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Numerical study of the solution of the Burgers and coupled Burgers equations by a differential transformation method

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ABSTRACT

In this paper, the Differential Transformation Method (DTM) is employed to obtain the numerical/analytical solutions of the Burgers and coupled Burgers equations. We begin by showing how the differential transformation method applies to the linear and nonlinear parts of any PDE and give some examples to illustrate the sufficiency of the method for solving such nonlinear partial differential equations. We also compare it against three famous methods, namely the homotopy perturbation method, the homotopy analysis method and the variational iteration method. These results show that the technique introduced here is accurate and easy to apply.

1. Introduction

The purpose of this paper is to apply the differential transformation method to the Burgers and coupled Burgers equations. The one-dimensional nonlinear partial differential equation

$$\frac{\partial u}{\partial t} + \mu u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 \le x \le 1,\ t > 0 \tag{1.1}$$

and two-dimensional nonlinear partial differential equation

$$\begin{aligned} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \end{aligned} \qquad 0 \le x, y \le 1,\ t > 0 \tag{1.2}$$

are known as the one-dimensional Burgers equation and the two-dimensional Burgers equation, respectively. The Burgers model of turbulence is a very important fluid dynamic model and the study of this model and the theory of shock waves has been considered by many authors, both to obtain a conceptual understanding of a class of physical flows and for testing various numerical methods. The distinctive feature of Eq. (1.1) is that it is the simplest mathematical formulation of the competition between nonlinear advection and viscous diffusion. It contains the simplest forms of the nonlinear advection term $\mu\, uu_x$ and the dissipation term νu_{xx} where $\mu \in \mathbb{R}$, and $\nu = 1/R$ is an arbitrary constant (ν: kinematic viscosity; R: Reynolds number) for simulating the physical phenomena of wave motion, and thus determines the behavior of the solution. The mathematical properties of Eq. (1.1) have been studied by Cole [1]. Nonlinear phenomena play a crucial role

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in applied mathematics and physics. The importance of obtaining the exact or approximate solutions of PDEs in physics and mathematics is still a hot topic as regards seeking new methods for obtaining new exact or approximate solutions [2–5]. For that purpose, different methods have been put forward for seeking various exact solutions of multifarious physical models described using nonlinear PDEs. A well-known model was first introduced by Bateman [6], who found its steady solutions, descriptive of certain viscous flows. It was later proposed by Burgers [1] as one of a class of equations describing mathematical models of turbulence. In the context of gas dynamics, it was discussed by Hopf [7] and Cole [8]. They also illustrated independently that the Burgers equation can be solved exactly for an arbitrary initial condition. Benton and Platzman [9] have surveyed the analytical solutions of the one-dimensional Burgers equation. It can be considered as a simplified form of the Navier–Stokes equation [10] due to the form of the nonlinear convection term and the occurrence of the viscosity term.

The numerical solution of the Burgers equation is of great importance due to the application of the equation in the approximate theory of flow through a shock wave travelling in a viscous fluid [8] and in the Burgers model of turbulence [11]. It can be solved analytically for arbitrary initial conditions [7]. Finite-element methods have been applied to fluid problems: Galerkin and Petrov–Galerkin finite-element methods involving a time-dependent grid [12,13].

The $(1+1)$-coupled Burgers system was derived by Esipov [14]:

$$\begin{aligned}
&\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}-2u\frac{\partial u}{\partial x}+\alpha\frac{\partial(uv)}{\partial x}=0,\\
&\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}-2v\frac{\partial v}{\partial x}+\beta\frac{\partial(uv)}{\partial x}=0,
\end{aligned}\tag{1.3}$$

and the $(2+1)$-coupled Burgers system is

$$\begin{aligned}
&\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}-2u\frac{\partial u}{\partial x}-2v\frac{\partial u}{\partial y}=0,\\
&\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 v}{\partial y^2}-2u\frac{\partial v}{\partial x}-2v\frac{\partial v}{\partial x}=0,
\end{aligned}\tag{1.4}$$

and these are simple models of the sedimentation or the evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids, under the effect of gravity [15,16].

The differential transform method (DTM) is a semi-numerical–analytic technique that formalizes the Taylor series in a totally different manner. The DTM was first introduced by Zhou in a study concerning electrical circuits [17]. The differential transform method obtains an analytical solution in the form of a polynomial. It is different from the traditional high order Taylor series method, which requires symbolic competition of the necessary derivatives of the data functions. The Taylor series method is computationally time-consuming for large orders. With this method, it is possible to obtain highly accurate results or exact solutions for differential equations. With this technique, the given partial differential equation and related initial conditions are transformed into a recurrence equation, that finally leads to the solution of a system of algebraic equations as coefficients of a power series solution. This method is useful for obtaining exact and approximate solutions of linear and nonlinear ordinary and partial differential equations. There is no need for linearization or perturbations, and large amounts of computational work and round-off errors are avoided. It has been used to solve effectively, easily and accurately a large class of linear and nonlinear problems with approximations. It is possible to solve a system of differential equations [18–20], differential algebraic equations [21], difference equations [22], differential difference equations [23], partial differential equations [24–27], fractional differential equations [28,29], pantograph equations [30], one-dimensional Volterra integral and integro-differential equations [31,32] and matrix differential equations [33] by using this method.

The aim of this paper is to extend the differential transformation method to solve three different equation types: the one-dimensional Burgers equation, the $(1+1)$-coupled Burgers system and the $(2+1)$-coupled Burgers system. The accuracy of the numerical results will be compared with that of the analytical ones.

2. The basic idea of the differential transform method

2.1. The one-dimensional differential transform

With reference to the articles [17–33], we introduce in this section the basic definition of the one-dimensional differential transformation:

Definition 2.1. If $u(t)$ is analytic in the domain T, then it can be differentiated continuously with respect to time t,

$$\frac{\mathrm{d}^k u(t)}{\mathrm{d}t^k}=\phi(t,k),\quad \forall t\in T\tag{2.1}$$

for $t=t_i$, where $\phi(t,k)=\phi(t_i,k)$, where k belongs to the set of non-negative integers, denoted as the K domain. Therefore,

Table 1
The fundamental operations of the two-dimensional differential transform method.

Original function	Transformed function
$w(x,t)=u(x,t)\pm v(x,t)$	$W(k,h)=U(k,h)\pm V(k,h)$
$w(x,t)=c\,u(x,t)$	$W(k,h)=c\,U(k,h)$
$w(x,t)=\frac{\partial}{\partial x}u(x,t)$	$W(k,h)=(k+1)U(k+1,h)$
$w(x,t)=\frac{\partial}{\partial t}u(x,t)$	$W(k,h)=(h+1)U(k,h+1)$
$w(x,t)=\frac{\partial^{r+s}}{\partial x^r \partial t^s}u(x,t)$	$W(k,h)=\frac{(k+r)!(h+s)!}{k!h!}U(k+r,h+s)$
$w(x,t)=u(x,t)v(x,t)$	$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)V(k-r,s)$
$w(x,t)=x^m t^n$	$W(k,h)=\delta(k-m,h-n)=\begin{cases}1 & k=m,h=n\\ 0 & \text{otherwise}\end{cases}$
$w(x,t)=\frac{\partial}{\partial x}u(x,t)\frac{\partial}{\partial t}v(x,t)$	$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)(h-s+1)U(k-r+1,s)V(r,h-s+1)$

Eq. (2.1) can be written as

$$U_i(k)=\phi(t_i,k)=\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_i}\quad \forall k\in K, \tag{2.2}$$

where $U_i(k)$ is called the spectrum of $u(t)$ at $t=t_i$, in the K domain.

Definition 2.2. If $u(t)$ can be expressed as a Taylor series, then $u(t)$ can be represented as

$$u(t)=\sum_{k=0}^{\infty}\frac{(t-t_i)^k}{k!}U(k). \tag{2.3}$$

Eq. (2.3) is known as the inverse transformation of $U(k)$. If $U(k)$ is defined as

$$U(k)=M(k)\left[\frac{d^k q(t)u(t)}{dt^k}\right]_{t=t_i}, \tag{2.4}$$

where $k=0,1,2,\ldots,\infty$, then the function $u(t)$ can be described as

$$u(t)=\frac{1}{q(t)}\sum_{k=0}^{\infty}\frac{(t-t_i)^k}{k!}\frac{U(k)}{M(k)} \tag{2.5}$$

where $M(k)\neq 0$, $q(t)\neq 0$. The function $M(k)$ is called the weighting factor and $q(t)$ is regarded as a kernel corresponding to $u(t)$. If $M(k)=1$ and $q(t)=1$, then Eqs. (2.3) and (2.5) are equivalent. In this way, Eq. (2.3) can be treated as a special case of Eq. (2.5). In this paper, the transformation with $M(k)=1/k!$ and $q(t)=1$ is applied. Then Eq. (2.4) becomes

$$U(k)=\frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_i},\quad \text{where } k=0,1,\ldots,\infty. \tag{2.6}$$

Using the differential transform, a differential equation in the domain of interest can be transformed to be an algebraic equation in the K domain and $u(t)$ can be obtained as a finite-term Taylor series plus a remainder:

$$u(t)=\frac{1}{q(t)}\sum_{k=0}^{n}\frac{(t-t_0)^k}{k!}\frac{U(k)}{M(k)}+R_{n+1}(t)=\sum_{k=0}^{n}(t-t_0)^k U(k)+R_{n+1}(t). \tag{2.7}$$

In order to speed up the convergence rate and improve the accuracy of calculation, the entire domain of t needs to be split into sub-domains [9].

Remark 2.1. In this paper, the symbol $\otimes$ is used to denote the differential transform version of multiplication.

2.2. The two-dimensional differential transform

Consider a function of two variables $w(x,t)$ and suppose that it can be represented as a product of two single-variable functions, i.e., $w(x,t)=f(x)g(t)$. On the basis of the properties of the one-dimensional differential transform, function $w(x,t)$ can be represented as

$$w(x,t)=\sum_{i=0}^{\infty}F(i)x^i\sum_{j=0}^{\infty}G(j)t^j=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}W(i,j)x^i t^j \tag{2.8}$$

where $W(i,j)=F(i)G(j)$ is called the spectrum of $w(x,t)$.

The basic definitions and operations for the two-dimensional differential transform are introduced as follows:

Definition 2.3. If $w(x,t)$ is analytic and continuously differentiable with respect to time t in the domain of interest, then

$$W(k,h)=\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k\partial t^h}w(x,t)\right]_{\substack{x=x_0\\t=t_0}}, \tag{2.9}$$

where the spectrum function $W(k,h)$ is the transformed function, which is also called the T-function for short.

In this paper, (lower case) $w(x,t)$ represents the original function while (upper case) $W(k,h)$ stands for the transformed function (T-function).

The differential inverse transform of $W(k,h)$ is defined as

$$w(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)(x-x_0)^k(t-t_0)^h. \tag{2.10}$$

Combining Eqs. (2.9) and (2.10), and assuming $x_0=t_0=0$, it can be obtained that

$$w(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k\partial t^h}w(x,t)\right]_{\substack{x=0\\t=0}}x^k t^h=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}W(k,h)x^k t^h. \tag{2.11}$$

From the above definitions, it can be found that the concept of the two-dimensional differential transform is derived from the two-dimensional Taylor series expansion. With Eqs. (2.10) and (2.11), the fundamental mathematical operations performed using the two-dimensional differential transform can be readily obtained and these are listed in Table 1.

2.3. The three-dimensional differential transform

By using the same theory as for the two-dimensional differential transform, we can reach the three-dimensional case. The basic definitions for the three-dimensional differential transform are shown below.

Definition 2.4. Given a w function which has three components such as x, y, t, the three-dimensional differential transform of $w(x,y,t)$ is defined as

$$W(k,h,m)=\frac{1}{k!h!m!}\left[\frac{\partial^{k+h+m}w(x,y,t)}{\partial x^k\partial y^h\partial t^m}\right]_{\substack{x=x_0\\y=y_0\\t=t_0}}, \tag{2.12}$$

where $w(x,y,t)$ is the original function and $W(k,h,m)$ is the transformed function. Again, the transformation can be called the T-function and the lower case and upper case letters represent the original and transformed functions respectively.

Definition 2.5. The differential inverse transform of $W(k,h,m)$ is defined as

$$w(x,y,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}W(k,h,m)(x-x_0)^k(y-y_0)^h(t-t_0)^m, \tag{2.13}$$

and from Eqs. (2.12) and (2.13), it can be concluded that

$$w(x,y,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{k!h!m!}\left[\frac{\partial^{k+h+m}w(x,y,t)}{\partial x^k\partial y^h\partial t^m}\right]_{\substack{x=x_0\\y=y_0\\t=t_0}}(x-x_0)^k(y-y_0)^h(t-t_0)^m. \tag{2.14}$$

Fundamental theorems for the three-dimensional case are now given.

Theorem 2.1. *Assume that $W(k,h,m)$, $U(k,h,m)$ and $V(k,h,m)$ are the differential transforms of the functions $w(x,y,t)$, $u(x,y,t)$ and $v(x,y,t)$, respectively; then:*

(a) *If $w(x,y,t)=u(x,y,t)\pm v(x,y,t)$, then $W(k,h,m)=U(k,h,m)\pm V(k,h,m)$.*
(b) *If $w(x,y,t)=c\,u(x,y,t)$, where $c\in\mathbf{R}$, then $W(k,h,m)=c\,U(k,h,m)$.*
(c) *If $w(x,y,t)=\frac{\partial}{\partial x}u(x,y,t)$, then $W(k,h,m)=(k+1)U(k+1,h,m)$.*
(d) *If $w(x,y,t)=\frac{\partial}{\partial y}u(x,y,t)$, then $W(k,h,m)=(h+1)U(k,h+1,m)$.*
(e) *If $w(x,y,t)=\frac{\partial^{r+s+m}}{\partial x^r\partial y^s\partial t^p}u(x,y,t)$, then*

$$W(k,h,m)=\frac{(k+r)!(h+s)!(m+p)!}{k!h!m!}U(k+r,h+s,m+p).$$

(f) *If* $w(x, y, t) = u(x, y, t)v(x, y, t)$, *then*

$$W(k, h, m) = \sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m} U(r, h-s, m-p)V(k-r, s, p).$$

(g) *If* $w(x, y, t) = x^r y^s t^p$, *then*

$$W(k, h, m) = \delta(k-r, h-s, m-p) = \begin{cases} 1 & k = r, h = s, m = p \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See [25–27, and their references]. □

3. Description of the DTM and its application

In this section, two different solutions of the Burgers equation and two different solutions of the coupled Burgers system will be investigated by using the DTM.

3.1. The one-dimensional Burgers equation

Consider the one-dimensional Burgers equation (1.1),

$$\frac{\partial u}{\partial t} + \mu u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 \le x \le 1,\ t > 0 \tag{3.1}$$

subject to initial conditions (1.2):

$$u(x, 0) = f(x), \quad 0 \le x \le 1.$$

Using the operations of Table 1 and Eq. (1.1), the differential transform version of Eq. (3.1) will be

$$(h+1)U(k, h+1) + \mu\, u \otimes \left.\frac{\partial u}{\partial x}\right|_{\substack{x=k\\ t=h}} - \nu \frac{(k+2)!}{k!} U(k+2, h) = 0, \tag{3.2}$$

where $U(k, h)$ is the differential transform of $u(x, t)$, and we suppose that $x_0 = t_0 = 0$ in Definition 2.3; then from the initial conditions, we have

$$\sum_{k=0}^{\infty} U(k, 0)x^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k \tag{3.3}$$

and therefore from Eq. (3.2), for $k,\ h = 0, 1, 2, \ldots, N$, we get

$$U(k, h+1) = \frac{1}{(h+1)}\left\{\nu \frac{(k+2)!}{k!} U(k+2, h) - \mu \sum_{r=0}^{k}\sum_{s=0}^{h} (k-r+1)U(r, h-s)U(k-r+1, s)\right\}. \tag{3.4}$$

Example 3.1. Consider the following one-dimensional Burgers equation (3.1), when $\mu = 1$ [34,35]:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 \le x \le 1,\ t > 0$$

subject to the initial conditions

$$u(x, 0) = c\left[1 - \tanh\left(\frac{c}{2\nu} x\right)\right], \quad 0 \le x \le 1$$

where the parameters c and $\nu > 0$ are arbitrary constants. From Eq. (3.3), and for finite values of N, we get

$$\sum_{k=0}^{N} U(k, 0)x^k = c - \frac{c^2}{2\nu} x + \frac{c^4}{24\nu^3} x^3 - \frac{c^6}{240\nu^5} x^5 + \frac{17c^8}{40320\nu^7} x^7 - \frac{31c^{10}}{725760\nu^9} x^9 + \cdots + R_{N+1}(x), \tag{3.5}$$

and then from Eq. (3.4), for $k,\ h = 0, 1, 2, \ldots, N$, we get

$$
\begin{aligned}
U(0,1) &= 2\nu\, U(2,0) - U(0,0)U(1,0) = \frac{c^3}{2\nu},\\
U(1,1) &= 6\nu\, U(3,0) - 2U(0,0)U(2,0) - U(1,0)^2 = 0,\\
U(2,1) &= 12\nu\, U(4,0) - 3U(0,0)U(3,0) - 3U(1,0)U(2,0) = -\frac{c^5}{8\nu^3},\\
U(3,1) &= 20\nu\, U(5,0) - 4U(0,0)U(4,0) - 4U(1,0)U(3,0) - 2U(2,0)^2 = 0,\\
&\vdots\\
U(0,2) &= \frac{1}{2}\big\{2\nu\, U(2,1) - U(0,1)U(1,0) - U(0,0)U(1,1)\big\} = 0,\\
U(1,2) &= \frac{1}{2}\big\{6\nu\, U(3,1) - 2U(0,1)U(2,0) - 2U(0,0)U(2,1) - 2U(1,1)U(1,0)\big\} = \frac{c^6}{8\nu^3},\\
U(2,2) &= \frac{1}{2}\big\{12\nu\, U(4,1) - 3U(0,1)U(3,0) - 3U(0,0)U(3,1) - 3U(1,1)U(2,0) - 3U(1,0)U(2,1)\big\} = 0,\\
U(3,2) &= \frac{1}{2}\big\{20\nu\, U(5,1) - 4U(0,1)U(4,0) - 4U(0,0)U(4,1) - 4U(1,1)U(3,0) - 4U(1,0)U(3,1)\\
&\quad - 4U(2,1)U(2,0)\big\} = -\frac{c^8}{24\nu^5},\\
&\ddots
\end{aligned}
$$

In the same manner, the rest of the components were obtained by using the recursive method of (3.4).

Substituting the above quantities in Eq. (2.11), the approximation solution in the series form of Example 3.1 is

$$
u(x,t) \simeq c - \frac{c^2}{2\nu}x + \frac{c^3}{2\nu}t + \frac{c^4}{24\nu^3}x^3 - \frac{c^5}{8\nu^3}x^2\, t + \frac{c^6}{8\nu^3}x\, t^2 - \frac{c^7}{24\nu^3}t^3 - \frac{c^6}{240\nu^5}x^5 + \frac{c^7}{48\nu^5}x^4\, t + \cdots,
$$

which is the same as the Taylor expansion of the exact solutions:

$$
u(x,t) = c\left[1 - \tanh\!\left(\frac{c}{2\nu}(x - c\, t)\right)\right]
$$

and is exactly the HAM solution [34,35].

3.2. The two-dimensional Burgers equation

Consider the two-dimensional Burgers equation (1.2),

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),\\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right),
\end{aligned}
\qquad 0 \le x, y \le 1,\ t > 0 \tag{3.6}
$$

subject to initial conditions

$$
u(x,y,0) = f(x,y), \qquad v(x,y,0) = g(x,y), \quad 0 \le x, y \le 1
$$

where R is the Reynolds number.

Using the operations listed in Theorem 2.1, the differential transform of Eq. (3.6) will be

$$
\begin{aligned}
&(m+1)U(k,h,m+1) + u \otimes \frac{\partial u}{\partial x}\bigg|_{\substack{x=k\\ y=h\\ t=m}} + v \otimes \frac{\partial u}{\partial y}\bigg|_{\substack{x=k\\ y=h\\ t=m}}\\
&\quad = \frac{1}{R}\left(\frac{(k+2)!}{k!}U(k+2,h,m) + \frac{(h+2)!}{h!}U(k,h+2,m)\right),\\
&(m+1)V(k,h,m+1) + u \otimes \frac{\partial v}{\partial x}\bigg|_{\substack{x=k\\ y=h\\ t=m}} + v \otimes \frac{\partial v}{\partial y}\bigg|_{\substack{x=k\\ y=h\\ t=m}}\\
&\quad = \frac{1}{R}\left(\frac{(k+2)!}{k!}V(k+2,h,m) + \frac{(h+2)!}{h!}V(k,h+2,m)\right),
\end{aligned} \tag{3.7}
$$

where $U(k,h,m)$ and $V(k,h,m)$ are the differential transforms of $u(x,y,t)$ and $v(x,y,t)$ respectively, and using Definitions 2.4 and 2.5, and from the initial conditions, we have

$$\begin{aligned} \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h,0)x^k y^h &= \sum_{r=0}^{\infty}\frac{1}{r!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^r f(x,y)\Big|_{\substack{x=0\\y=0}},\\ \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} V(k,h,0)x^k y^h &= \sum_{r=0}^{\infty}\frac{1}{r!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^r g(x,y)\Big|_{\substack{x=0\\y=0}} \end{aligned} \tag{3.8}$$

and therefore from Eq. (3.7), for $k, h, m = 0, 1, 2, \ldots$, we get

$$\begin{aligned} U(k,h,m+1) &= \frac{1}{(m+1)}\Big\{-\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(r+1)U(k-r,s,p)U(r+1,h-s,m-p)\\ &\quad -\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(h-s+1)V(r,h-s+1,m-p)U(k-r,s,p)\\ &\quad +\frac{1}{R}\left(\frac{(k+2)!}{k!}U(k+2,h,m)+\frac{(h+2)!}{h!}U(k,h+2,m)\right)\Big\},\\ V(k,h,m+1) &= \frac{1}{(m+1)}\Big\{-\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(r+1)U(k-r,s,p)V(r+1,h-s,m-p)\\ &\quad -\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(h-s+1)V(r,h-s+1,m-p)V(k-r,s,p)\\ &\quad +\frac{1}{R}\left(\frac{(k+2)!}{k!}V(k+2,h,m)+\frac{(h+2)!}{h!}V(k,h+2,m)\right)\Big\}. \end{aligned} \tag{3.9}$$

Example 3.2. Consider the two-dimensional Burgers equation (3.6), subject to the initial conditions [36,37]

$$u(x,y,0) = \frac{3}{4} - \frac{1}{4\left(1+\mathrm{Exp}\left(\frac{R}{32}(-4x+4y)\right)\right)},$$
$$v(x,y,0) = \frac{3}{4} + \frac{1}{4\left(1+\mathrm{Exp}\left(\frac{R}{32}(-4x+4y)\right)\right)}.$$

Then according to relationships (3.8), we get

$$\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h,0)x^k y^h = \frac{5}{8} - \frac{R}{128}x + \frac{R}{128}y + \frac{R^3}{98304}x^3 - \frac{R^3}{32768}x^2y + \frac{R^3}{32768}xy^2 - \frac{R^3}{98304}y^3 + \cdots,$$
$$\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} V(k,h,0)x^k y^h = \frac{7}{8} + \frac{R}{128}x - \frac{R}{128}y - \frac{R^3}{98304}x^3 + \frac{R^3}{32768}x^2y - \frac{R^3}{32768}xy^2 + \frac{R^3}{98304}y^3 + \cdots,$$

where the parameter R is the Reynolds number. From recurrence relation (3.9), we get

$$\begin{aligned} U(0,0,1) &= -\frac{1}{R}\Big(-2U(2,0,0)-2U(0,2,0)+U(0,0,0)U(1,0,0)R+V(0,0,0)U(0,1,0)R\Big) = -\frac{1}{512}R^3,\\ V(0,0,1) &= -\frac{1}{R}\Big(-2V(2,0,0)-2V(0,2,0)+U(0,0,0)V(1,0,0)R+V(0,0,0)V(0,1,0)R\Big) = \frac{1}{512}R^3,\\ U(1,0,1) &= -\frac{1}{R}\Big(-6U(3,0,0)-2U(1,2,0)+U(1,0,0)^2R+2U(0,0,0)U(2,0,0)R+V(1,0,0)U(0,1,0)R\\ &\quad +V(0,0,0)U(1,1,0)R\Big) = 0,\\ V(1,0,1) &= -\frac{1}{R}\Big(-6V(3,0,0)-2V(1,2,0)+U(1,0,0)V(1,0,0)R+2U(0,0,0)V(2,0,0)R\\ &\quad +V(1,0,0)V(0,1,0)R+V(0,0,0)V(1,1,0)R\Big) = 0,\\ U(2,0,1) &= -\frac{1}{R}\Big(-12U(4,0,0)-2U(2,2,0)+3U(2,0,0)U(1,0,0)R+3U(0,0,0)U(3,0,0)R\\ &\quad +V(2,0,0)U(0,1,0)R+V(1,0,0)U(1,1,0)R+V(0,0,0)U(2,1,0)R\Big) = \frac{1}{131072}R^3, \end{aligned}$$

$$V(2,0,1) = -\frac{1}{R}\Big(-12V(4,0,0) - 2V(2,2,0) + U(2,0,0)V(1,0,0)R + 2U(1,0,0)V(2,0,0)R + 3U(0,0,0)V(3,0,0)R + V(2,0,0)V(0,1,0)R + V(1,0,0)V(1,1,0)R + V(0,0,0)V(2,1,0)R\Big) = -\frac{1}{131072}R^3,$$

$$\vdots$$

And so on. We can calculate other values of $U(k, h, m)$ using recurrence relation (3.9).

Substituting the above quantities in Eq. (2.14), the approximation solution in the series form of Example 3.2 will be

$$u(x,y,t) \simeq \frac{5}{8} - \frac{R}{128}x + \frac{R}{128}y - \frac{R}{512}t + \frac{R^3}{98304}x^3 - \frac{R^3}{32768}x^2y + \frac{R^3}{131072}x^2t + \frac{R^3}{32768}xy^2 - \frac{R^3}{65536}xyt + \frac{R^3}{524288}xt^2 + \frac{R^3}{131072}y^2t - \frac{R^3}{524288}yt^2 + \cdots,$$

$$v(x,y,t) \simeq \frac{7}{8} + \frac{R}{128}x - \frac{R}{128}y + \frac{R}{512}t - \frac{R^3}{98304}x^3 + \frac{R^3}{32768}x^2y - \frac{R^3}{131072}x^2t - \frac{R^3}{32768}xy^2 + \frac{R^3}{65536}xyt - \frac{R^3}{524288}xt^2 - \frac{R^3}{131072}y^2t + \frac{R^3}{524288}yt^2 + \cdots,$$

which are the same as the Taylor expansions of the exact solutions

$$u(x,y,t) = \frac{3}{4} - \frac{1}{4\Big(1 + \mathrm{Exp}\left(\frac{R}{32}(-4x+4y-t)\right)\Big)},$$

$$v(x,y,t) = \frac{3}{4} + \frac{1}{4\Big(1 + \mathrm{Exp}\left(\frac{R}{32}(-4x+4y-t)\right)\Big)},$$

and are exactly the same as the results obtained by VIM [36] and using the finite-difference method [37].

3.3. The $(1+1)$-coupled Burgers equations

Consider the $(1+1)$-coupled Burgers equations

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - 2u\frac{\partial u}{\partial x} + \alpha\frac{\partial (uv)}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - 2v\frac{\partial v}{\partial x} + \beta\frac{\partial (uv)}{\partial x} &= 0, \end{aligned} \tag{3.10}$$

with the initial conditions

$$u(x,0) = f(x), \qquad v(x,0) = g(x).$$

Using the operations of Table 1, we get the differential transforms of Eq. (3.10) as follows:

$$\begin{aligned} (h+1)U(k,h+1) - \frac{(k+2)!}{k!}U(k+2,h) - 2u \otimes \frac{\partial u}{\partial x}\bigg|_{\substack{x=k\\t=h}} + \alpha\, v \otimes \frac{\partial u}{\partial x}\bigg|_{\substack{x=k\\t=h}} + \alpha\, u \otimes \frac{\partial v}{\partial x}\bigg|_{\substack{x=k\\t=h}} &= 0, \\ (h+1)V(k,h+1) - \frac{(k+2)!}{k!}V(k+2,h) - 2v \otimes \frac{\partial v}{\partial x}\bigg|_{\substack{x=k\\t=h}} + \beta\, v \otimes \frac{\partial u}{\partial x}\Big|_{\substack{x=k\\t=h}} + \beta\, u \otimes \frac{\partial v}{\partial x}\Big|_{\substack{x=k\\t=h}} &= 0, \end{aligned} \tag{3.11}$$

where $U(k, h)$, and $V(k, h)$ are the differential transformations of $u(x, t)$, and $v(x, t)$ respectively. Suppose that $x_0 = t_0 = 0$, in Definition 2.3; then from the initial conditions, we have

$$\begin{aligned} \sum_{k=0}^{\infty} U(k,0)x^k &= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k, \\ \sum_{k=0}^{\infty} V(k,0)x^k &= \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!}x^k. \end{aligned} \tag{3.12}$$

By Eq. (3.11), we get

$$
\begin{aligned}
U(k,h+1) &= \frac{1}{(h+1)}\Big\{\frac{(k+2)!}{k!}U(k+2,h) + 2\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)U(k-r+1,s) \\
&\quad -\alpha\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)V(r,h-s)U(k-r+1,s) - \alpha\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)V(k-r+1,s)\Big\}, \\
V(k,h+1) &= \frac{1}{(h+1)}\Big\{\frac{(k+2)!}{k!}V(k+2,h) + 2\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)V(r,h-s)V(k-r+1,s) \\
&\quad -\beta\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)V(r,h-s)U(k-r+1,s) - \beta\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)V(k-r+1,s)\Big\}.
\end{aligned} \tag{3.13}
$$

Example 3.3. Consider the $(1+1)$-coupled Burgers equations [38]:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - 2u\frac{\partial u}{\partial x} + \frac{5}{2}\frac{\partial (uv)}{\partial x} = 0, \\
&\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - 2v\frac{\partial v}{\partial x} + \frac{5}{2}\frac{\partial (uv)}{\partial x} = 0,
\end{aligned} \tag{3.14}
$$

with the initial conditions

$$
u(x,0) = v(x,0) = \lambda\left(1 - \tanh\left(\frac{3}{2}\lambda x\right)\right), \tag{3.15}
$$

where λ is an arbitrary constant. By Eq. (3.12), we get

$$
\sum_{k=0}^{\infty} U(k,0)x^k = \sum_{k=0}^{\infty} V(k,0)x^k = \lambda - \frac{3\lambda^2}{2}x + \frac{9\lambda^4}{8}x^3 - \frac{81\lambda^6}{80}x^5 + \frac{4131\lambda^8}{4480}x^7 - \frac{7533\lambda^{10}}{8960}x^9 + \cdots \tag{3.16}
$$

and then from Eq. (3.13), we get

$$
\begin{aligned}
U(0,1) = V(0,1) &= \Big\{2U(2,0) + 2U(0,0)U(1,0) - \frac{5}{2}V(0,0)U(1,0) - \frac{5}{2}U(0,0)V(1,0)\Big\} = \frac{9}{2}\lambda^3, \\
U(1,1) = V(1,1) &= \big\{6U(3,0) + 4U(0,0)U(2,0) + 2U(1,0)^2 - 5V(0,0)U(2,0) \\
&\quad - 5V(1,0)U(1,0) - 5U(0,0)V(2,0)\big\} = 0, \\
U(2,1) = V(2,1) &= \Big\{12U(4,0) + 6U(0,0)U(3,0) + 6U(1,0)U(2,0) - \frac{15}{2}V(0,0)U(3,0) - \frac{15}{2}V(1,0)U(2,0) \\
&\quad - \frac{15}{2}V(2,0)U(1,0) - \frac{15}{2}U(0,0)V(3,0)\Big\} = -\frac{81}{8}\lambda^5, \\
U(3,1) = V(3,1) &= \big\{20U(5,0) + 8U(0,0)U(4,0) + 8U(1,0)U(3,0) + 4U(2,0)^2 - 10V(0,0)U(4,0) \\
&\quad - 10V(1,0)U(3,0) - 10V(2,0)U(2,0) - 10V(3,0)U(1,0) - 10U(0,0)V(4,0)\big\} = 0, \\
U(4,1) = V(4,1) &= \Big\{30U(6,0) + 10U(0,0)U(5,0) + 10U(1,0)U(4,0) + 10U(2,0)U(3,0) - \frac{25}{2}V(0,0)U(5,0) \\
&\quad - \frac{25}{2}V(1,0)U(4,0) - \frac{25}{2}V(2,0)U(3,0) - \frac{25}{2}V(3,0)U(2,0) - \frac{25}{2}V(4,0)U(1,0) \\
&\quad - \frac{25}{2}U(0,0)V(5,0)\Big\} = \frac{243}{16}\lambda^7, \\
&\;\vdots
\end{aligned}
$$

In the same manner, the rest of components can be obtained using the recurrence relation (3.13).

Substituting the quantities obtained in Eq. (2.11), the approximation solution in the series form of Example 3.3 is

$$
u(x,t) = v(x,t) \simeq \lambda - \frac{3\lambda^2}{2}x + \frac{9\lambda^3}{2}t + \frac{9\lambda^4}{8}x^3 - \frac{81\lambda^5}{8}x^2 t + \frac{243\lambda^6}{8}x t^2 - \frac{243\lambda^7}{8}t^3 - \frac{81\lambda^6}{80}x^5 + \cdots,
$$

which is the same as the Taylor expansion of the exact solutions

$$u(x,t) = v(x,t) = \lambda\left[1 - \tanh\left(\frac{3}{2}\lambda(x - 3\lambda\, t)\right)\right]$$

and is exactly the same as the results obtained by VIM [38].

3.4. *The* $(2+1)$*-coupled Burgers equations*

Example 3.4. We finally consider the $(2+1)$-coupled Burgers equations [16]:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2u\frac{\partial u}{\partial x} - 2v\frac{\partial u}{\partial y} = 0,\\ &\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial yZ^2} - 2u\frac{\partial v}{\partial x} - 2v\frac{\partial v}{\partial x} = 0. \end{aligned} \tag{3.17}$$

The exact solutions to Eq. (3.17) are given by

$$\begin{aligned} &u(x,y,t) = \lambda - \alpha\mu\ \tanh\left(-\alpha\mu\, x + (\alpha\lambda + \beta)y + 2\beta\mu\, t + \gamma\right),\\ &v(x,y,t) = \mu + (\alpha\lambda + \beta)\ \tanh\left(-\alpha\mu\, x + (\alpha\lambda + \beta)y + 2\beta\mu\, t + \gamma\right) \end{aligned} \tag{3.18}$$

where $\alpha, \beta, \gamma, \lambda$, and μ are any arbitrary parameters. For simplicity, $\alpha = 1, \beta = 1, \gamma = 1, \lambda = 1, \mu = 1$ are used for the arbitrary variables. Then, Eq. (3.18) takes the following form:

$$\begin{aligned} &u(x,y,t) = 1 - \tanh\left(-x + 2y + 2t + 1\right),\\ &v(x,y,t) = 1 + 2\ \tanh\left(-x + 2y + 2t + 1\right). \end{aligned} \tag{3.19}$$

Then according to the exact solution (3.19), the initial conditions are

$$\begin{aligned} &u(x,y,0) = 1 - \tanh\left(-x + 2y + 1\right),\\ &v(x,y,0) = 1 + 2\ \tanh\left(-x + 2y + 1\right). \end{aligned}$$

Using the operations of Table 1, the differential transforms of Eq. (3.17) are

$$\begin{aligned} &(m+1)U(k,h,m+1) - \frac{(k+2)!}{k!}U(k+2,h,m) - \frac{(h+2)!}{h!}U(k,h+2,m) - 2u\otimes\left.\frac{\partial u}{\partial x}\right|_{\substack{x=k\\y=h\\t=m}} - 2v\otimes\left.\frac{\partial u}{\partial y}\right|_{\substack{x=k\\y=h\\t=m}} = 0,\\ &(m+1)V(k,h,m+1) - \frac{(k+2)!}{k!}V(k+2,h,m) - \frac{(h+2)!}{h!}V(k,h+2,m) - 2u\otimes\left.\frac{\partial v}{\partial x}\right|_{\substack{x=k\\y=h\\t=m}} - 2v\otimes\left.\frac{\partial v}{\partial x}\right|_{\substack{x=k\\y=h\\t=m}} = 0, \end{aligned} \tag{3.20}$$

where $U(k,h,m)$, and $V(k,h,m)$ are the differential transformations of $u(x,y,t)$, and $v(x,y,t)$ respectively. Then from the initial conditions, we have

$$\begin{aligned} \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h,0)x^k y^h &= 1 + \frac{(A-1)}{(A+1)} + \frac{4Ax}{(A+1)^2} - \frac{8Ay}{(A+1)^2} - \frac{4(A-1)Ax^2}{(A+1)^3} + \frac{16(A-1)Ayx}{(A+1)^3}\\ &\quad - \frac{16(A-1)Ay^2}{(A+1)^3} + \frac{8}{3}\frac{A(A^2-4A+1)x^3}{(A+1)^4} - \frac{16A(A^2-4A+1)yx^2}{(A+1)^4} + \cdots,\\ \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} V(k,h,0)x^k y^h &= 1 - \frac{2(A-1)}{(A+1)} - \frac{8Ax}{(A+1)^2} + \frac{16Ay}{(A+1)^2} + \frac{8(A-1)Ax^2}{(A+1)^3} - \frac{32(A-1)Ayx}{(A+1)^3}\\ &\quad + \frac{32(A-1)Ay^2}{(A+1)^3} - \frac{16}{3}\frac{A(A^2-4A+1)x^3}{(A+1)^4} + \frac{32A(A^2-4A+1)yx^2}{(A+1)^4} + \cdots, \end{aligned}$$

where $A = e^{(-2)}$. By using Theorem 2.1 and for $k, h, m = 0, 1, 2, \ldots$, we can rewrite Eq. (3.20) as follows:

$$\begin{aligned}
U(k,h,m+1) \;=\; & \frac{1}{(m+1)}\Big\{2\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(r+1)U(k-r,s,p)U(r+1,h-s,m-p) \\
& +2\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(h-s+1)V(r,h-s+1,m-p)U(k-r,s,p) \\
& +(k+1)(k+2)U(k+2,h,m)+(h+1)(h+2)U(k,h+2,m)\Big\}, \\
V(k,h,m+1) \;=\; & \frac{1}{(m+1)}\Big\{2\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(r+1)U(k-r,s,p)V(r+1,h-s,m-p) \\
& +2\sum_{r=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(h-s+1)V(r,h-s+1,m-p)V(k-r,s,p) \\
& +(k+1)(k+2)V(k+2,h,m)+(h+1)(h+2)V(k,h+2,m)\Big\}
\end{aligned} \tag{3.21}$$

and then from recurrence relation (3.21), we can obtain the quantities of $U(k,h,m)$ and $V(k,h,m)$, for $k,h,m=0,1,2,\ldots$:

$$\begin{aligned}
U(0,0,1) &= 2U(2,0,0)+2U(0,2,0)+2U(0,0,0)U(1,0,0)+2V(0,0,0)U(0,1,0) = -\frac{8A}{(A+1)^2}, \\
V(0,0,1) &= 2V(2,0,0)+2V(0,2,0)+2U(0,0,0)V(1,0,0)+2V(0,0,0)V(0,1,0) = \frac{16A}{(A+1)^2}, \\
U(1,0,1) &= 6U(3,0,0)+2U(1,2,0)+2U(1,0,0)^2+4U(0,0,0)U(2,0,0)+2V(1,0,0)U(0,1,0) \\
&\quad +2V(0,0,0)U(1,1,0) = \frac{16A(A-1)}{(A+1)^3}, \\
V(1,0,1) &= 6V(3,0,0)+2V(1,2,0)+2U(1,0,0)V(1,0,0)+4U(0,0,0)V(2,0,0)+2V(1,0,0)V(0,1,0) \\
&\quad +2V(0,0,0)V(1,1,0) = -\frac{32A(A-1)}{(A+1)^3}, \\
U(2,0,1) &= 12U(4,0,0)+2U(2,2,0)+6U(2,0,0)U(1,0,0)+6U(0,0,0)U(3,0,0)+2V(2,0,0)U(0,1,0) \\
&\quad +2V(1,0,0)U(1,1,0)+2V(0,0,0)U(2,1,0) = -\frac{16A(A^2-4A+1)}{(A+1)^4}, \\
V(2,0,1) &= 12V(4,0,0)+2V(2,2,0)+2U(2,0,0)V(1,0,0)+4U(1,0,0)V(2,0,0)+6U(0,0,0)V(3,0,0) \\
&\quad +2V(2,0,0)V(0,1,0)+2V(1,0,0)V(1,1,0)+2V(0,0,0)V(2,1,0) = \frac{32A(A^2-4A+1)}{(A+1)^4}.
\end{aligned}$$

In the same manner, the rest of the components can be obtained using the recurrence relation (3.21). Substituting the quantities obtained in Eq. (2.14), the approximation solution in the series form of Example 3.4 will be

$$\begin{aligned}
u(x,y,t) \;=\; & 1+\frac{(A-1)}{(A+1)}+\frac{4Ax}{(A+1)^2}-\frac{8Ay}{(A+1)^2}-\frac{8At}{(A+1)^2}-\frac{4A(A-1)x^2}{(A+1)^3}+\frac{16A(A-1)xy}{(A+1)^3}+\frac{16A(A-1)xt}{(A+1)^3} \\
& -\frac{16A(A-1)y^2}{(A+1)^3}-\frac{32A(A-1)yt}{(A+1)^3}-\frac{16A(A-1)t^2}{(A+1)^3}+\frac{8}{3}\frac{A(A^2-4A+1)x^3}{(A+1)^4}+\cdots, \\
v(x,y,t) \;=\; & 1-\frac{2(A-1)}{(A+1)}-\frac{8Ax}{(A+1)^2}+\frac{16Ay}{(A+1)^2}+\frac{16At}{(A+1)^2}+\frac{8A(A-1)x^2}{(A+1)^3}-\frac{32A(A-1)xy}{(A+1)^3}-\frac{32A(A-1)xt}{(A+1)^3} \\
& +\frac{32A(A-1)y^2}{(A+1)^3}+\frac{64A(A-1)yt}{(A+1)^3}+\frac{32A(A-1)t^2}{(A+1)^3}-\frac{16}{3}\frac{A(A^2-4A+1)x^3}{(A+1)^4}+\cdots
\end{aligned}$$

which is the same as the Taylor expansion of the exact solutions and is exactly the HPM solution [16]:

$$\begin{aligned}
u(x,y,t) &= 1-\tanh\big(-x+2y+2t+1\big), \\
v(x,y,t) &= 1+2\,\tanh\big(-x+2y+2t+1\big).
\end{aligned}$$

4. Conclusions

In this paper, the differential transformation method was applied to the Burgers and coupled Burgers equations. The results of the test examples show that the differential transformation method results are equal to VIM, HPM and HAM results. The advantage of the differential transform method over other methods, such as VIM, HPM and HAM, is that the differential transform method is exact. Nonetheless, it is rather straightforward to apply. The present method reduces the computational difficulties of the other methods and all the calculations can be made with simple manipulations. On the other hand the results are quite reliable. Therefore, this method can be applied to many complicated linear and nonlinear PDEs and systems of PDEs and does not require linearization, discretization or perturbation. Also, as can be seen in the Examples 3.1, 3.2, 3.3 and Example 3.4, the accuracy of the series solution increases when the number of terms in the series solution is increased.

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