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GROUP INVARIANTS OF LINKS

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EVERY oriented link can be realized as a closed braid $L(b)$ for some braid b. If b is an n-string braid, the link group $\pi_1(S^3 - L(b))$ has a presentation

$$\langle X_1, \ldots, X_n | X_i\,\alpha(b) = X_i \quad (i = 1, \ldots, n)\rangle,$$

where α is a homomorphism of the braid group B_n to the group of automorphisms of the free group $\langle x_1, \ldots, x_n \rangle$ called the Artin representation (see §1).

By a computer experiment, we have found some other representations of the braid group B_n on the free groups which, in the same way as above, give group invariants of links. We will study these group invariants in this paper.

In §1, we describe the link group via the Artin representation. This shows the motivation of our computer experiment. The result of the experiment is given in §2. We will analyze the group invariants obtained by this method in the following sections.

§1. THE LINK GROUP

We fix our notation about braids first. For more detailed arguments, we refer to [3].

The braid group B_n of order n is generated by the elements σ_i $(i = 1, \ldots, n-1)$ shown in Fig. 1. These generators satisfy the following fundamental relations ([2]).

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad (|i - j| \geq 2) \tag{1}$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad (i = 1, \ldots, n-2) \tag{2}$$

We will write the composition of braids from left to right. Thus, all the actions of the braid group will be from the right.

The braid group B_n can also be regarded as the mapping class group of (D, d), where D denotes a disk with n holes, and d its outer boundary. Therefore, the braid group B_n naturally acts on $\pi_1(D)$, and it induces a representation $\alpha: B_n \to \mathrm{Aut}(\pi_1(D))$ called the Artin representation. The fundamental group $\pi_1(D)$ is a free group of rank n generated by $x_1, \ldots, x_n$, where x_i is represented by the boundary of the i-th hole. The action of the generator $\sigma_i \in B_n$ on D is shown in Fig. 2. Therefore, the Artin representation is given by

$$\begin{aligned} x_i\alpha(\sigma_i) &= x_i x_{i+1} x_i^{-1}, \\ x_{i+1}\alpha(\sigma_i) &= x_i, \\ x_j\alpha(\sigma_i) &= x_j \quad (j \neq i, i+1). \end{aligned}$$

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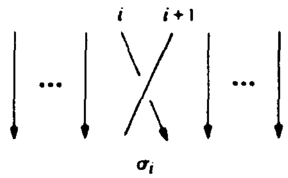

Fig. 1.

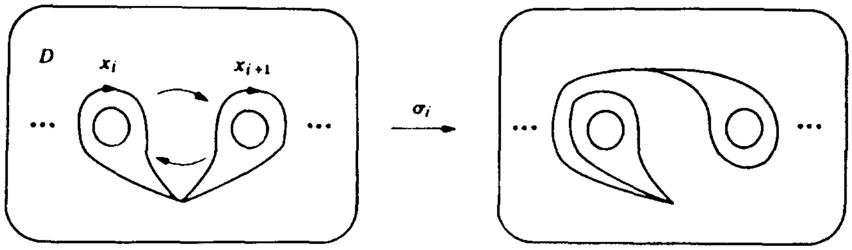

Fig. 2.

Let $L(b)$ be a closed braid obtained by closing a braid $b \in B_n$. An application of the van Kampen's theorem shows that the link group $\pi_1(S^3 - L(b))$ has a presentation

$$\langle x_1, \ldots, x_n \mid x_i \alpha(b) = x_i \quad (i = 1, \ldots, n) \rangle. \tag{3}$$

This gives an alternative definition of the link group, since every oriented link can be realized as a closed braid $L(b)$ for some braid $b \in B_n$ (Alexander [1]). That the isomorphism type of the group presentation (3) does not depend on the choice of the braid b is obvious from the fact that it is a presentation of the fundamental group of the complement of $L(b)$. However, there is another way of verifying this, which allows us to look for a modification of the group invariant.

Let us denote the disjoint union of the braid groups as

$$B = \{(b, n) \mid b \in B_n \quad (n = 1, 2, \ldots)\},$$

Markov [4] proved that two closed braids $L(b, n)$ and $L(b', n')$ are isotopic as oriented links in S^3 if and only if $(b, n) \sim (b', n')$, where the equivalence relation $\sim$ on B is generated by

$$\text{I.} \quad (b, n) \sim (\sigma_i b \sigma_i^{-1}, n)$$
$$\text{II.} \quad (b, n) \sim (b\sigma_n^{\pm 1}, n + 1)$$

which are referred to as Markov moves of type I and type II respectively.

Now, we can prove that the isomorphism type of the group given by (3) is independent of the choice of the braid b as follows. (See also [5].) First, notice that the group (3) is the group of co-invariants of b, namely the maximal quotient of the free group $\langle x_i, \ldots, x_n \rangle$ on which b acts trivially. It is a general fact that the group of co-invariants is invariant under conjugation by an automorphism. Hence, we only need to deal with the Markov moves of type II. Consider the presentation

$$\langle x_1, \ldots, x_{n+1} \mid x_i \alpha(b\sigma_n) = x_i \quad (i = 1, \ldots, n + 1) \rangle,$$

where $b \in B_n$. Notice that the last relation is just

$$x_{n+1} = x_n.$$

By replacing all other occurrences of x_{n+1} by x_n, one can easily see that the presentation reduces to (3). It is left to the reader to verify that the presentation

$$\langle x_1, \ldots, x_{n+1} \mid x_i \alpha(b\sigma_n^{-1}) = x_i \quad (i = 1, \ldots, n+1)\rangle$$

also reduces to (3).

§2. THE EXPERIMENT

Let F_n denote the free group of rank n generated by $x_1, \ldots, x_n$. We look for representations

$$\rho: B_n \to \mathrm{Aut}(F_n)$$

of the special form:

$$\begin{aligned} x_i\rho(\sigma_i) &= u(x_i, x_{i+1}),\\ x_{i+1}\rho(\sigma_i) &= v(x_i, x_{i+1}),\\ x_j\rho(\sigma_i) &= x_j \quad (j \neq i, i+1),\end{aligned}$$

where $u, v \in \langle a, b\rangle$ are words in two letters. We call such a representation ρ a shift type representation. The relations (1) for the braid group are immediately satisfied, and the relations (2) reduce to the condition

$$\text{C1.} \begin{cases} u(u(a,b), u(v(a,b),c)) = u(a, u(b,c))\\ v(u(a,b), u(v(a,b),c)) = u(v(a,u(b,c)), v(b,c))\\ v(v(a,b),c) = v(v(a,u(b,c)), v(b,c))\end{cases}$$

For the representation ρ above to be well-defined, the pair (u, v) also needs to satisfy the condition

C2. The assignment $a \to u$, $b \to v$ defines an automorphism of the free group $\langle a, b\rangle$.

Our computer program produced all the possible pairs of words $u, v \in \langle a, b\rangle$ such that the sum of the word lengths of u and v is no more than 10, and checked the condition C1 for each pair.

Notice that there are two symmetries among shift type representations, one corresponding to the involution of the free group F_n given by simultaneously replacing x_i with x_i^{-1}, and another which comes from the symmetry of the braid group B_n which is given by interchanging σ_i and σ_i^{-1}.

Up to these two symmetries, the pairs (u, v) which satisfy both C1 and C2 fall into the following seven types:

Type 1. (a, b).
Type 2. (b, a^{-1}).
Type 3. (b^{-1}, a^{-1}).
Type 4. $(a^m b a^{-m}, a)$ where m is an integer.
Type 5. $(ab^{-1}a, a)$.
Type 6. (aba, a^{-1}).
Type 7. $(a^2b, b^{-1}a^{-1}b)$.

We suspect that those given above are, up to the two symmetries, the only pairs which give shift type representations of the braid group on the free group of rank n.

Using a shift type representation $\rho: B_n \to \mathrm{Aut}(F_n)$ determined by such a pair (u, v), we would like to define a group invariant of a link $L = L(b, n)$ by

$$G_{u,v}(L) = \langle x_1, \ldots, x_n | x_i\rho(b) = x_i \quad (i = 1, \ldots, n)\rangle. \tag{4}$$

As noted in §1, this group is always invariant under Markov moves of type I. For this group to be invariant under Markov moves of type II, it suffices for the pair (u, v) to satisfy the condition

C3. The relation $v = b$ implies $u = a$.

To see this, consider the presentation

$$\langle x_1, \ldots, x_{n+1} | x_i\rho(b\sigma_n) = x_i \quad (i = 1, \ldots, n+1)\rangle. \tag{5}$$

Notice that the last relation is

$$v(x_n, x_{n+1}) = x_{n+1}.$$

By the condition C3, we also get

$$u(x_n, x_{n+1}) = x_n.$$

This shows that the generator $\sigma_n \in B_n$ acts trivially on the quotient group (5). Therefore, we can replace all the other relations $x_i\rho(b\sigma_n) = x_i$ by $x_i\rho(b) = x_i$. Then, by dropping the last relation together with the generator x_{n+1}, we reduce the presentation to (4). We also need to show that the presentation

$$\langle x_1, \ldots, x_{n+1} | x_i\rho(b\sigma_n^{-1}) = x_i \quad (i = 1, \ldots, n+1)\rangle$$

is reducible to (4). Since $\rho(\sigma_n)$ is an automorphism of F_n, the relations can be rewritten as

$$x_i\rho(b) = x_i\rho(\sigma_n) \qquad (i = 1, \ldots, n+1).$$

Then, the last relation is

$$x_{n+1} = v(x_n, x_{n+1}).$$

From here, we can proceed as before, and show that the presentation reduces to (4).

Of the seven types given above, types 3–7 satisfy the condition C3, therfore define group invariants of a link. The groups corresponding to types 1 and 2 are not invariant under Markov moves of type II.

It should be pointed out that for each pair (u, v) of type 3–7, the group invariant $G_{u,v}(L)$ also admits Wirtinger type presentations. Consider a regular projection P of an oriented link L. Regard P as a directed graph, and denote the edges of P by $x_1, \ldots, x_n$. These are the generators of the group $G_{u,v}(L)$. For each crossing of P, we give two relations as follows. If the crossing is right-handed as in Fig. 3(a), the relations are $x_i = u(x_k, x_l)$ and $x_j = v(x_k, x_l)$, and if the crossing is left-handed as in Fig. 3(b), we give $x_k = u(x_i, x_j)$ and $x_l = v(x_i, x_j)$.

It is not difficult to see that the isomorphism type of the group presentation defined this way is invariant under the Reidemeister moves for oriented links, thus defines a group invariant of an oriented link L. When P is the projection of a closed braid, this definition of $G_{u,v}(L)$ coincides with our previous definition.

The representation $\rho: B_n \to \mathrm{Aut}(F_n)$ of type 3, given by the pair (b^{-1}, a^{-1}), maps each generator $\sigma_i \in \mathrm{b}_n$ to an involution of F_n. Therefore, the representation ρ factors through the symmetric group of order n, and the group invariant $G_{b^{-1},a^{-1}}(L)$ carries no information about the link L other than the number of components of L. In fact, $G_{b^{-1},a^{-1}}(L)$ is a free group whose rank is equal to the number of components of L. The same remark applies to the type 4 representation when $m = 0$.

The representation of type 6, given by (aba, a^{-1}), can be reduced to type 5 by conjugating the representation by the involution τ of F_n defined by $x_{2i}\tau = x_{2i}$ and $x_{2i+1}\tau = x_{2i+1}^{-1}$.

In the rest of this paper, we analyze the group invariants given by the representations of types 4, 5, and 7.

§3. GROUP INVARIANTS OF TYPE 4

Let us denote the representation of type 4 by

$$\alpha_m : B_n \to \mathrm{Aut}(F_n).$$

Namely, α_m is defined by

$$x_i \alpha_m(\sigma_i) = x_i^m x_{i+1} x_i^{-m},$$
$$x_{i+1} \alpha_m(\sigma_i) = x_i,$$
$$x_j \alpha_m(\sigma_i) = x_j \qquad (j \neq i, i+1).$$

The Artin representation $\alpha = \alpha_1$ is a special case of this.

For an oriented link L isotopic to a closed braid $L(b, n)$, denote the corresponding group invariant by

$$G_m(L) = \langle x_1, \ldots, x_n | x_i \alpha_m(b) = x_i \quad (i = 1, \ldots, n) \rangle.$$

To understand this group geometrically, consider a disk with n holes, D, as in §1. But, this time, imagine that the loop x_i is "hidden inside the i-th hole", and the boundary of the i-th hole is represented by x_i^m. The homeomorphism of D to itself which represents $\sigma_i \in B_n$ now induces $\alpha_m \in \mathrm{Aut}(F_n)$ (Fig. 4).

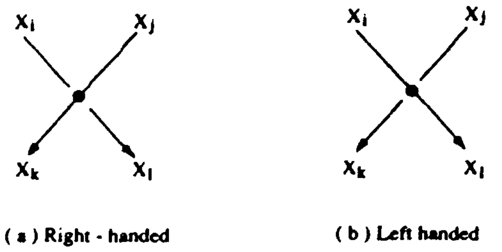

Fig. 3.

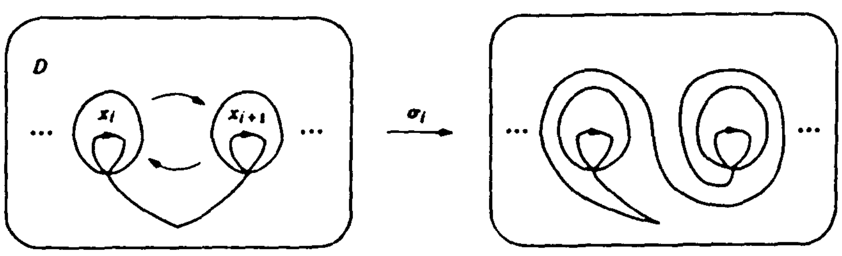

Fig. 4.

To be more precise, consider the exterior X of an oriented link L. Each boundary component of X is a torus $S^1 \times S^1$ whose first coordinate we assume represents the meridian and the second coordinate the longitude. We paste this boundary torus to another torus by the map $f: S^1 \times S^1 \to S^1 \times S^1$ defined by $f(z_1, z_2) = (z_1^m, z_2)$. Do this pasting to each boundary component of X. Then, the fundamental group of the resulting space is isomorphic to $G_m(L)$.

Generalizing this idea, we can imagine more than one loops hidden inside each hole of D. This leads to the following construction. Let Y be the wedge of r circles, and $w: S^1 \to Y$ be any loop in Y. We paste each boundary torus $S^1 \times S^1$ of X to a copy of $Y \times S^1$ by the map $w \times \mathrm{id}: S^1 \times S^1 \to Y \times S^1$. The fundamental group of the resulting space

$$W = X \bigcup_{w \times \mathrm{id}} Y \times S^1 \dots \bigcup_{w \times \mathrm{id}} Y \times S^1$$

has the following description.

The fundamental group of Y is the free group generated by the r circls $Y_1, \dots, Y_r$. The loop w represents a word $w(y_1, \dots, y_r)$. We define a shift type representation α_w of the braid group B_n on the free group $F_{nr} = \langle x_{1,1}, \dots, x_{1,r}, x_{2,1}, \dots, x_{n,r} \rangle$ by

$$\begin{aligned} x_{i,k}\alpha_w(\sigma_i) &= w(X_i)w^{-1}(X_{i+1})x_{i+1,k}w(X_{i+1})w^{-1}(X_i), \\ x_{i+1,k}\alpha_w(\sigma_i) &= x_{i,k}, \\ x_{j,k}\alpha_w(\sigma_i) &= x_{j,k} \qquad (j \neq i, i+1), \end{aligned}$$

where X_i abbreviates $(x_{i,1}, \dots, x_{i,r})$. If the link L is isotopic to a closed braid $L(b, n)$, then the fundamental group $\pi_1(W)$ is isomorphic to the group

$$G_w(L) = \langle x_{1,1}, \dots, x_{n,r} \mid x_{i,j}\alpha_w(\sigma_i) = x_{i,j} \quad (\forall i, j) \rangle.$$

For instance, if w is the word $y_1 y_2, \dots, y_r$, this group is the group of the r-parallel link of L, since we can take a disk with r holes instead of Y in this case.

§4. GROUP INVARIANTS OF TYPE 5

Recall that the representation of type 5,

$$\beta: B_n \to \mathrm{Aut}(F_n)$$

is defined by

$$\begin{aligned} x_i\beta(\sigma_i) &= x_i x_{i+1}^{-1} x_i, \\ x_{i+1}\beta(\sigma_i) &= x_i, \\ x_j\beta(\sigma_i) &= x_j \qquad (j \neq i, i+1). \end{aligned}$$

Then, for an oriented link L isotopic to a closed braid $L(b, n)$, we define

$$G_\beta(L) = \langle x_1, \dots, x_n \mid x_i\beta(b) = x_i \quad (i = 1, \dots, n) \rangle.$$

We show that this group $G_\beta(L)$ is isomorphic to the free product of the fundamental group of the 2-fold branched covering space of L and an infinite cyclic group.

Let X denote the exterior of the link $L = L(b, n)$, and $p: \tilde{X} \to X$ its r-fold cyclic covering space. Identify in $\tilde{X}$ the inverse image $p^{-1}(x_0)$ of the base point $x_0 \in X$, and denote the quotient space by $\tilde{X}/p^{-1}(x_0)$. The fundamental group $\pi_1(\tilde{X}/p^{-1}(x_0))$ is just the free product of $\pi_1(\tilde{X})$ and the free group of rank $r - 1$. There is a natural $\mathbf{Z}_r$-action on $\tilde{X}$, hence on $\pi_1(\tilde{X}/p^{-1}(x_0))$. The group of co-invariants of the $\mathbf{Z}_r$-action on $\pi_1(\tilde{X}/p^{-1}(x_0))$ is naturally identified with $\pi_1(X)$. The group $\pi_1(\tilde{X}/p^{-1}(x_0))$ has the following description via a representation of the braid group.

Let F_{nr} be the free group of rank nr generated by

$$\{x_{i,k} \mid i = 1, \ldots, n, k \in \mathbf{Z}_r\}.$$

We define the representation

$$\beta_r : B_n \to \mathrm{Aut}(F_{nr})$$

by

$$\begin{aligned} x_{i,k}\beta_r(\sigma_i) &= x_{i,k}x_{i+1,k+1}x_{i,k+1}^{-1}, \\ x_{i+1,k}\beta_r(\sigma_i) &= x_{i,k}, \\ x_{j,k}\beta_r(\sigma_i) &= x_{j,k} \qquad (j \neq i, i+1). \end{aligned}$$

One can show that for a closed braid $L = L(b, n)$, the group

$$G_{\beta_r}(L) = \langle x_{i,k} \mid x_{i,k}\beta_r = x_{i,k} \quad (i = 1, \ldots, n, k \in Z_r)\rangle$$

is isomorphic to $\pi_1(\tilde{X}/p^{-1}(x_0))$. Notice that the $\mathbf{Z}_r$-action on $G_{\beta_r}(L)$ is given by $x_{i,k} \to x_{i,k+1}$. The lifts of (the r-th power of) the meridians of L in $\tilde{X}$ are represented by $x_{i,0}x_{i,1} \ldots x_{i,r-1}$ (which are conjugate to each other if L is a knot).

Now, consider the case $r = 2$. The group $\pi_1(\tilde{X}/p^{-1}(x_0))$ is the free product of $\pi_1(\tilde{X})$ and an infinite cyclic group. Sewing back the link L to the 2-fold covering space $\tilde{X}$ introduces the relations $x_{1,0}x_{1,1} = x_{2,0}x_{2,1} = \ldots = x_{n,0}x_{n,1} = 1$. If we replace $x_{1,1}$ by $x_{1,0}^{-1}$ in the definition of the representation β_r, we obtain β. This shows that the group $G_\beta(L)$ is the free product of the fundamental group of the 2-fold branched covering space of the link L, and an infinite cyclic group.

§5. GROUP INVARIANTS OF TYPE 7

The representation of type 7,

$$\gamma : B_n \to \mathrm{Aut}(F_n)$$

is defined by

$$\begin{aligned} x_i\gamma(\sigma_i) &= x_i^2 x_{i+1}, \\ x_{i+1}\gamma(\sigma_i) &= x_{i+1}^{-1}x_i^{-1}x_{i+1}, \\ x_j\gamma(\sigma_i) &= x_j \quad (j \neq i, i+1). \end{aligned}$$

For an oriented link L isotopic to a closed braid $L(b, n)$, the associated group invariant is defined by

$$G_\gamma(L) = \langle x_1, \ldots, x_n \mid x_i\gamma(b) = x_i \quad (i = 1, \ldots, n)\rangle.$$

When abelianized, the representation γ gives the same homomorphism of the braid group B_n to $GL(2, \mathbf{Z})$ as the type 6 representation (which reduces to type 5). Therefore, the abelianization $H_1(G_\gamma(L))$ is isomorphic to the direct sum $H_1(\tilde{X}) \oplus \mathbf{Z}$, where $\tilde{X}$ denotes the 2-fold branched covering space of L.

However, some experimental computations by Fox differential calculus indicates that the behaviour of the group $G_\gamma(L)$ is quite different from that of the fundamental group of the 2-fold branched covering space of L.

In fact, we do not know what the group $G_\gamma(L)$ represents. Therefore, we merely ask the following question.

Question. Can one interpret the group invariant $G_\gamma(L)$ as the fundamental group of some space associated to the link L?

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