# The ground state energy of the three-dimensional Ginzburg–Landau model in the mixed phase

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**Abstract**

We consider the Ginzburg–Landau functional defined over a bounded and smooth three-dimensional domain. Supposing that the strength of the applied magnetic field varies between the first and second critical fields, in such a way that $H_{C_1} \ll H \ll H_{C_2}$, we estimate the ground state energy to leading order as the Ginzburg–Landau parameter tends to infinity.
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## 1. Introduction and main result

In the last two decades, much progress has been done in the analysis of the celebrated Ginzburg–Landau model of superconductivity. The Ginzburg–Landau model successfully describes the behavior of a superconductor subject to an external applied magnetic field, and has a similar structure to other models from condensed matter physics, that are being analyzed currently, like superfluidity, Bose–Einstein condensates and liquid crystals (see [1,18] and the references therein).

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The Ginzburg–Landau model is a rich mathematical structure and a diversity of different mathematical tools, including linear analysis, potential theory and topological nonlinear analysis, all in a variational setting, are needed in its analysis. In the pioneering work [7], Bethuel, Brezis and Helein study a Ginzburg–Landau model without an applied magnetic field. In a special singular limit, they identify specific singularities of the solution and call them *vortices*. The approach in [7] gives what is currently the standard definition of *vortices*. A rich mathematical literature develops the analysis in [7] to fit other analogous problems, especially the one in superconductivity, as one can see the monograph [21] and the references therein. Linear analysis, especially the important semi-classical methods in spectral theory, are also crucial in the understanding of the Ginzburg–Landau model when the intensity of the external magnetic field is very strong. The monograph [10] (and references therein) is totally devoted to this subject (further details and comments on these results will be given below).

The subject we discuss in this paper is the energy of the Ginzburg–Landau model in a three-dimensional domain and in an asymptotic regime where *vortices* are expected to exist. Unlike the model in a two-dimensional domain, the identification of vortices in the presence of a magnetic field remains unclear. In [21], a very detailed study of vortices has been given in a two-dimensional domain, and it is mentioned, in a list of open problems, that the generalization to three-dimensional domains is quite interesting and difficult. In this paper, we obtain a result concerning the energy of the model by a rather 'short' proof that does not involve the explicit construction of 'three-dimensional' *vortices*. The proof is by combining techniques from linear analysis that we learn from [10], together with results concerning the two-dimensional model that we borrow from [21]. Since the Ginzburg–Landau model in three dimensions is similar to that of the Landau–de Gennes model of liquid crystals [15,17], it is expected that the approach of this paper will serve in the analysis of the Landau–de Gennes model as well.

Consider a bounded open and simply connected domain $\Omega \subset \mathbb{R}^3$ with smooth boundary. Suppose that $\Omega$ models a superconducting sample subject to an applied external magnetic field. The energy of the sample is given by the Ginzburg–Landau functional

$$\mathcal{E}^{3D}(\psi, \mathbf{A}) = \mathcal{E}^{3D}_{\kappa, H}(\psi, \mathbf{A}) = \int_{\Omega} \left( \left| (\nabla - i\kappa H \mathbf{A})\psi \right|^2 + \frac{\kappa^2}{2}(1 - |\psi|^2)^2 \right) dx$$

$$+ \kappa^2 H^2 \int_{\mathbb{R}^3} |\mathrm{curl}\, \mathbf{A} - \beta|^2 \, dx. \tag{1.1}$$

Here $\kappa$ and $H$ are two positive parameters, the wave function (order parameter) $\psi \in H^1(\Omega; \mathbb{C})$, the induced magnetic potential $\mathbf{A} \in \dot{H}^1_{\mathrm{div}, \mathbf{F}}(\mathbb{R}^3)$, where $\dot{H}^1_{\mathrm{div}, \mathbf{F}}(\mathbb{R}^3)$ is the space introduced in (1.2) below. Finally, $\beta$ is the direction of the external magnetic field that we choose constant, $\beta = (0, 0, 1)$. Here, the scaling in the functional (1.1) and its variational space are taken as in [10]. In [21], the scaling for the intensity of the external magnetic field (denoted $h_{\mathrm{ex}}$) is different and selected so that $h_{\mathrm{ex}} = \kappa H$. We choose the scaling from [10] for purely convenience reasons when estimating the ground state energy of the functional.

Let $\dot{H}^1(\mathbb{R}^3)$ be the homogeneous Sobolev space, i.e. the closure of $C_c^\infty(\mathbb{R}^3)$ under the norm $u \mapsto \|u\|_{\dot{H}^1(\mathbb{R}^3)} := \|\nabla u\|_{L^2(\mathbb{R}^3)}$. Let further $\mathbf{F}(x) = (-x_2/2, x_1/2, 0)$. Clearly $\mathrm{div}\,\mathbf{F} = 0$.

We define the space

$$\dot{H}^1_{\mathrm{div}, \mathbf{F}}(\mathbb{R}^3) = \left\{ \mathbf{A} \colon \mathrm{div}\,\mathbf{A} = 0, \text{ and } \mathbf{A} - \mathbf{F} \in \dot{H}^1(\mathbb{R}^3) \right\}. \tag{1.2}$$

Critical points $(\psi, \mathbf{A}) \in H^2(\Omega; \mathbb{C}) \times \dot{H}^1_{\mathrm{div}, \mathbf{F}}(\mathbb{R}^3)$ of $\mathcal{E}^{3D}$ satisfy the Ginzburg–Landau equations

$$
\begin{cases}
-(\nabla - i\kappa H\mathbf{A})^2\psi = \kappa^2\big(1 - |\psi|^2\big)\psi & \text{in } \Omega, \\[4pt]
\operatorname{curl}^2 \mathbf{A} = -\dfrac{1}{\kappa H}\operatorname{Im}\big(\overline{\psi}(\nabla - i\kappa H\mathbf{A})\psi\big)\mathbf{1}_\Omega & \text{in } \mathbb{R}^3, \\[4pt]
\nu \cdot (\nabla - i\kappa H\mathbf{A})\psi = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{1.3}
$$

where $\mathbf{1}_\Omega$ is the characteristic function of the domain $\Omega$, and $\nu$ is the pointing interior unit normal vector of $\partial\Omega$.

For a solution $(\psi, \mathbf{A})$ of (1.3), the function $\psi$ describes the superconducting properties of the material and $H \operatorname{curl} \mathbf{A}$ gives the induced magnetic field. The number $\kappa$ is a material parameter, and the number $H$ is the intensity of a constant magnetic field externally applied to the sample. As in [10,21], we focus on the regime of large values of $\kappa$, $\kappa \to \infty$. In this regime one distinguishes three critical values $H_{C_1}$, $H_{C_2}$ and $H_{C_3}$ for the applied field. Those critical fields are roughly described as follows. If $H < H_{C_1}$, the material is in the superconducting phase. Mathematically, this corresponds to $|\psi| > 0$ for any minimizer $(\psi, \mathbf{A})$ of (1.1). If $H_{C_1} < H < H_{C_2}$, the magnetic field penetrates the sample in quantized vortices (corresponding to zeros of $\psi$). If $H_{C_2} < H < H_{C_3}$, superconductivity is confined to the surface of the sample (corresponding to $|\psi|$ very small in the bulk). Finally, if $H > H_{C_3}$, superconductivity is lost, which is reflected by $\psi = 0$ everywhere in $\Omega$. In this paper, we will focus on the regime when the applied magnetic field varies between $H_{C_1}$ and $H_{C_2}$. In the scaling we choose in this paper, this regime corresponds to $\ln\kappa/\kappa \ll H \ll \kappa$ as $\kappa \to \infty$. Here, if $a(\kappa)$ and $b(\kappa)$ are two positive functions, the notation $a(\kappa) \ll b(\kappa)$ means that $a(\kappa)/b(\kappa) \to 0$ as $\kappa \to \infty$.

In the case of two-dimensional domains, which correspond to infinite cylindrical superconducting samples, there exists a quite satisfactory analysis of the critical fields $H_{C_1}$, $H_{C_2}$ and $H_{C_3}$. As we cannot give an exhaustive list of references, we invite the reader to see the monographs [10,21], where a detailed review of the material is present. Still in the two-dimensional setting, the most accurate available characterization of the critical field $H_{C_2}$ is given in [12,11].

The situation is less understood in three dimensions, especially the regime of magnetic fields close to the first critical field $H_{C_1}$. For a superconductor occupying a ball domain, a candidate for the expression of the critical field $H_{C_1}$ is given in [3]. Related results are obtained for superconducting shells in [8]. For general domains, the analysis of the critical field $H_{C_3}$ started in [16], then a sharp characterization of $H_{C_3}$ is given in [9]. In the papers [4,17], it is proved that superconductivity is confined to the surface of the domain, provided that the magnetic field is close to and below $H_{C_3}$. A fine characterization of the critical field $H_{C_2}$ together with leading order estimates of the ground state energy in large magnetic fields are recently obtained in [13,14]. This paper is complementary to those in [13,14].

The ground state energy of the functional in (1.1) is defined as follows:

$$
C_0(\kappa, H) = \inf\big\{\mathcal{E}^{3D}(\psi, \mathbf{A}): (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times \dot{H}^1_{\mathrm{div}, \mathbf{F}}(\mathbb{R}^3)\big\}.
\tag{1.4}
$$

The main result of this paper is Theorem 1.1 below. It is a generalization of an analogous result proved for the two-dimensional functional in [19].

**Theorem 1.1.** *Suppose that the magnetic field $H$ is a function of $\kappa$ and satisfies*

$$\frac{\ln \kappa}{\kappa} \ll H \ll \kappa, \quad as \ \kappa \to \infty.$$

*Then, the ground state energy in* (1.4) *satisfies*

$$C_0(\kappa, H) = |\Omega| \kappa H \ln \sqrt{\frac{\kappa}{H}} + o\left( \kappa H \ln \sqrt{\frac{\kappa}{H}} \right), \quad as \ \kappa \to \infty. \tag{1.5}$$

As immediate consequences of Theorem 1.1 we obtain that, if $(\psi, \mathbf{A})$ is a minimizer of (1.1), then the induced magnetic field $\operatorname{curl} \mathbf{A}$ is close to the applied magnetic field $\beta$, and that the magnitude of the order parameter $|\psi|$ is close to 1 almost everywhere in $\Omega$. The physical meaning of this is that the applied magnetic field penetrates the sample almost everywhere and concentrates along 'vortex lines'. On these vortex lines the order parameter $\psi$ is expected to have zeros (this is not rigorously proved in this paper), but away of them, the sample remains in the superconducting phase ($|\psi|$ is close to 1). Therefore, the regime considered in Theorem 1.1 corresponds to what is actually named in the physics literature as the *mixed phase*.

In the course of the proof of Theorem 1.1, we obtain the following conclusions as immediate corollaries.

**Corollary 1.2.** *Under the assumptions made in Theorem* 1.1, *if* $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times \dot{H}^1_{\mathrm{div}}(\mathbb{R}^3)$ *is a minimizer of the energy in* (1.1), *then, as* $\kappa \to \infty$,

$$\operatorname{curl} \mathbf{A} - \beta \to 0 \quad in \ H^1(\mathbb{R}^3; \mathbb{R}^3), \tag{1.6}$$

$$e_{\kappa, H}(\psi, \mathbf{A}) \to dx \quad in \ \mathcal{D}'(\Omega), \tag{1.7}$$

$$\mu_{\kappa, H}(\psi, \mathbf{A}) \to (0, 0, dx) \quad in \ \mathcal{D}'(\Omega; \mathbb{R}^3). \tag{1.8}$$

*Here, $dx$ is the Lebesgue measure in $\Omega$, the measure $e_{\kappa, H}(\psi, \mathbf{A})$ and the current $\mu_{\kappa, H}(\psi, \mathbf{A})$ are respectively*

$$e_{\kappa, H}(\psi, \mathbf{A}) = \frac{(|(\nabla - i \kappa H \mathbf{A}) \psi|^2 + \frac{\kappa^2}{2}(1 - |\psi|^2)^2)}{\kappa H \ln \sqrt{\kappa / H}} \, dx, \tag{1.9}$$

$$\mu_{\kappa, H}(\psi, \mathbf{A}) = \operatorname{curl}\left( -\frac{1}{\kappa H} \operatorname{Im}\left( \overline{\psi} (\nabla - i \kappa H \mathbf{A}) \psi \right) \right) + \operatorname{curl} \mathbf{A}. \tag{1.10}$$

In two dimensions, $\mu_{\kappa, H}$ is a measure and it is proved that it gives the density of vortices, hence it is called the *vorticity measure*, see [21].

The proof of Theorem 1.1 is obtained as follows. First we start by the analysis of an approximate problem in a 'large' cube. The cube geometry allows us to link this problem to another two-dimensional problem in a square. The later is analyzed using tools from [21].

Using a ground state of the approximate problem, we construct a test configuration whose energy provides an upper bound of the ground state energy $C_0(\kappa, H)$. As a consequence of this upper bound, we obtain that, for a minimizer $(\psi, \mathbf{A})$ of (1.1), the induced magnetic field $\operatorname{curl} \mathbf{A}$ is close to the applied field $\beta$ in $L^2$-norm. Using this and the regularity of the curl–div system in $\mathbb{R}^3$, we get an estimate of $\mathbf{A} - \mathbf{F}$ in $C^{0,1/2}$-norm.

The *a priori* estimates obtained for minimizers allow us to determine a lower bound of the energy that matches with the obtained upper bound. Actually, we use the 'semi-classical' localization techniques developed in [10] to reduce the problem to that of the approximate problem in a cube. Then the analysis of the later problem is used to obtain the matching lower bound.

An interesting aspect of the analysis is that we do not use constructions involving vortices, i.e. we do not localize the set where $\{x \in \Omega : |\psi(x)| \leqslant 1/2\}$ (as this is certainly difficult in three dimensions). This is a significant difference between the strategy of our proof and the one given in [19] for the two-dimensional functional. However, the construction of 'vortex-balls' for the two-dimensional functional 'implicitly' appears in the analysis of the three-dimensional approximate problem, as we refer to results of [21,19]. In the context of the Ginzburg–Landau model, the implementation of 'semi-classical' techniques to address situations where vortices exist seems rather new.

The analysis presented in Section 2 combined with a recently proved estimate in [13] enables us to prove a theorem of independent interest (Theorem 2.4 below), which concerns the asymptotic behavior of a limiting constant appearing in [20], thereby answering a question raised by the authors of the aforementioned paper.

The paper is organized as follows. Section 2 is devoted to the analysis of the approximate problem. In Section 3, an upper bound of the ground state energy is obtained. In Section 4, interesting estimates are obtained for minimizers of (1.1). Section 5 is devoted to the proof of the lower bound.

**Remark on notation.**

- The letter $C$ denotes a positive constant that is independent of the parameters $\kappa$ and $H$, and whose value may change from line to line.
- If $a(\kappa)$ and $b(\kappa)$ are two functions with $b(\kappa) \neq 0$, we write $a(\kappa) \sim b(\kappa)$ if $a(\kappa)/b(\kappa) \to 1$ as $\kappa \to \infty$.

## 2. The approximate problem

### 2.1. Two-dimensional energy

Let $K = (-1/2, 1/2) \times (-1/2, 1/2)$ be a square of unit side length, $h_{\mathrm{ex}}$ and $\varepsilon$ be two positive parameters. Consider the functional defined for all $u \in H^1(K; \mathbb{C})$,

$$E^{2D}(u) = \int_K \left( \left| (\nabla - ih_{\mathrm{ex}} \mathbf{A}_0) u \right|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 \right) dx. \tag{2.1}$$

Here $\mathbf{A}_0$ is the vector potential:

$$\mathbf{A}_0(x_1, x_2) = \frac{1}{2}(-x_2, x_1), \quad (x_1, x_2) \in \mathbb{R}^2, \tag{2.2}$$

whose curl is equal to 1.

Notice that the functional $E^{2D}$ is a simplified version of the full Ginzburg–Landau functional considered in [19], as the magnetic potential in (2.1) is given and *not* an unknown of the problem.

We introduce the ground state energy

$$m_0(h_{\mathrm{ex}}, \varepsilon) = \inf\{E^{2\mathrm{D}}(u) \colon u \in H^1(K; \mathbb{C})\}. \tag{2.3}$$

Since $E^{2\mathrm{D}}$ is bounded from below, there exists a ground state (minimizer) associated to $m_0(h_{\mathrm{ex}}, \varepsilon)$. If $u$ is such a ground state, then it results from a standard application of the maximum principle that

$$|u| \leqslant 1 \quad \text{in } K. \tag{2.4}$$

Consider the regime of magnetic fields $h_{\mathrm{ex}}$ as in Theorem 2.1 below. We can obtain a lower bound of $m_0(h_{\mathrm{ex}}, \varepsilon)$ (or rather of $E^{2\mathrm{D}}(u)$, with $u$ a ground state) exactly as in [21, Section 8.2], by using a scaling argument that reduces the situation to magnetic fields of lower order (precisely of order $|\ln \varepsilon|$). In this way, we get the following theorem.

**Theorem 2.1.** *Assume that $h_{\mathrm{ex}}$ is a function of $\varepsilon$ such that*

$$|\ln \varepsilon| \ll h_{\mathrm{ex}} \ll \frac{1}{\varepsilon^2}, \quad \text{as } \varepsilon \to 0.$$

*Then the ground state energy $m_0(h_{\mathrm{ex}}, \varepsilon)$ satisfies*

$$m_0(h_{\mathrm{ex}}, \varepsilon) \geqslant h_{\mathrm{ex}} \ln \frac{1}{\varepsilon \sqrt{h_{\mathrm{ex}}}}(1 + o(1)),$$

*as $\varepsilon \to 0$.*

Minimization of the functional $E^{2\mathrm{D}}$ over 'magnetic periodic' functions appears naturally as well. Let us introduce the following space

$$E_{h_{\mathrm{ex}}} = \{u \in H^1_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{C}) \colon u(x_1 + 1, x_2) = e^{ih_{\mathrm{ex}}x_2/2}u(x_1, x_2),$$
$$u(x_1, x_2 + 1) = e^{-ih_{\mathrm{ex}}x_1/2}u(x_1, x_2)\}, \tag{2.5}$$

together with the ground state energy

$$m_{\mathrm{p}}(h_{\mathrm{ex}}, \varepsilon) = \inf\{E^{2\mathrm{D}}(u) \colon u \in E_{h_{\mathrm{ex}}}\}. \tag{2.6}$$

**Theorem 2.2.** *Assume that $h_{\mathrm{ex}}$ is a function of $\varepsilon$ such that*

$$|\ln \varepsilon| \ll h_{\mathrm{ex}} \ll \frac{1}{\varepsilon^2}, \quad \text{as } \varepsilon \to 0.$$

*Then the ground state energy $m_{\mathrm{p}}(h_{\mathrm{ex}}, \varepsilon)$ satisfies*

$$m_{\mathrm{p}}(h_{\mathrm{ex}}, \varepsilon) = h_{\mathrm{ex}} \ln \frac{1}{\varepsilon \sqrt{h_{\mathrm{ex}}}}(1 + o(1)),$$

*as $\varepsilon \to 0$.*

**Proof.** Since the restriction of a function in $E_{h_{ex}}$ to $K$ is a function in $H^1(K)$, we get that $m_p(h_{ex}, \varepsilon) \geqslant m_0(h_{ex}, \varepsilon)$, where $m_0(h_{ex}, \varepsilon)$ is the ground state energy in (2.3). Theorem 2.1 then gives us a lower bound of $m_p(h_{ex}, \varepsilon)$.

We prove the upper bound by computing the energy of a test function $u$ constructed in [6]. Let $N$ be the largest positive integer satisfying $N \leqslant \sqrt{h_{ex}/2\pi} < N+1$. Divide the square $K$ into $N^2$ disjoint squares $(K_j)_{0 \leqslant j \leqslant N^2-1}$ each of side length equal to $1/N$ and center $a_j$.

Let $h$ be the unique solution of the problem

$$\begin{cases} -\Delta h + h_{ex} = 2\pi \delta_{a_0} & \text{in } K_0, \\ \dfrac{\partial h}{\partial \nu} = 0 & \text{on } \partial K_0, \\ \displaystyle\int_{K_0} h\, dx = 0. \end{cases}$$

Here $\nu$ is the unit outward normal vector of $K_0$. By uniqueness of $h$ as solution of the afore-mentioned problem, $h$ is symmetric with respect to the axes of the square $K_0$ and hence satisfies periodic conditions on the boundary of $K_0$. Moreover, the function $v(x) = h(x) - \ln|x - a_0|$ is smooth in $K_0$, since $-\Delta v + h_{ex} = 0$. Consequently, through a scaling argument, it is easy to check that, as $\varepsilon \to 0$,

$$\int_{K_0 \setminus B(a_0, \varepsilon)} |\nabla h|^2\, dx \leqslant 2\pi \ln \frac{1}{\varepsilon N} + \mathcal{O}(1)$$

$$\leqslant 2\pi \ln \frac{1}{\varepsilon \sqrt{h_{ex}}} + \mathcal{O}(1).$$

We extend $h$ by periodicity in the square $K$. Let $\phi$ be a function (defined modulo $2\pi$) satisfying in $K \setminus \{a_j : 0 \leqslant j \leqslant N^2 - 1\}$,

$$\nabla \phi = -\nabla^\perp h + h_{ex} \mathbf{A}_0.$$

Here $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$ and $\mathbf{A}_0$ is the magnetic potential in (2.2).

If $x \in K_0$, let $\rho(x) = \min(1, |x - a_0|/\varepsilon)$. We extend the function $\rho$ by periodicity in the square $K$. We put $u(x) = \rho(x)e^{i\varphi(x)}$ for all $x \in K$. Then $u$ can be extended as a function in the space $E_{h_{ex}}$ in (2.5), see [5, Lemma 5.11] for details.

The energy of $u$ is easily computed, since $u$ is 'magnetic periodic'. Actually,

$$E^{2D}(u) = N^2 \times \int_{K_0} \left( \rho^2 |\nabla h|^2 + |\nabla \rho|^2 + \frac{1}{2\varepsilon^2}(1-\rho)^2 \right) dx$$

$$\leqslant N^2 \times \left( 2\pi \ln \frac{1}{\varepsilon \sqrt{h_{ex}}} + \mathcal{O}(1) \right).$$

Since, $N = \sqrt{h_{ex}/2\pi}(1 + o(1))$ as $\varepsilon \to 0$, and $m_p(h_{ex}, \varepsilon) \leqslant E^{2D}(u)$, we deduce that

$$m_p(h_{ex}, \varepsilon) \leqslant h_{ex} \ln \frac{1}{\varepsilon \sqrt{h_{ex}}} (1 + o(1)),$$

as $\varepsilon \to 0$. □

Since $m_0(h_{ex}, \varepsilon) \leqslant m_p(h_{ex}, \varepsilon)$, we get as a corollary of Theorems 2.1 and 2.2:

**Corollary 2.3.** *Let $m_0(h_{ex}, \varepsilon)$ be the ground state energy introduced in* (2.3) *above. Suppose that $h_{ex}$ is a function of $\varepsilon$ and $|\ln \varepsilon| \ll h_{ex} \ll 1/\varepsilon^2$ as $\varepsilon \to 0$. Then*

$$m_0(h_{ex}, \varepsilon) = h_{ex} \ln \frac{1}{\varepsilon \sqrt{h_{ex}}} \left(1 + o(1)\right),$$

*as $\varepsilon \to 0$.*

Theorem 2.2 serves in answering a question of independent interest arising in [20]. Consider two constants $b \in (0, 1)$ and $R > 0$. Let $K_R = (-R/2, R/2) \times (-R/2, R/2)$. If $u \in H^1(K_R)$, we define the energy

$$F_{K_R}(u) = \int\limits_{K_R} \left(b \left|(\nabla - i\mathbf{A}_0)u\right|^2 + \frac{1}{2}\left(1 - |u|^2\right)^2\right) dx,$$

together with the ground state energy

$$e_p(b, R) = \inf\left\{F_{K_R}(u) \colon u \in E_R\right\}.$$

Here $\mathbf{A}_0$ is the magnetic potential introduced in (2.2) and $E_R$ is the space introduced in (2.5) (with $h_{ex} = R$ and $x_i + 1$ replaced by $x_i + R$). It is proved that, for all $b \in (0, 1)$, there exists a constant $f(b)$ such that

$$f(b) = \frac{1}{2} \lim_{R \to \infty} \frac{e_p(b, R)}{R^2}. \tag{2.7}$$

The limiting constant $f(b)$ appeared in [20,2], then it is recently studied with different tools in [13]. This limiting constant describes the ground state energy of both two- and three-dimensional superconductors subject to high magnetic fields (see [13]).

The behavior of the function $f(b)$ as $b \to 1_-$ is analyzed in details in [13]. However, the behavior as $b \to 0_+$ remains open. Only a non-optimal estimate on $f(b)$ is given as $b \to 0_+$ in [20]. Here, using Theorem 2.2 and an estimate in [13], we describe the leading order asymptotic behavior of $f(b)$ as $b \to 0_+$.

**Theorem 2.4.** *Let $f(b)$ be as defined in* (2.7). *Then, as $b \to 0_+$, $f(b)$ satisfies*

$$f(b) = \frac{b}{2} \ln \frac{1}{\sqrt{b}} \left(1 + o(1)\right).$$

**Remark 2.5.** In [20], it is proved that

$$b - \frac{b^2}{2} \leqslant f(b) \leqslant \frac{b}{2} \ln \frac{1}{\sqrt{b}} \left(1 + o(1)\right),$$

as $b \to 0_+$.

**Proof of Theorem 2.4.** It is proved in [13, Theorem 2.1 and Proposition 2.8] that there exist universal constants $C$ and $R_0$ such that

$$\forall b \in (0, 1), \ \forall R \geqslant R_0, \quad \left| 2f(b) - \frac{e_{\mathrm{p}}(b, R)}{R^2} \right| \leqslant \frac{C}{R}. \tag{2.8}$$

Let $h_{\mathrm{ex}} = R^2$ and $\varepsilon = \sqrt{b}/R$. A scaling argument shows that

$$e_{\mathrm{p}}(b, R) = b m_{\mathrm{p}}(h_{\mathrm{ex}}, \varepsilon). \tag{2.9}$$

We select $R = 1/b$ so that as $b \to 0_+$ we have $\varepsilon \to 0$ and $|\ln \varepsilon| \ll h_{\mathrm{ex}} \ll \varepsilon^{-2}$. Theorem 2.2 then tells us that

$$m_{\mathrm{p}}(h_{\mathrm{ex}}, \varepsilon) = h_{\mathrm{ex}} \ln \frac{1}{\varepsilon \sqrt{h_{\mathrm{ex}}}} \big(1 + o(1)\big). $$

We insert this estimate into (2.9) then we substitute the values $h_{\mathrm{ex}} = R^2$ and $\varepsilon \sqrt{h_{\mathrm{ex}}} = \sqrt{b}$. Finally, inserting the resulting estimate into (2.8) finishes the proof of the proposition. $\quad\square$

### 2.2. Three-dimensional energy

If $\mathcal{D}$ is an open set of $\mathbb{R}^3$ and $u \in H^1(\mathcal{D}; \mathbb{C})$, we define the energy

$$G_{\mathcal{D}}(u) = \int_{Q_R} \left( b \big| (\nabla - i\mathbf{F}) u \big|^2 + \frac{1}{2} \big(1 - |u|^2\big)^2 \right) dx. \tag{2.10}$$

Here $\mathbf{F}$ is the magnetic potential:

$$\mathbf{F}(x_1, x_2, x_3) = (-x_2/2, x_1/2, 0), \quad (x_1, x_2, x_3) \in \mathbb{R}^3, \tag{2.11}$$

whose curl is equal to 1.

Let $b$ and $R$ be two positive parameters. Consider a cube $Q_R$ of side length $R$ defined as follows:

$$Q_R = (-R/2, R/2) \times (-R/2, R/2) \times (-R/2, R/2). \tag{2.12}$$

We introduce the ground state energy

$$M_0(b, R) = \inf\big\{ G_{Q_R}(u) \colon u \in H^1(Q_R; \mathbb{C}) \big\}. \tag{2.13}$$

In the next theorem, we give an asymptotic lower bound of the ground state energy $M_0(b, R)$ as $b \to 0$ and $R \to \infty$ simultaneously, in such a way that $\ln(Rb^{-1/2}) \ll R^2$.

**Theorem 2.6.** *Suppose that the positive parameters $b = b(\epsilon)$ and $R = R(\epsilon)$ are functions of a parameter $\epsilon$ such that*

$$b(\epsilon) \to 0, \qquad R(\epsilon) \to \infty, \quad \text{and} \quad \frac{1}{R(\epsilon)^2} \ln \frac{R(\epsilon)}{\sqrt{b(\epsilon)}} \to 0,$$

*as $\epsilon \to 0$.*

*Then, the ground state energy $M_0(b, R)$ satisfies*

$$\frac{M_0(b, R)}{R^3} = b \ln \frac{1}{\sqrt{b}} \big(1 + o(1)\big),$$

*as $\epsilon \to 0$.*

**Proof.** Let $h_{\mathrm{ex}} = R^2$ and $\varepsilon = \sqrt{b}/R$. By the assumption on $b$ and $R$, it is easy to see that $\varepsilon \to 0$ and $|\ln \varepsilon| \ll h_{\mathrm{ex}} \ll 1/\varepsilon^2$.

Consequently, Theorem 2.1 tells us that the ground state energy $m_0(h_{\mathrm{ex}}, \varepsilon)$ in (2.3) satisfies

$$m_0(h_{\mathrm{ex}}, \varepsilon) = h_{\mathrm{ex}} \ln \frac{1}{\varepsilon \sqrt{h_{\mathrm{ex}}}} \big(1 + o(1)\big).$$

We will prove that

$$M_0(b, R) = b R m_0(h_{\mathrm{ex}}, \varepsilon),$$

which will immediately give us the asymptotic estimate in Theorem 2.6.

Let $u \in H^1(Q_R; \mathbb{C})$, $K = (-1/2, 1/2) \times (-1/2, 1/2)$ and $Q_1 = K \times (-1/2, 1/2)$. Define the rescaled function $\widetilde{u} \in H^1(Q_1; \mathbb{C})$ as follows:

$$\forall x \in Q_1, \quad \widetilde{u}(x) = u(Rx).$$

It is easy to check that

$$G_{Q_R}(u) = bR \int_{-1/2}^{1/2} \left( \int_K \left( |(\nabla - i h_{\mathrm{ex}} \mathbf{F}) \widetilde{u}|^2 + \frac{1}{2\varepsilon^2} \big(1 - |\widetilde{u}|^2\big)^2 \right) dx_\perp \right) dx_3$$

$$\geqslant bR \int_{-1/2}^{1/2} \left( \int_K \left( |(\nabla_{x_\perp} - i h_{\mathrm{ex}} \mathbf{F}) \widetilde{u}|^2 + \frac{1}{2\varepsilon^2} \big(1 - |\widetilde{u}|^2\big)^2 \right) dx_\perp \right) dx_3.$$

Here, if $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we write $x_\perp = (x_1, x_2)$ and $\nabla_{x_\perp} = (\partial_{x_1}, \partial_{x_2})$. Then, recalling the definition of $m_0(h_{\mathrm{ex}}, \varepsilon)$, we get

$$G_{Q_R}(u) \geqslant bR \int_{-1/2}^{1/2} m_0(h_{\mathrm{ex}}, \varepsilon) \, dx_3 = bR m_0(h_{\mathrm{ex}}, \varepsilon).$$

Taking the infimum over all functions $u \in H^1(Q_R; \mathbb{C})$, we get that $M_0(b, R) \geqslant bR m_0(h_{\mathrm{ex}}, \varepsilon)$.

Let $u_{h_{\mathrm{ex}}, \varepsilon}$ be a ground state of $E^{2D}$, i.e. $E^{2D}(u_{h_{\mathrm{ex}}, \varepsilon}) = m_0(h_{\mathrm{ex}}, \varepsilon)$. Define the function

$$u : Q_R \ni x \mapsto u_{h_{\mathrm{ex}}, \varepsilon}(x_\perp / R).$$

Then, $G_{Q_R}(u) = bR E^{2D}(u_{h_{\mathrm{ex}}, \varepsilon})$, thereby showing that $M_0(b, R) \leqslant bR m_0(h_{\mathrm{ex}}, \varepsilon)$. $\quad \square$

## 3. Upper bound of the energy

The aim of this section is to give an upper bound on the ground state energy $C_0(\kappa, H)$ in (1.4).

**Theorem 3.1.** *Assume that the magnetic field $H$ satisfies $\ln \kappa / \kappa \ll H \ll \kappa$ as $\kappa \to \infty$. Then the ground state energy $C_0(\kappa, H)$ in (1.4) satisfies*

$$C_0(\kappa, H) \leqslant |\Omega| \kappa H \ln \sqrt{\frac{\kappa}{H}} \left(1 + o(1)\right), \tag{3.1}$$

*as $\kappa \to \infty$.*

*Furthermore, there exists a constant $\kappa_0$ such that, if $\kappa \geqslant \kappa_0$ and $(\psi, \mathbf{A})$ is a minimizer of the functional in (1.1), then*

$$\left\| \operatorname{curl}(\mathbf{A} - \mathbf{F}) \right\|_{L^2(\mathbb{R}^3)} \leqslant \frac{2|\Omega|}{\sqrt{\kappa H}} \sqrt{\ln \sqrt{\frac{\kappa}{H}}}. \tag{3.2}$$

**Proof.** Notice that if $(\psi, \mathbf{A})$ is a minimizer of (1.1), then $\mathcal{E}^{3D}(\psi, \mathbf{A}) = C_0(\kappa, H)$. Consequently, the estimate in (3.2) follows immediately from the upper bound in (3.1).

Let $b = H/\kappa$ and $\ell = \left(\frac{\kappa H}{\ln \kappa}\right)^{1/4} \frac{1}{\sqrt{\kappa H}}$. Then, as $\kappa \to \infty$, we have

$$b \ll 1, \qquad \ell \ll 1, \qquad \ell \sqrt{\kappa H} \gg 1.$$

Let $h_{\text{ex}} = 1/\ell^2$ and $\varepsilon = \sqrt{b}\ell$. Then, as $\kappa \to \infty$, we have $\varepsilon \ll 1$ and $|\ln \varepsilon| \ll h_{\text{ex}} \ll 1/\varepsilon^2$.

Recall the ground state energy $m_p(h_{\text{ex}}, \varepsilon)$ and the space $E_{h_{\text{ex}}}$ introduced in (2.6) and (2.5) respectively. Let $u \in E_{h_{\text{ex}}}$ be a ground state corresponding to $m_p(h_{\text{ex}}, \varepsilon)$, i.e.

$$\int_K \left( \left| (\nabla - i h_{\text{ex}} \mathbf{A}_0) u \right|^2 + \frac{1}{2\varepsilon^2} \left(1 - |u|^2\right)^2 \right) dx = m_p(h_{\text{ex}}, \varepsilon).$$

For all $x = (x_\perp, x_3) \in \mathbb{R}^3$, we introduce the function

$$v(x) = u(\ell \sqrt{\kappa H} x_\perp).$$

Let $(Q_j)$ be a lattice of $\mathbb{R}^3$ generated by the cube

$$Q = \left( -\frac{1}{2\ell\sqrt{\kappa H}}, \frac{1}{2\ell\sqrt{\kappa H}} \right) \times \left( -\frac{1}{2\ell\sqrt{\kappa H}}, \frac{1}{2\ell\sqrt{\kappa H}} \right) \times \left( -\frac{1}{2\ell\sqrt{\kappa H}}, \frac{1}{2\ell\sqrt{\kappa H}} \right).$$

It is easy to check that

$$\int_Q \left( \left| (\nabla - i\kappa H \mathbf{F}) v \right|^2 + \frac{\kappa^2}{2} \left(1 - |v|^2\right)^2 \right) dx = \frac{1}{\ell\sqrt{\kappa H}} m_p(h_{\text{ex}}, \varepsilon).$$

Here $\mathbf{F}$ is the magnetic potential in (2.11). Let $\mathcal{J} = \{Q_j \colon Q_j \cap \Omega \neq \emptyset\}$ and $N = \mathrm{Card}\,\mathcal{J}$. Then, as $\kappa \to \infty$, we have

$$N = |\Omega| \times (\ell\sqrt{\kappa H})^3 (1 + o(1)).$$

Recall the functional $\mathcal{E}^{3\mathrm{D}}$ in (1.1). We compute the energy of the test configuration $(v, \mathbf{F})$. Since $\mathrm{curl}\,\mathbf{F} = \beta$ and the function $v$ is magnetic periodic with respect to the lattice $Q_j$, we get

$$\mathcal{E}^{3\mathrm{D}}(v, \mathbf{F}) = N \times \int_Q \left( \left| (\nabla - i\kappa H \mathbf{F}) v \right|^2 + \frac{\kappa^2}{2} (1 - |v|^2)^2 \right) dx$$

$$= N \times \frac{1}{\ell\sqrt{\kappa H}} m_{\mathrm{p}}(h_{\mathrm{ex}}, \varepsilon).$$

We use Theorem 2.2, the definitions of $h_{\mathrm{ex}}$ and $\varepsilon$, and the asymptotic behavior of $N$ to get

$$N \times \frac{1}{\ell\sqrt{\kappa H}} m_{\mathrm{p}}(h_{\mathrm{ex}}, \varepsilon) = \kappa H \ln\sqrt{\frac{\kappa}{H}} (1 + o(1)),$$

as $\kappa \to \infty$. This proves the upper bound of Theorem 3.1. $\quad\square$

## 4. A priori estimates of minimizers

The aim of this section is to give *a priori* estimates on the solutions of the Ginzburg–Landau equations (1.3). Those estimates play an essential role in controlling the error resulting from various approximations.

The starting point is the following $L^\infty$-bound resulting from the maximum principle. Actually, if $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times \dot{H}^1_{\mathrm{div}, \mathbf{F}}(\mathbb{R}^3)$ is a solution of (1.3), then

$$\|\psi\|_{L^\infty(\Omega)} \leqslant 1. \tag{4.1}$$

Next we prove an estimate on the induced magnetic potential.

**Proposition 4.1.** *Suppose that the magnetic field $H$ is a function of $\kappa$ such that $\ln\kappa \ll \kappa H \ll \kappa^2$ as $\kappa \to \infty$. There exist positive constants $\kappa_0$ and $C$ such that, if $\kappa \geqslant \kappa_0$ and $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times \dot{H}^1_{\mathrm{div}, \mathbf{F}}(\mathbb{R}^3)$ is a minimizer of the energy in (1.1), then*

$$\|\mathbf{A} - \mathbf{F}\|_{H^2(\Omega)} \leqslant \frac{C}{\sqrt{\kappa H}} \sqrt{\ln\sqrt{\frac{\kappa}{H}}},$$

$$\|\mathbf{A} - \mathbf{F}\|_{C^{0,1/2}(\Omega)} \leqslant \frac{C}{\sqrt{\kappa H}} \sqrt{\ln\sqrt{\frac{\kappa}{H}}}.$$

*Here $\mathbf{F}$ is the magnetic potential introduced in* (2.11).

**Proof.** The estimate in $C^{0,1/2}$-norm is a consequence of the Sobolev embedding of $H^2(\Omega)$ in $C^{0,1/2}(\Omega)$.

Notice that it follows from Theorem 3.1 that

$$\big\| \text{curl}(\mathbf{A} - \mathbf{F}) \big\|_{L^2(\mathbb{R}^3)} \leqslant \frac{2|\Omega|}{\sqrt{\kappa H}} \sqrt{\ln \sqrt{\frac{\kappa}{H}}},$$

$$\big\| (\nabla - i\kappa H \mathbf{A})\psi \big\|_{L^2(\Omega)} \leqslant 2|\Omega| \sqrt{\kappa H} \sqrt{\ln \sqrt{\frac{\kappa}{H}}}. \tag{4.2}$$

Let $a = \mathbf{A} - \mathbf{F}$. We will prove that $\|a\|_{H^2(\Omega)} \leqslant \frac{C}{\sqrt{\kappa H}} \sqrt{\ln \sqrt{\frac{\kappa}{H}}}$. Since $\text{div}\, a = 0$, we get by regularity of the curl–div system (see e.g. [10, Theorem D.3.1]),

$$\|a\|_{L^6(\mathbb{R}^3)} \leqslant C \|\text{curl}\, a\|_{L^2(\mathbb{R}^3)}. \tag{4.3}$$

The second equation in (1.3) reads as follows:

$$-\Delta a = \frac{1}{\kappa H} \text{Im}\big( \overline{\psi}(\nabla - i\kappa H \mathbf{A})\psi \big) \mathbf{1}_\Omega.$$

Select a positive constant $M$ such that the open ball $K = B(0, M)$ contains $\Omega$. By elliptic estimates (see e.g. [10, Theorem E.4.2]),

$$\|a\|_{H^2(\Omega)} \leqslant C \big( \|a\|_{L^2(K)} + \|\Delta a\|_{L^2(K)} \big).$$

Using the embedding of $L^2(K)$ into $L^6(K)$, the estimate in (4.3) and the bound $|\psi| \leqslant 1$, we get that

$$\|a\|_{H^2(\Omega)} \leqslant C \bigg( \|\text{curl}\, a\|_{L^2(\mathbb{R}^3)} + \frac{1}{\kappa H} \big\| (\nabla - i\kappa H)\psi \big\|_{L^2(\Omega)} \bigg).$$

Inserting the estimates in (4.2) into this upper bound finishes the proof of the proposition. □

## 5. Lower bound of the energy

In this section, we suppose that $D$ is an open set with smooth boundary such that $D \subset \Omega$. We will give a lower bound of the energy

$$\mathcal{E}_0(\psi, \mathbf{A}; D) = \int\limits_D \bigg( \big| (\nabla - i\kappa H \mathbf{A})\psi \big|^2 + \frac{\kappa^2}{2}\big(1 - |\psi|^2\big)^2 \bigg) dx, \tag{5.1}$$

where $(\psi, \mathbf{A})$ is a minimizer of the functional in (1.1). The precise statement is the subject of the next theorem.

**Theorem 5.1.** *Suppose that the magnetic field $H$ is a function of $\kappa$ such that $\ln \kappa \ll \kappa H \ll \kappa^2$ as $\kappa \to \infty$. If $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times \dot{H}^1_{\mathrm{div}, \mathbf{F}}(\mathbb{R}^3)$ is a minimizer of the function in (1.1), then*

$$\mathcal{E}_0(\psi, \mathbf{A}; D) \geqslant |D| \kappa H \ln \sqrt{\frac{\kappa}{H}} + o\left( \kappa H \ln \sqrt{\frac{\kappa}{H}} \right),$$

*as $\kappa \to \infty$. Here $\mathcal{E}_0(\psi, \mathbf{A}; D)$ is introduced in (5.1).*

**Proof.** Let $\ell \in (0, 1)$ be a parameter (depending on $\kappa$) that will be chosen later in such a way that $(\sqrt{\kappa H})^{-1} \ll \ell \ll 1$ as $\kappa \to \infty$. Consider a lattice $(\mathcal{Q}_j)_j$ of $\mathbb{R}^3$ generated by the cube

$$Q_\ell = (-\ell/2, \ell/2) \times (-\ell/2, \ell/2) \times (-\ell/2, \ell/2).$$

Let $\mathcal{J} = \{j\colon \mathcal{Q}_j \subset D\}$ and $N = \mathrm{Card}\,\mathcal{J}$. Then, as $\kappa \to \infty$, the natural number $N$ satisfies

$$N = \frac{|D|}{\ell^3} + o\left( \frac{1}{\ell^3} \right). \tag{5.2}$$

Moreover, we have the lower bound

$$\mathcal{E}_0(\psi, \mathbf{A}; D) \geqslant \sum_{j \in \mathcal{J}} \mathcal{E}_0(\psi, \mathbf{A}; \mathcal{Q}_j). \tag{5.3}$$

For each $j \in \mathcal{J}$, we will bound from below the term $\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{Q}_j)$. Let $x_j$ be the center of the cube $\mathcal{Q}_j$. Using the estimate of $\|\mathbf{A} - \mathbf{F}\|_{C^{0,1/2}(\Omega)}$ given in Proposition 4.1, we may write for all $x \in \mathcal{Q}_j$,

$$\left| \mathbf{A}(x) - \mathbf{F}(x) - \left( \mathbf{A}(x_j) - \mathbf{F}(x_j) \right) \right| \leqslant C \lambda \ell^{1/2},$$

where $C$ is a constant that is independent of $j$, $x$ and $\kappa$, and the parameter $\lambda$ is defined by

$$\lambda = \frac{1}{\sqrt{\kappa H}} \sqrt{\ln \sqrt{\frac{\kappa}{H}}}. \tag{5.4}$$

We define $\varphi_j(x) = (\mathbf{A}(x_j) - \mathbf{F}(x_j)) \cdot x$, $u_j(x) = e^{i\varphi(x)} \psi(x)$ and $\mathbf{a}_j(x) = \mathbf{A}(x) - \nabla \varphi_j(x)$. Then we may write

$$\forall x \in \mathcal{Q}_j, \quad \left| \mathbf{a}_j(x) - \mathbf{F}(x) \right| \leqslant C \lambda \ell^{1/2}, \tag{5.5}$$

and

$$\mathcal{E}_0(\psi, \mathbf{A}; \mathcal{Q}_j) = \mathcal{E}_0(u_j, \mathbf{a}_j; \mathcal{Q}_j). \tag{5.6}$$

We may write, for all $\delta \in (0, 1)$,

$$\left| (\nabla - i \kappa H \mathbf{a}_j) u_j \right|^2 \geqslant (1 - \delta) \left| (\nabla - i \kappa H \mathbf{F}) u_j \right|^2 - 2 \delta^{-1} (\kappa H)^2 |\mathbf{a}_j - \mathbf{F}_j|^2 |u_j|^2.$$

We insert this estimate into the expression of $\mathcal{E}_0(u_j, \mathbf{a}_j; \mathcal{Q}_j)$ then we use the estimate in (5.5) and that $|u_j| = |\psi|$ to get

$$\mathcal{E}_0(u_j, \mathbf{a}_j; \mathcal{Q}_j) \geqslant (1 - \delta)\mathcal{E}_0(u_j, \mathbf{F}; \mathcal{Q}_j) - C\delta^{-1}(\kappa H)^2 \lambda^2 \ell \int_{\mathcal{Q}_j} |\psi|^2 \, dx. \tag{5.7}$$

Let $R = \ell \sqrt{\kappa H}$ and $b = H/\kappa$. For all $x \in \mathbb{R}^3$ such that $|x| \leqslant R$, we define

$$v_j(x) = u\left(x_j + \frac{x}{\sqrt{\kappa H}}\right).$$

Then a simple change of variable shows that

$$\mathcal{E}_0(u_j, \mathbf{F}; \mathcal{Q}_j) = \frac{1}{b\sqrt{\kappa H}} G_{Q_R}(v_j), \tag{5.8}$$

where $G_{Q_R}$ is the functional in (2.10) and $Q_R$ is the cube in (2.12).

We select $\ell$ in the following way:

$$\ell = \left(\frac{\kappa H}{\ln \kappa}\right)^{1/4} \frac{1}{\sqrt{\kappa H}}. \tag{5.9}$$

With this choice, we have $(\sqrt{\kappa H})^{-1} \ll \ell \ll 1$, $1 \ll R$ and $\frac{1}{R^2} \ln \frac{R}{\sqrt{b}} \ll 1$ as $\kappa \to \infty$. Consequently, Theorem 2.6 tells us that the ground state $M_0(b, R)$ in (2.13) satisfies

$$M_0(b, R) = bR^3 \ln \frac{1}{\sqrt{b}}(1 + o(1)).$$

Since $v_j \in H^1(Q_R)$, we get $G_{Q_R}(v_j) \geqslant M_0(b, R)$. Substituting this into (5.8) and using the aforementioned asymptotic expansion of $M_0(b, R)$, we get

$$\mathcal{E}_0(u_j, \mathbf{F}; \mathcal{Q}_j) = \frac{R^3}{\sqrt{\kappa H}} \ln \frac{1}{\sqrt{b}}(1 + o(1)). \tag{5.10}$$

By inserting (5.10) into (5.7) and using (5.6), we get for all $j \in \mathcal{J}$,

$$\mathcal{E}(\psi, \mathbf{A}; \mathcal{Q}_j) \geqslant (1 - \delta)\frac{R^3}{\sqrt{\kappa H}} \ln \frac{1}{\sqrt{b}}(1 + o(1)) - C\delta^{-1}\ell(\kappa H)^2 \lambda^2 \int_{\mathcal{Q}_j} |\psi|^2 \, dx.$$

Taking the sum over $j \in \mathcal{J}$ and using (5.3), we get

$$\mathcal{E}(\psi, \mathbf{A}; D) \geqslant (1 - \delta)N \times \frac{R^3}{\sqrt{\kappa H}} \ln \frac{1}{\sqrt{b}}(1 + o(1)) - C\delta^{-1}\ell(\kappa H)^2 \lambda^2 \int_D |\psi|^2 \, dx, \tag{5.11}$$

where $N = \mathrm{Card}\, \mathcal{J}$. To finish the proof, we use the bound $|\psi| \leqslant 1$, the definition of $\lambda$ in (5.4), and we choose $\delta = \ell^{1/2}$. This gives that the remainder term in (5.11) is equal to $o(\kappa H \ln \sqrt{\kappa/H})$. For

the leading order term in (5.11), we use the asymptotic expansion of $N$ in (5.2), that $R = \ell\sqrt{\kappa H}$, and we observe that it is equal to

$$|D|\kappa H \ln\sqrt{\frac{\kappa}{H}}\left(1 + o(1)\right). \qquad \Box$$

**Proof of Theorem 1.1.** Combining the upper bound in Theorem 3.1 and the lower bound in Theorem 5.1 with $D = \Omega$, we get the estimate of the ground state energy in Theorem 1.1. $\quad\Box$

**Proof of Corollary 1.2.** The convergence of $\mathrm{curl}\,\mathbf{A} - \beta$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$ is proved in Theorem 3.1. Since $\mathrm{div}(\mathbf{A} - \mathbf{F}) = 0$ and $\mathbf{A} - \mathbf{F} \in \dot{H}^1(\mathbb{R}^3; \mathbb{R}^3)$, we get that

$$\left\|\nabla \mathrm{curl}(\mathbf{a} - \mathbf{F})\right\|_{L^2(\mathbb{R}^3)} = \left\|\mathrm{curl}(\mathbf{A} - \mathbf{F})\right\|_{L^2(\mathbb{R}^3)}.$$

Consequently, it results from the convergence of $\mathrm{curl}\,\mathbf{A}$ in $L^2(\mathbb{R}^3)$ that $\mathrm{curl}\,\mathbf{A} \to \beta$ in $H^1(\mathbb{R}^3; \mathbb{R}^3)$.

We prove the convergence of $\mu_{\kappa,H}(\psi, \mathbf{A})$. Let $\mathbf{B}(x) = \mathrm{curl}\,\mathbf{A}(x)$. Since $\mathrm{div}\,\mathbf{A} = 0$, it results by taking the curl on both sides of the second equation in (1.3),

$$-\Delta\mathbf{B} + \mathbf{B} = \mu_{\kappa,H}(\psi, \mathbf{A}) \quad \text{in } \Omega.$$

Since $\mathbf{B} \to \beta$ in $H^1(\mathbb{R}^3; \mathbb{R}^3)$, we get that $-\Delta\mathbf{B} + \mathbf{B} \to \beta\,dx$ in $\mathcal{D}'(\mathbb{R}^3; \mathbb{R}^3)$.

It remains to prove the convergence of the measure $e_{\kappa,H}(\psi, \mathbf{A})$. It suffices to prove that $e_{\kappa,H}(\psi, \mathbf{A}) \to dx$ in the sense of measures. If $D$ is any open set in $\Omega$ with smooth boundary, then we have by Theorem 5.1,

$$\mathcal{E}_0(\psi, \mathbf{A}; D) \geqslant |D|\kappa H \ln\sqrt{\frac{\kappa}{H}}\left(1 + o(1)\right),$$

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega \setminus D) \geqslant |\Omega \setminus D|\kappa H \ln\sqrt{\frac{\kappa}{H}}\left(1 + o(1)\right).$$

Here $\mathcal{E}_0(\psi, \mathbf{A}; D)$ is introduced in (5.1). Recall the functional $\mathcal{E}^{3D}$ in (1.1). Since

$$\mathcal{E}_0(\psi, \mathbf{A}; D) + \mathcal{E}_0(\psi, \mathbf{A}; \Omega \setminus D) = \mathcal{E}^{3D}(\psi, \mathbf{A})$$

$$\leqslant \kappa H |\Omega| \ln\sqrt{\frac{\kappa}{H}}\left(1 + o(1)\right),$$

we infer from Theorem 3.1,

$$\mathcal{E}_0(\psi, \mathbf{A}; D) = |D|\kappa H \ln\sqrt{\frac{\kappa}{H}}\left(1 + o(1)\right).$$

This is sufficient to conclude the convergence of $e_{\kappa,H}(\psi, \mathbf{A})$ to $dx$ in the sense of measures. $\quad\Box$

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## References

[1] A. Aftalion, Vortices in Bose Einstein Condensates, Progr. Nonlinear Differential Equations Appl., vol. 67, Birkhäuser, 2006.

[2] A. Aftalion, S. Serfaty, Lowest Landau level approach in superconductivity for the Abrikosov lattice close to $H_{c_2}$, Selecta Math. (N.S.) 13 (2) (2007) 183–202.

[3] S. Alama, L. Bronsard, J.A. Montero, On the Ginzburg–Landau model of a superconducting ball in a uniform field, Ann. Inst. H. Poincaré Anal. Non Linéaire 23 (2) (2006) 237–267.

[4] Y. Almog, Non-linear surface superconductivity in three dimensions in the large $\kappa$ limit, Commun. Contemp. Math. 6 (4) (2004) 637–652.

[5] H. Aydi, Doctoral dissertation, Université Paris-XII, 2004, http://hal.archives-ouvertes.fr/docs/00/29/71/36/PDF/these-aydi.pdf.

[6] H. Aydi, E. Sandier, Vortex analysis of the periodic Ginzburg–Landau model, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (4) (2009) 1223–1236.

[7] F. Bethuel, H. Brezis, F. Helein, Ginzburg–Landau Vortices, Progr. Nonlinear Differential Equations Appl., vol. 13, Birkhäuser, 1994.

[8] A. Contreras, P. Sternberg, Gamma-convergence and the emergence of vortices for Ginzburg–Landau on thin shells and manifolds, Calc. Var. Partial Differential Equations 38 (1–2) (2010) 243–274.

[9] S. Fournais, B. Helffer, On the Ginzburg–Landau critical field in three dimensions, Comm. Pure Appl. Math. 62 (2) (2009) 215–241.

[10] S. Fournais, B. Helffer, Spectral Methods in Surface Superconductivity, Progr. Nonlinear Differential Equations Appl., vol. 77, Birkhäuser, Boston, 2010.

[11] S. Fournais, A. Kachmar, Strength of superconductivity close to critical magnetic field, in: XVIth International Congress on Mathematical Physics, 2009, pp. 625–629, doi:10.1142/9789814304634_0058.

[12] S. Fournais, A. Kachmar, Nucleation of bulk superconductivity close to critical magnetic field, Adv. Math. 226 (2011) 1213–1258.

[13] S. Fournais, A. Kachmar, The ground state energy of the three dimensional Ginzburg–Landau functional. Part I: Bulk regime, preprint.

[14] S. Fournais, A. Kachmar, M. Persson, The ground state energy of the three dimensional Ginzburg–Landau functional. Part II: Surface regime, preprint.

[15] B. Helffer, X.B. Pan, Reduced Landau–de Gennes functional and surface smectic state of liquid crystals, J. Funct. Anal. 255 (11) (2008) 3008–3069.

[16] K. Lu, X.-B. Pan, Surface nucleation of superconductivity in 3-dimensions, J. Differential Equations 168 (2) (2000) 386–452.

[17] X.B. Pan, Surface superconductivity in 3 dimensions, Trans. Amer. Math. Soc. 356 (10) (2004) 3899–3937.

[18] X.B. Pan, Analogies between superconductors and liquid crystals: nucleation and critical fields, in: Asymptotic Analysis and Singularities. Elliptic and Parabolic PDEs and Related Problems, in: Adv. Stud. Pure Math., vol. 47-2, Mathematical Society Japan, Tokyo, 2007, pp. 479–518.

[19] E. Sandier, S. Serfaty, On the energy of type-II superconductors in the mixed phase, Rev. Math. Phys. 12 (9) (2000) 1219–1257.

[20] S. Sandier, S. Serfaty, The decrease of bulk-superconductivity close to the second critical field in the Ginzburg–Landau model, SIAM J. Math. Anal. 34 (4) (2003) 939–956.

[21] S. Sandier, S. Serfaty, Vortices in the Magnetic Ginzburg–Landau Model, Progr. Nonlinear Differential Equations Appl., vol. 70, Birkhäuser, Boston, 2007.