{x_p}^{x_{p+1}} \phi_{n+1}(x - x_{p-n}) \,\mathrm{d}x \\ & + h^{n-1} \int_{x_p}^{x_{p+1}} \phi_n(x - x_{p-n}) (\tilde{L}_n f)(x) \,\mathrm{d}x \\ & \text{if } 0 < (n+1)kh < \pi. \end{aligned} \tag{3.5}$$

We mention explicitly the results for $n = 3$. The (explicit) extended Adams–Bashforth formula becomes in backward difference form:

$$\begin{aligned} y_{p+1} = & y_p + hf(x_p, y_p) + \tfrac{1}{2} h \nabla f(x_p, y_p) \\ & + h \left[\frac{2 \sin^2 kh - 1}{kh \sin kh} + \frac{3 \cos kh - 2}{2(1 - \cos kh)} \right] \nabla^2 f(x_p, y_p) \\ & + h \left[\frac{-1 - 2 \cos kh}{2kh \sin kh} + \frac{3}{4(1 - \cos kh)} \right] \nabla^3 f(x_p, y_p), \end{aligned} \tag{3.6}$$

which is the counterpart of (3.3). On account of the midvalue theorem and of property (2.17) the local truncation error takes on following interesting form:

$$\begin{aligned} & \frac{1}{h} h^2 \int_{x_p}^{x_{p+1}} \phi_3(x - x_{p-3}) (\tilde{L}_3 f)(x) \,\mathrm{d}x \\ & = \frac{h^2}{k^2} \left[\frac{23}{12} - \frac{kh \sin kh - 2 \cos 2kh (1 - \cos kh)}{2(1 - \cos kh) kh \sin kh} \right] (L_3 f)(\eta_1), \\ & x_{p-3} < \eta_1 < x_{p+1}, \quad \text{if } 0 < 4kh < \pi, \end{aligned} \tag{3.7}$$

since $\phi_3(x - x_{p-3})$ does not change sign in the integration interval.

We can also derive implicit methods of the Adams' type, which are called Adams–Moulton formulae. The classical ones are obtained by approximating $f(x, y(x))$ in (3.2) by the interpolation polynomial based on the $n+1$ interpolation nodes $x_{p+1}, x_p, \ldots, x_{p-n+1}$. For $n = 3$, the Adams–Moulton method gives the equations:

$$\begin{aligned} y_{p+1} - y_p = & h \big[f(x_{p+1}, y_{p+1}) - \tfrac{1}{2} \nabla f(x_{p+1}, y_{p+1}) \\ & - \tfrac{1}{12} \nabla^2 f(x_{p+1}, y_{p+1}) - \tfrac{1}{24} \nabla^3 f(x_{p+1}, y_{p+1}) \big], \end{aligned} \tag{3.8}$$

with a local truncation error of the same order as (3.4):

$$-\tfrac{19}{720} h^4 f^{\mathrm{iv}}(\xi), \quad x_{p-2} < \eta < x_{p+1}. \tag{3.9}$$

The same approach, but using our mixed type of interpolation, gives the equations:

$$y_{p+1}-y_p=\sum_{j=0}^{n}(-1)^j h\,\nabla^j f(x_{p+1}, y_{p+1})\int_0^1\binom{t}{j}\,\mathrm{d}t$$
$$-k^2\,\nabla^{n-1}f(x_{p+1}, y_{p+1})\int_{x_p}^{x_{p+1}}\phi_n(x-x_{p-n+1})\,\mathrm{d}x$$
$$-k^2\,\nabla^{n}f(x_{p+1}, y_{p+1})\int_{x_p}^{x_{p+1}}\phi_{n+1}(x-x_{p-n+1})\,\mathrm{d}x, \qquad (3.10)$$

with local truncation error

$$\frac{1}{h}h^{n-1}\int_{x_p}^{x_{p+1}}\phi_n(x-x_{p-n+1})(\tilde{L}_n f)(x)\,\mathrm{d}x, \quad \text{if } 0<nkh<\pi. \qquad (3.11)$$

In particular, the "extended" Adams–Moulton method for $n=3$ reads

$$y_{p+1}=y_p+hf(x_{p+1}, y_{p+1})-\tfrac{1}{2}h\,\nabla f(x_{p+1}, y_{p+1})$$
$$+h\left[-\frac{\cos kh}{kh\sin kh}+\frac{\cos kh}{2(1-\cos kh)}\right]\nabla^2 f(x_{p+1}, y_{p+1})$$
$$+h\left[-\frac{1}{2kh\sin kh}+\frac{1}{4(1-\cos kh)}\right]\nabla^3 f(x_{p+1}, y_{p+1}), \qquad (3.12)$$

with local truncation error

$$\frac{h^2}{k^2}\left[\frac{5}{12}-\frac{kh\sin kh-2(1-\cos kh)\cos kh}{2(1-\cos kh)kh\sin kh}\right](L_3 f)(\eta_2),$$
$$x_{p-2}<\eta_2<x_{p+1}, \quad \text{if } 0<3kh<\pi, \qquad (3.13)$$

which is completely analogous to (3.7).

The appearance of the particular linear combination of higher-order derivatives taken in one intermediate point η_1 in (3.7) or η_2 in (3.13), allows us to make an optimal choice of k^2 for the prediction and correction of y_{p+1}. Indeed, if we choose k^2 such that

$$f^{\mathrm{iv}}(\eta_1)+k^2 f''(\eta_1)=0, \qquad (3.14)$$

to predict y_{p+1} and

$$f^{\mathrm{iv}}(\eta_2)+k^2 f''(\eta_2)=0, \qquad (3.15)$$

to correct this estimate, then we make no additional error in the calculation of y_{p+1} starting from previously calculated values. Of course, we do not know the intermediate points, but if we calculate k^2 by demanding for instance that

$$f^{\mathrm{iv}}(x_p, y_p)+k^2 f''(x_p, y_p)=0, \qquad (3.16)$$

the local truncation error will be small if the derivatives of $f(x, y(x))$ do not behave badly. The derivatives required in (3.16) are calculated by means of finite-difference formulae which only involve previously calculated values y_{p-j} ($j \geqslant 0$). Therefore, no additional function evaluations in comparison with the classical methods are needed for the calculation of k^2. Numerical experiments show that for ODEs, with a solution carrying some oscillatory character in it, this procedure gives much better results than those obtained by the classical methods [5,6]. The same conclusion holds for second-order ODEs when linear multistep methods of the Numerov type are extended on the basis of mixed type interpolation [4]. These methods can also be used for solving systems of differential equations. One then has to use, according to the prescription (3.16), different values of k for different components.

It can happen that the calculated value of k does not satisfy the bounds which guarantee the particular form (e.g., (3.7) or (3.13)) of the local truncation error. It should, however, be emphasized that these bounds on k are only sufficient conditions, and in practice one finds that the accuracy of the results obtained by our method does not depend on whether or not the bounds are systematically satisfied.

Finally, it should be mentioned that in numerical applications the value of k^2 which is the solution of an equation of the type (3.16) can become negative. If such is the case, it is allowed to formally replace in all foregoing formulae k by $\mathrm{i}k$ ($\mathrm{i}^2 = -1$), an operation which in particular turns all trigonometric functions into their hyperbolic equivalents. Moreover, all restrictions on the value of k reduce to the unique condition $k \neq 0$.

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