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Block Gauss elimination followed by a classical iterative method for the solution of linear systems

Maria Alanelli, Apostolos Hadjidimos*,[1]

Department of Mathematics, University of Crete, GR-714 09 Heraklion, Greece

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Abstract

In the last two decades many papers have appeared in which the application of an iterative method for the solution of a linear system is preceded by a step of the Gauss elimination process in the hope that this will increase the rates of convergence of the iterative method. This combination of methods has been proven successful especially when the matrix A of the system is an M-matrix. The purpose of this paper is to extend the idea of one to more Gauss elimination steps, consider other classes of matrices A, e.g., p-cyclic consistently ordered, and generalize and improve the asymptotic convergence rates of some of the methods known so far.

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1. Introduction and preliminaries

For the numerical solution of the linear system

$$Ax = b, \quad A \in \mathbb{C}^{n,n}, \quad b \in \mathbb{C}^n, \tag{1.1}$$

where A $(=(a_{ij}),\ i,j=1(1)n)$, we apply a number of Gauss elimination steps followed by an iterative method. The idea of applying one elimination step preceding an iterative method has been given by

* Corresponding author. Tel.: +30-81-393865; fax: +30-81-393881.
E-mail address: hadjidim@math.uoc.gr (A. Hadjidimos).
[1] Also at Institute of Applied and Computational Mathematics, Foundation of Research and Technology, GR-711 10 Heraklion, Greece (hadjidim@math.uoc.gr). For the academic year 2002–2003: Visiting Department of Mathematics and Statistics, University of Cyprus, CY-1678 Nicosia, Cyprus (hadjidim@ucy.ac.cy).

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Juncosa and Mulliken [12]. Based on this idea and on similar ones (see, e.g., [25,26,6]), Milaszewicz [19–21] improved the then known results. Similar works, in a different direction, by Gunawardena et al. [10], Kohno et al. [13] and recently by Li and Sun [14] and Hadjidimos et al. [10] appeared.

To perform the elimination most of the researchers used preconditioners on (1.1) and then applied a Jacobi or a Gauss–Seidel-type iterative method to the preconditioned system. Specifically, Milaszewicz [21], assuming $a_{11} = 1$, considered essentially the preconditioner

$$P_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ -a_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & 0 & \dots & 1 \end{bmatrix} \tag{1.2}$$

to eliminate the elements of the first column below the diagonal of the nonsingular M-matrix A. The preconditioner of Gunawardena et al. [8] eliminates the elements of the first upper co-diagonal of the (non)singular M-matrix A. Kohno et al. [13], Li and Sun [14] and Hadjidimos et al. [10] introduced parameters in the above preconditioners to accelerate the asymptotic convergence rates of the subsequent iterative method. In [14,10] regular, weak regular and M-splittings (see, e.g., [28]) were considered to compare the spectral radii of the various iteration matrices involved.

In all the previous works one step of Gauss elimination was applied followed by a "point" iterative method. In this work we apply more than one elimination steps followed by a "block" iterative method. Specifically, in Section 2, a block partitioning of the nonsingular M-matrix A is considered, where

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1p} \\ A_{21} & A_{22} & \dots & A_{2p} \\ \vdots & \vdots & \ddots & \\ A_{p1} & A_{p2} & \dots & A_{pp} \end{bmatrix}, \tag{1.3}$$

and $A_{ij} \in \mathbb{R}^{n_i,n_j}$, $i,j = 1(1)p$, $\sum_{i=1}^{p} n_i = n$, and $\det(A_{ii}) \neq 0$, $i = 1(1)p$. To (1.1) we apply a block preconditioner P that eliminates the first n_1 columns of A below its diagonal. In Section 3, $A \in \mathbb{C}^{n,n}$ is block p-cyclic consistently ordered and it is proved that applying P to A is equivalent to a block cyclic repartitioning from the p-cyclic to the $(p-1)$-cyclic case. So, problems that researchers like Markham et al. [18], Pierce [23], Pierce et al. [24], Eiermann et al. [4], Galanis and Hadjidimos [7] and Hadjidimos and Plemmons [11] dealt with reappear. In Section 4, the case of a singular A is discussed. Finally, in Section 5, a number of numerical examples are presented.

Two points before we conclude this introductory section:

(i) It would be interesting to introduce the idea of block elimination of this work in the preconditioner of [8]. This has already been done for the point case in [10]. Although some relative numerical examples are given and comparisons are made in Section 5 this is not done here because there are a number of unanswered theoretical questions in the point case that have to be answered first before one moves on to the block case. These examples show the superiority of our method compared to previous similar ones and also that there are cases where our method competes very well with more state-of-the-art methods like, e.g., Incomplete $LU(0)$ Factorization applied as a preconditioning technique to GMRES method.

(ii) It is understood that the preconditioners of this work can be useful as preconditioners with Krylov subspace methods, or as smoothers for multigrid and multilevel methods or even for providing the conceptual framework for the analysis of the domain decomposition methods. Among others, a real numerical experiment is given in Section 5 comparing our method against the Incomplete LU Factorization one.

2. Nonsingular M-matrices

2.1. Basic theory

Consider the partitioning (1.3) for the nonsingular M-matrix A. It is known that the diagonal blocks $A_{ii} \in \mathbb{R}^{n_i,n_i}$, $i = 1(1)p$, of A are nonsingular M-matrices while the off-diagonal blocks $A_{ij} \in \mathbb{R}^{n_i,n_j}$, $i \neq j = 1(1)p$, are nonpositive matrices ($A_{ij} \leqslant 0$, $i \neq j = 1(1)p$) (see [3]). (Note: It is reminded that a matrix $A \in \mathbb{R}^{n,n}$ is a Z-matrix if all its off-diagonal elements are nonpositive. A Z-matrix A is an M-matrix if $A = sI - B$, where $s \geqslant \rho(B)$, $s > 0$ and $B \geqslant 0$ (see, e.g., [3]), with $\rho(\cdot)$ denoting spectral radius.) In [3], it is said that in the triangular decomposition of a nonsingular M-matrix A $(=LU)$, L, U are lower and upper triangular matrices, have positive diagonal elements and are nonsingular M-matrices. We may assume that L has unit diagonal elements.

For our results the following lemma attributed to Fan [5], for nonsingular M-matrices, and to Funderlic and Plemmons [6], for singular ones, is used.

Lemma 2.1. *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$, $n \geqslant 2$, be a nonsingular M-matrix, and let*

$$L_1^{-1} = \begin{bmatrix} 1 & 0 & & \\ \dfrac{-a_{21}}{a_{11}} & 1 & & \\ \vdots & \vdots & \ddots & \\ \dfrac{-a_{n1}}{a_{11}} & 0 & \dots & 1 \end{bmatrix}.$$

Then, the matrices $\tilde{A} = L_1^{-1}A$ and $\tilde{A}_1$, obtained from $\tilde{A}$ by deleting its first row and column, are nonsingular M-matrices. (Note: *If A is irreducible then so is $\tilde{A}_1$ and if, in addition, A is singular then so are $\tilde{A}$ and $\tilde{A}_1$.*)

Based on Lemma 2.1 we prove our first result.

Theorem 2.1. *Let $A \in \mathbb{R}^{n,n}$ be a nonsingular M-matrix partitioned as in* (1.3). *Then n_1 successive applications of the Gauss elimination process on A are equivalent to premultiplying A by the* (*preconditioning*) *matrix*

$$P = \begin{bmatrix} L_{11}^{-1} & O_{12} & \dots & O_{1p} \\ -A_{21}A_{11}^{-1} & I_{22} & \dots & O_{2p} \\ \vdots & \vdots & \ddots & \\ -A_{p1}A_{11}^{-1} & O_{p2} & \dots & I_{pp} \end{bmatrix} = Q + S, \tag{2.1}$$

$$Q = \mathrm{diag}(L_{11}^{-1}, I_{22}, \dots, I_{pp}) \geqslant 0, \quad I_{ii} \in \mathbb{R}^{n_i,n_i}, \quad i = 2(1)p, \tag{2.2}$$

$$S = \begin{bmatrix} O_{11} & O_{12} & \dots & O_{1p} \\ -A_{21}A_{11}^{-1} & O_{22} & \dots & O_{2p} \\ \vdots & \vdots & \ddots & \\ -A_{p1}A_{11}^{-1} & O_{p2} & \dots & O_{pp} \end{bmatrix} \geqslant 0, \tag{2.3}$$

with L_{11} *being the lower triangular matrix in the LU triangular decomposition of* A_{11}. *Moreover,* $\tilde{A} = PA$ *and the matrix* $\tilde{A}_1$, *obtained from* $\tilde{A}$ *by deleting its first* n_1 *rows and columns, are also nonsingular* M*-matrices.* (Note : *If* A *is irreducible then so is* $\tilde{A}_1$ *while if* A *is, in addition, singular then so are* $\tilde{A}$ *and* $\tilde{A}_1$.)

Proof. We apply Gauss elimination to the first n_1 columns of A. By Lemma 2.1 each elimination step, $k = 1(1)n_1$, yields a matrix $\tilde{A}^{(k)}$ ($\tilde{A}^{(0)} = A$) which is a nonsingular M-matrix and whose the bottom right corner submatrix $\tilde{A}_{n_1+1-k} \in \mathbb{R}^{n-k,n-k}$, $k = 1(1)n_1$, is also a nonsingular M-matrix. So, $\tilde{A} := \tilde{A}^{(n_1)}$ and $\tilde{A}_1$ are nonsingular M-matrices. (Note: By the same lemma, if A is irreducible then so will be $\tilde{A}_1$ and if A is, in addition, singular so will be $\tilde{A}$ and $\tilde{A}_1$.) To express the above process in matrix form, note that in the kth elimination step, $k = 1(1)n_1$, we multiply $\tilde{A}^{(k-1)}$ on the left by a lower triangular matrix with units on the diagonal and only nonzero elements in its kth column. The product P of all these n_1 matrices will be a lower triangular matrix with units on the diagonal and only nonzero elements on the first n_1 columns. So, P, in a form consistent with A, will be

$$P = \begin{bmatrix} P_{11} & O_{12} & \dots & O_{1p} \\ P_{21} & I_{22} & \dots & O_{2p} \\ \vdots & \vdots & \ddots & \\ P_{p1} & O_{p2} & \dots & I_{pp} \end{bmatrix}, \tag{2.4}$$

where I_{jj}, $j = 2(1)p$, is the unit matrix of order n_j. To determine the block elements of P, we use $PA = \tilde{A}$, and note that $\tilde{A}_{ij} = P_{i1}A_{1j} + A_{ij}$, $i,j = 1(1)p$. Since $\tilde{A}_{i1} = O_{i1}$, $i = 2(1)p$, then $P_{i1} = -A_{i1}A_{11}^{-1}$, $i=2(1)p$. Observe that the first n_1 diagonal elements of $\tilde{A}$ are the pivots of the elimination process. Since, after the elimination, $\tilde{A}_{11}$ has zeros below its diagonal we conclude that $\tilde{A}_{11}$ is the matrix U_{11} yielded after the elimination is applied to A_{11}. Thus, if $L_{11}U_{11}$ is the LU factorization of A_{11}, $P_{11} = L_{11}^{-1}$ and the proof is complete. □

Note: The nonnegativity of Q and S in (2.2) and (2.3) is based on the fact that the inverse of the nonsingular M-matrices L_{11} and A_{11} are nonnegative matrices.

Corollary 2.1. *The application of* P, *of Theorem* (2.1), *to* A *of* (1.3) *results the matrix* $\tilde{A}$ *whose block elements are given by the following expressions*:

$$\tilde{A}_{ij} = \begin{cases} L_{11}^{-1}A_{1j}, & j = 1(1)p, \\ O_{i1}, & i = 2(1)p, \\ A_{ij} - A_{i1}A_{11}^{-1}A_{1j}, & i,j = 2(1)p. \end{cases} \tag{2.5}$$

Using the preconditioner P we premultiply system (1.1) to obtain equivalently

$$\tilde{A}x = \tilde{b}\ (=Pb), \tag{2.6}$$

where $x^{\mathrm{T}} = [x_1^{\mathrm{T}}\ x_2^{\mathrm{T}}\ \dots\ x_p^{\mathrm{T}}]$, $\tilde{b}^{\mathrm{T}} = [\tilde{b}_1^{\mathrm{T}}\ \tilde{b}_2^{\mathrm{T}}\ \dots\ \tilde{b}_p^{\mathrm{T}}]$, $x_i, \tilde{b}_i \in \mathbb{R}^{n_i}$, $i = 1(1)p$, and $\tilde{b}_1 = L_{11}^{-1}b_1$, $\tilde{b}_i = b_i - A_{i1}A_{11}^{-1}b_1$, $i = 2(1)p$. For the solution of (2.6) we consider then a classical block iterative method applied to (2.6) or, equivalently, applied to

$$\tilde{A}_1[x_2^{\mathrm{T}}\ x_3^{\mathrm{T}}\ \dots\ x_p^{\mathrm{T}}]^{\mathrm{T}} = [\tilde{b}_2^{\mathrm{T}}\ \tilde{b}_3^{\mathrm{T}}\ \dots\ \tilde{b}_p^{\mathrm{T}}]^{\mathrm{T}}, \tag{2.7}$$

followed by a back substitution applied to

$$U_{11}x_1 = \tilde{b}_1. \tag{2.8}$$

2.2. Jacobi and Gauss–Seidel-type iterative methods

Let

$$A = D - L - U, \tag{2.9}$$

$$D = \mathrm{diag}(A_{11}, A_{22}, \dots, A_{pp}),$$

$$L = \begin{bmatrix} O_{11} & O_{12} & \dots & O_{1p} \\ -A_{21} & O_{22} & \dots & O_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{p1} & -A_{p2} & \dots & O_{pp} \end{bmatrix}, \quad U = \begin{bmatrix} O_{11} & -A_{12} & \dots & -A_{1p} \\ O_{21} & O_{22} & \dots & -A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ O_{p1} & O_{p2} & \dots & O_{pp} \end{bmatrix}.$$

To solve (2.6) using a classical iterative method we consider various splittings of $\tilde{A}$. For this we define the following matrices in a way analogous to the point case in [10]:

$$SU = \hat{L} + \hat{D} + \hat{U}, \tag{2.10}$$

where

$$\hat{D} = \mathrm{diag}(O_{11}, A_{21}A_{11}^{-1}A_{12}, \dots, A_{p1}A_{11}^{-1}A_{1p})\ (\geqslant 0), \tag{2.11}$$

$$\hat{L} = \begin{bmatrix} O_{11} & O_{12} & \dots & O_{1p} \\ O_{21} & O_{22} & \dots & O_{2p} \\ O_{31} & A_{31}A_{11}^{-1}A_{12} & \dots & O_{3p} \\ \vdots & \vdots & \ddots & \vdots \\ O_{p1} & A_{p1}A_{11}^{-1}A_{12} & \dots & O_{pp} \end{bmatrix} \quad (\geqslant 0), \tag{2.12}$$

$$\hat{U} = \begin{bmatrix} O_{11} & O_{12} & O_{13} & \dots & O_{1p} \\ O_{21} & O_{22} & A_{21}A_{11}^{-1}A_{13} & \dots & A_{21}A_{11}^{-1}A_{1p} \\ O_{31} & O_{32} & O_{33} & \dots & A_{31}A_{11}^{-1}A_{1p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{p1} & O_{p2} & O_{p3} & \dots & O_{pp} \end{bmatrix} \quad (\geqslant 0). \tag{2.13}$$

Having in mind (2.10) and (2.9), we consider the following splittings of $\tilde{A}$:

$$\tilde{A} = (Q+S)(D-L-U) = \begin{cases} QD - (PL - SD + \hat{L} + \hat{D} + QU + \hat{U}), \\ (QD - \hat{D}) - (PL - SD + \hat{L} + QU + \hat{U}). \end{cases} \tag{2.14}$$

The block Jacobi and Gauss–Seidel as well as the block Jacobi and Gauss–Seidel-type iteration matrices associated with the two splittings in (2.14) are:

$$B = D^{-1}(L+U) \quad (\text{for } A), \tag{2.15}$$

$$B' = (QD)^{-1}(PL - SD + \hat{L} + \hat{D} + QU + \hat{U}), \tag{2.16}$$

$$B'' = (QD - \hat{D})^{-1}(PL - SD + \hat{L} + QU + \hat{U}), \tag{2.17}$$

$$H = (D-L)^{-1}U \quad (\text{for } A), \tag{2.18}$$

$$H' = (P(D-L) - \hat{L})^{-1}(\hat{D} + QU + \hat{U}), \tag{2.19}$$

$$H'' = (P(D-L) - \hat{L} - \hat{D})^{-1}(QU + \hat{U}). \tag{2.20}$$

Theorem 2.2. *Under the notation and the definitions so far, suppose that A in* (1.3) *is a nonsingular M-matrix and let $\rho(B) > 0$. Let B_1', B''_1, H_1, H_1', H''_1 denote the $(n-n_1)\times(n-n_1)$ bottom right corner submatrices of B', B'', H, H', H'', respectively. Then the following relationships hold*:

$$\rho(B_1'') \equiv \rho(B'') \leqslant \rho(B') \equiv \rho(B_1') < 1, \tag{2.21}$$

$$\rho(H_1'') \equiv \rho(H'') \leqslant \rho(H') \equiv \rho(H_1') \leqslant \rho(H) \equiv \rho(H_1) < 1. \tag{2.22}$$

Also, there exists a vector $y \in \mathbb{R}^n$, *with* $y \geqslant 0$, *such that*

$$B'y \leqslant By. \tag{2.23}$$

Moreover, the spectral radii of the iteration matrices involved satisfy the relationships below:

$$\rho(H'') \leqslant \rho(B''), \quad \rho(H') \leqslant \rho(B'), \quad 0 < \rho(H) < \rho(B) < 1. \tag{2.24}$$

If, in addition, A *is irreducible, then all the inequalities in* (2.21)–(2.24) *are strict. Also, in* (2.23), $y > 0$, *implying that*

$$\rho(B_1') \;\equiv\; \rho(B') < \rho(B). \tag{2.25}$$

Proof. By Theorem 2.1, $\tilde{A}$ and $\tilde{A}_1$ are M-matrices and if A is irreducible then so is $\tilde{A}_1$. Based on properties of M-matrices [28,3] and on the fact that Q, S, $\hat{L}$, $\hat{U} \geqslant 0$, we conclude that D, QD, $QD-\hat{D}$, $D-L$, $P(D-L)-\hat{L}$ and $P(D-L)-\hat{L}-\hat{D}$ are nonsingular M-matrices. Also, the second matrix factors in (2.15)–(2.20) are nonnegative. Therefore, the splittings from which the iterative matrices B, B', B'', H, H' and H'' are produced are M-splittings [27] hence all these matrices are convergent. From this point onwards the proof of the present theorem duplicates that in [10], where all the parameters α_i, $i=2(1)n$, in it, are equal to 1. The difference is that instead of "point" we deal with "block" iteration matrices. So, as in [25], use of the Perron–Frobenius theory for nonnegative matrices [28] and of regular splittings, weak regular splittings [3] and M-splittings [27] is made. The complete proof can be found in [1]; it is very long and so is not possible to give here. □

The results of the "block" case of Theorem 2.2 can be compared with the ones of the corresponding "point" case in [21]. In the statement below we show that ours have better asymptotic convergence rates.

Theorem 2.3. *Under the notation and the definitions used in Theorem* 2.2, *suppose that* A *is a nonsingular* M*-matrix. Let* $B'^{(k)}$, $B''^{(k)}$, $H'^{(k)}$ *and* $H''^{(k)}$, $k=1(1)n_1$, *denote the "point" iteration matrices* (*Jacobi and Gauss–Seidel type*) *associated with the matrix* $\tilde{A}^{(k)}$ ($\tilde{A}^{(0)}=A$) *of Theorems* 2.1 *and* 2.2 *after the kth elimination step* $k=1(1)n_1$. *Let also* $B^{(0)}$, $H^{(0)}$ *be the point Jacobi and Gauss–Seidel iteration matrices associated with* A. *Then, there will hold*

$$\rho(H'') \leqslant \rho(H''^{(n_1)}) \leqslant \rho(H''^{(1)}) \leqslant \rho(H^{(0)}) \quad (<1). \tag{2.26}$$

If, in addition, A *is irreducible, then there will also hold*

$$\rho(B'') < \rho(B''^{(n_1)}) < \rho(B''^{(1)}) < \rho(B^{(0)}) \quad (<1), \tag{2.27}$$

and all the inequalities in (2.26) *will be strict.*

Proof. Let us apply the preconditioners P_1, in (1.2), and P, in (2.1), to (1.1). The application of P_1 is equivalent to the first Gauss elimination step of Lemma 2.1. So, according to the notation of Theorem 2.1, $P_1A \equiv \tilde{A}^{(1)}$. For the corresponding matrices $B'^{(1)}$, $B''^{(1)}$, $H'^{(1)}$ and $H''^{(1)}$ (the point Jacobi and Gauss–Seidel-type matrices) relationships analogous to those in (2.21)–(2.25) hold

[21,10]. Specifically, we note that $\rho(H''^{(1)}) \leqslant \rho(H^{(0)}) < 1$ and that if A is irreducible the leftmost inequality will be strict and there will also hold that $\rho(B''^{(1)}) < \rho(B^{(0)}) < 1$. So, by induction, after the kth elimination step, $\tilde{A}^{(k)}$ will have point Jacobi and Gauss–Seidel iteration matrices $B''^{(k)}$ and $H''^{(k)}$, $k = 1(1)n_1$, whose spectral radii will be connected with those of the $(k-1)$st step in the same way the corresponding spectral radii after the first elimination step are connected with those of the initial point Jacobi, $B^{(0)}$, and Gauss–Seidel, $H^{(0)}$, iteration matrices. Therefore, we have shown that

$$\rho(H''^{(n_1)}) \leqslant \rho(H''^{(1)}) \quad (< 1). \tag{2.28}$$

If A is irreducible, inequality (2.28) is strict and the following relationships will also hold:

$$\rho(B''^{(n_1)}) < \rho(B''^{(1)}) \quad (< 1). \tag{2.29}$$

Observe that after the n_1th elimination step we apply "point" Jacobi and Gauss–Seidel iterative methods to $\tilde{A}^{(n_1)}$, which is nothing but the matrix $\tilde{A}$ of Theorem 2.1. If we apply the corresponding "block" iterative methods to $\tilde{A}$, as in Theorem 2.2, then because $\tilde{A}$ is a nonsingular M-matrix, all four "point" and "block" iterative methods correspond to M-splittings of the type $M - N$. It is checked that in these splittings the N matrices of the "point" Jacobi and Gauss–Seidel methods are greater than or equal to the corresponding ones of the "block" methods. Hence the "block" methods converge asymptotically at least as fast as the "point" ones. This convergence will be strictly faster if A is irreducible, which gives us the leftmost inequalities in (2.26) and (2.27). This and the results in (2.28) and (2.29) conclude the proof. □

2.3. SOR-type iterative methods

Based on (2.9)–(2.11) we consider the following SOR-type splittings for A and $\tilde{A}$:

$$\tilde{A} = \begin{cases} \frac{1}{\omega}P(D-\omega L) - \frac{1}{\omega}P((1-\omega)D+\omega U), \\ \frac{1}{\omega}(QD-\omega(PL-SD+\hat{L})) - \frac{1}{\omega}((1-\omega)QD+\omega(\hat{D}+QU+\hat{U})), \\ \frac{1}{\omega}((QD-\hat{D})-\omega(PL-SD+\hat{L})) - \frac{1}{\omega}((1-\omega)(QD-\hat{D})+\omega(QU+\hat{U})), \end{cases} \tag{2.30}$$

where $\hat{D}$, $\hat{L}$, $\hat{U}$ are given in (2.11), (2.12), (2.13), respectively. In view of (2.30) the block SOR and SOR-type iteration matrices associated with A and $\tilde{A}$ are

$$\begin{aligned} \mathscr{L}_\omega &= (D-\omega L)^{-1}((1-\omega)D+\omega U) \quad (\text{for } A \text{ and } \tilde{A}), \\ \mathscr{L}'_\omega &= (QD-\omega(PL-SD+\hat{L}))^{-1}((1-\omega)QD+\omega(\hat{D}+QU+\hat{U})), \\ \mathscr{L}''_\omega &= ((QD-\hat{D})-\omega(PL-SD+\hat{L}))^{-1}((1-\omega)(QD-\hat{D})+\omega(QU+\hat{U})). \end{aligned} \tag{2.31}$$

Below we give a statement due to Milaszewicz [19,21] and part of Theorem 3.5 of Marek and Szyld [15] which will be used in the proof of our main statement of this section.

Lemma 2.2. *Let* $V, T \in \mathbb{R}^{n,n}$, $V, T \geqslant 0$, *for which* $\rho(V) < \rho(V + T)$ *holds. Then,* $\rho(V + tT)$ *strictly increases with* $t \in [0, \infty)$ *and is unbounded. Moreover, if* $\rho(V) < 1$ *there exists a unique* $t = t_1 > 0$ *such that* $\rho(V + t_1 T) = 1$. *Also* $\rho((I - V)^{-1}T) = 1/t_1$.

Corollary 2.2. *If* $V + T$ *is irreducible Lemma* 2.2 *holds by the Perron–Frobenius theory of nonnegative matrices.*

Lemma 2.3. *Let* $A^{-1} \geqslant 0$. *Let* $A = M_1 - N_1 = M_2 - N_2$ *be two weak splittings with* $T_1 = M_1^{-1}N_1$, $T_2 = M_2^{-1}N_2$ *having property "d", and* $\rho(T_1) < 1$, $\rho(T_2) < 1$. *Let* $z > 0$ *such that* $T_2 z = \rho(T_2) z$. *If* $N_2 z \geqslant N_1 z$ *then* $\rho(T_1) \leqslant \rho(T_2)$.

Based on the splittings (2.30) and the matrices in (2.31) we will prove the statement below.

Theorem 2.4. *Under the assumptions of Theorem* 2.2, *for the block SOR and SOR-type matrices defined in* (2.31) *and* $\forall \omega \in (0, 1]$ *there hold*:

$$(\mathrm{a})\ \rho(\mathscr{L}_\omega) < 1, \quad (\mathrm{b})\ \rho(\mathscr{L}'_\omega) < 1, \quad (\mathrm{c})\ \rho(\mathscr{L}''_\omega) < 1, \tag{2.32}$$

and

$$\rho(\mathscr{L}''_\omega) \leqslant \rho(\mathscr{L}'_\omega). \tag{2.33}$$

Also, for ω_1, ω_2, *such that* $0 < \omega_1 < \omega_2 \leqslant 1$, *there hold*:

$$(\mathrm{a})\ \rho(\mathscr{L}_{\omega_2}) \leqslant \rho(\mathscr{L}_{\omega_1}), \quad (\mathrm{b})\ \rho(\mathscr{L}'_{\omega_2}) \leqslant \rho(\mathscr{L}'_{\omega_1}), \quad (\mathrm{c})\ \rho(\mathscr{L}''_{\omega_2}) \leqslant \rho(\mathscr{L}''_{\omega_1}). \tag{2.34}$$

Moreover, there exists a vector $z \in \mathbb{R}^n$, $z \geqslant 0$, *such that*

$$\mathscr{L}'_\omega z \leqslant \mathscr{L}_\omega z, \quad 0 < \omega \leqslant 1. \tag{2.35}$$

If, in addition, A *is irreducible and* $\mathscr{L}'_{\omega,1}$, $\mathscr{L}''_{\omega,1}$ *denote the SOR-type matrices corresponding to the last two splittings in* (2.30) *and are associated with the matrix* $\tilde{A}_1$ *of Theorem* 2.1, *then* $\tilde{A}_1$ *is irreducible and the corresponding relationships in* (2.33)–(2.34) *are strict. If* $B = D^{-1}(L + U)$ *is irreducible the vector* z *in* (2.35) *is positive and it is also implied that*

$$\rho(\mathscr{L}'_\omega) \leqslant \rho(\mathscr{L}_\omega), \quad 0 < \omega \leqslant 1. \tag{2.36}$$

(Note: *For* $\omega = 1$ (*Gauss–Seidel case*) *some of the above assertions are proved in Theorem* 2.2 *and will not be proved here although continuity arguments can cover this case as well.*)

Proof. (2.32): In view of the assumptions, the nonsingular M-matrix character of A and $\tilde{A}$ and the fact $0 < \omega \leqslant 1$ we observe the following:

(a) The iteration matrix $\mathscr{L}_\omega = (D - \omega L)^{-1}((1 - \omega)D + \omega U)$ is derived from the first splitting in (2.30) and is also derived from the splitting $1/\omega(D - \omega L) - 1/\omega((1 - \omega)D + \omega U)$. In the latter, the matrix $D - \omega L$ is a nonsingular M-matrix because D is a block nonsingular M-matrix, the block strictly lower triangular matrix $-\omega L$ is nonpositive and $1/\omega > 0$. Also $1/\omega((1 - \omega)I + \omega D^{-1}U) \geqslant 0$. Therefore, the splitting from which $\mathscr{L}_\omega$ is obtained is an M-splitting and thus it is convergent implying (a) of (2.32).

(b) In the same way it can be proved that the splitting $M' - N'$ for $\mathscr{L}'_\omega$ is an M-splitting and therefore convergent. Indeed, $M' = 1/\omega(QD)^{-1}(M_1 - \omega\hat{L})$, where

$$M_1 = QD - \omega(PL - SD) = \begin{bmatrix} U_{11} & O_{12} & O_{13} & \dots & O_{1p} \\ O_{21} & A_{22} & O_{23} & \dots & O_{2p} \\ O_{31} & \omega A_{32} & A_{33} & \dots & O_{3p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{p1} & \omega A_{p2} & \omega A_{p3} & \dots & A_{pp} \end{bmatrix}.$$

M_1 is a block lower triangular Z-matrix whose diagonal blocks are nonsingular M-matrices and so it is a nonsingular M-matrix itself since its inverse is nonnegative. For the same reason $M_1 - \omega\hat{L}$ is a nonsingular M-matrix. Since $QD = \mathrm{diag}(U_{11}, A_{22}, \dots, A_{pp})$ it can be checked by direct calculation that M' is a nonsingular M-matrix. On the other hand, N' is nonnegative and so the splitting yielding $\mathscr{L}'_\omega$ is an M-splitting.

(c) The proof goes along the same lines as in the previous one except that $QD - \hat{D}$ and $QU + \hat{U}$ play the roles of QD and $QU + \hat{D} + \hat{U}$, respectively.

(2.33): The last two splittings for $\tilde{A}$ are M-splittings and as can be checked from the second splitting $M'' - N''$ it is $M'' = (QD - \hat{D})^{-1} = (I - (QD)^{-1}\hat{D})^{-1}(QD)^{-1} = (I + (QD)^{-1}\hat{D} + \cdots + ((QD)^{-1}\hat{D})^{p-1})(QD)^{-1} \geqslant (QD)^{-1} = M'$ and according to [29], (2.33) holds true.

(2.34): As was seen all three splittings from which the three SOR-type matrices are produced are M-splittings of the form $M_{\omega_i} - N_{\omega_i}$, $i = 1, 2$. It is checked that in view of $0 < \omega_1 < \omega_2 \leqslant 1$ for each one of them there holds $N_{\omega_2} \leqslant N_{\omega_1}$. Consequently, (2.34) are valid.

(2.35): Use of Lemma 2.2 will be made. Let $V = \omega D^{-1}L \geqslant 0$ and $T = (1-\omega)I + \omega D^{-1}U \geqslant 0$. Since $\rho(V) = 0$ and $\rho(V+T) = 1 - \omega + \omega\rho(B) > 0$, because $0 < \omega \leqslant 1$ and $\rho(B) > 0$, the assumptions of the lemma are satisfied. Therefore, there exists a $t_1 > 0$ such that $\rho(V + t_1 T) = 1$. Note that $t_1 > 1$ because $0 < \rho(B) < 1$, $0 < \rho(V+T) = 1 - \omega(1 - \rho(B)) < 1$. Since $V + t_1 T \geqslant 0$ there exists a vector $z \geqslant 0$ (eigenvector) such that

$$(\omega D^{-1}L + t_1((1-\omega)I + \omega D^{-1}U))z = z, \tag{2.37}$$

from which

$$(\mathscr{L}_\omega z =)\ (I - \omega D^{-1}L)^{-1}((1-\omega)I + \omega D^{-1}U)z = \frac{1}{t_1}z. \tag{2.38}$$

From (2.37) we can readily obtain

$$(\omega PL + t_1((1-\omega)PD + \omega PU))z = PDz$$

or

$$(\omega PL + t_1(1-\omega)QD + t_1(1-\omega)SD + \omega t_1 QU + \omega t_1 SU)z = (QD + SD)z$$

or equivalently after some manipulation

$$\begin{aligned}(QD - \omega(PL - SD + \hat{L}))z \\ &= t_1((1-\omega)QD + \omega(QU + \hat{D} + \hat{U}))z + (t_1 - 1)((1-\omega)SD + \omega\hat{L})z \\ &\geqslant t_1((1-\omega)QD + \omega(QU + \hat{D} + \hat{U}))z, \end{aligned} \tag{2.39}$$

Because $(t_1 - 1)((1-\omega)SD + \omega\hat{L})z \geqslant 0$. Combining (2.38) and (2.39), (2.35) is proved.

If A is irreducible then so will be $\tilde{A}_1$. Therefore, $\tilde{A}_1^{-1} > 0$ and the M-splittings, from which $\mathscr{L}'_\omega$, $\mathscr{L}'_{\omega_1}$, $\mathscr{L}'_{\omega_2}$ and $\mathscr{L}''_\omega$, $\mathscr{L}''_{\omega_1}$, $\mathscr{L}''_{\omega_2}$ are yielded, give strict inequalities in (2.33) and (2.34).

(2.36): If B ($\geqslant 0$) is irreducible so will be $\mathscr{L}_\omega$, $\omega \in (0, 1)$, because after some algebra it is

$$\mathscr{L}_\omega = (1-\omega)I + (1-\omega)\omega D^{-1}L + \omega D^{-1}U + \text{nonnegative terms}$$

$$= (1-\omega)I + (1-\omega)\omega D^{-1}(L+U) + \omega^2 D^{-1}U + \text{nonnegative terms} \geqslant 0, \tag{2.40}$$

and the rightmost matrix expression is irreducible since $B = D^{-1}(L+U)$ is. Hence the eigenvector w corresponding to the spectral radius of $\mathscr{L}_\omega$ will be positive. The first splitting in (2.30) is a weak (nonnegative) convergent one for the nonsingular M-matrix $\tilde{A}$ and at the same time it is an M-splitting for the matrix A. Let $\tilde{A} = M_A - N_A = M_{\tilde{A}} - N_{\tilde{A}}$ denote the two splittings that give the iteration matrices $\mathscr{L}_\omega$ and $\mathscr{L}'_\omega$. We have that $N_A - N_{\tilde{A}} = 1/\omega(P(1-\omega)D + \omega PU) - 1/\omega((1-\omega)QD + \omega(\hat{D} + QU + \hat{U})) = 1/\omega((1-\omega)SD + \omega\hat{L}) \geqslant 0$ and therefore $(N_A - N_{\tilde{A}})w \geqslant 0$. So, according to Lemma 2.3, (2.36) is valid. □

3. Nonsingular *p*-cyclic consistently ordered matrices

We consider the nonsingular matrix $A \in \mathbb{C}^{n,n}$ of the special block form

$$A = \begin{bmatrix} A_{11} & O_{12} & O_{13} & \dots & A_{1p} \\ A_{21} & A_{22} & O_{23} & \dots & O_{2p} \\ O_{31} & A_{32} & A_{33} & \dots & O_{3p} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O_{p1} & O_{p2} & \dots & A_{p,p-1} & A_{pp} \end{bmatrix}, \tag{3.1}$$

with $p \geqslant 3$ and A_{ii}, $i = 1(1)p$, nonsingular matrices. The matrix A is block p-cyclic consistently ordered [28,30] and its block Jacobi matrix will be

$$B = \begin{bmatrix} O_{11} & O_{12} & O_{13} & \dots & -A_{11}^{-1}A_{1p} \\ -A_{22}^{-1}A_{21} & O_{22} & O_{23} & \dots & O_{2p} \\ O_{31} & -A_{33}^{-1}A_{32} & O_{33} & \dots & O_{3p} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O_{p1} & O_{p2} & \dots & -A_{pp}^{-1}A_{p,p-1} & O_{pp} \end{bmatrix}. \tag{3.2}$$

The matrix B is then weakly cyclic of index p and consistently ordered [28]. Applying the preconditioner P of (2.1) we obtain according to (2.5) that

$$\tilde{A} = \begin{bmatrix} U_{11} & O_{12} & O_{13} & \dots & L_{11}^{-1}A_{1p} \\ O_{21} & A_{22} & O_{23} & \dots & -A_{21}A_{11}^{-1}A_{1p} \\ O_{31} & A_{32} & A_{33} & \dots & O_{3p} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O_{p1} & O_{p2} & \dots & A_{p,p-1} & A_{pp} \end{bmatrix}.$$

The block Jacobi matrix B'' associated with $\tilde{A}$ (see (2.17)) will be

$$B'' = \begin{bmatrix} O_{11} & O_{12} & O_{13} & \dots & -A_{11}^{-1}A_{1p} \\ O_{21} & O_{22} & O_{23} & \dots & A_{22}^{-1}A_{21}A_{11}^{-1}A_{1p} \\ O_{31} & -A_{33}^{-1}A_{32} & O_{33} & \dots & O_{3p} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ O_{p1} & O_{p2} & \dots & -A_{pp}^{-1}A_{p,p-1} & O_{pp} \end{bmatrix}. \tag{3.3}$$

The matrix B''_1, yielded from B'' as $\tilde{A}_1$ is yielded from $\tilde{A}$ will be block weakly cyclic of index $p-1$ and consistently ordered and $\tilde{A}_1$ will be block $(p-1)$-cyclic consistently ordered.

Direct computations show that the powers B^p and ${B''}_1^{p-1}$ are diagonal matrices. Using the fact that *if* $E \in \mathbb{C}^{n,m}$, $F \in \mathbb{C}^{m,n}$ *and* $\lambda \in \mathbb{C} \setminus \{0\}$, *then* $\lambda \in \sigma(EF)$ *if and only if* $\lambda \in \sigma(FE)$, *with* $\sigma(\cdot)$ *denoting eigenvalue spectrum*, we find that the eigenvalue spectra of B and B''_1 are connected via the following relationship:

$$\sigma({B''}_1^{p-1}) \setminus \{0\} \equiv \sigma(B^p) \setminus \{0\}. \tag{3.4}$$

However, (3.4) and especially the expressions for B, $B1''$ and $B1''_1$ strongly remind us of the problem of the best block p-cyclic SOR repartitioning, first considered and studied by Markham et al. [18]. Indeed, it can be seen that if we repartition the matrix A into the following block $(p-1)$-cyclic consistently ordered form

$$A = \left[\begin{array}{cc|c|c|c} A_{11} & O_{12} & O_{13} & \dots & A_{1p} \\ A_{21} & A_{22} & O_{23} & \dots & O_{2p} \\ \hline O_{31} & A_{32} & A_{33} & \dots & O_{3p} \\ \hline \vdots & \vdots & \ddots & \ddots & \vdots \\ \hline O_{p1} & O_{p2} & \dots & A_{p,p-1} & A_{pp} \end{array}\right] \tag{3.5}$$

we can find out that the block Jacobi matrix, J, associated with A of (3.5) is identically the same with the matrix B'' associated with A without any repartitioning, since

$$J = \left[\begin{array}{cc|c|c|c} O_{11} & O_{12} & O_{13} & \dots & -A_{11}^{-1}A_{1p} \\ O_{21} & O_{22} & O_{23} & \dots & A_{22}^{-1}A_{21}A_{11}^{-1}A_{1p} \\ \hline O_{31} & -A_{33}^{-1}A_{32} & O_{33} & \dots & O_{3p} \\ \hline \vdots & \vdots & \ddots & \ddots & \vdots \\ \hline O_{p1} & O_{p2} & \dots & -A_{pp}^{-1}A_{p,p-1} & O_{pp} \end{array}\right].$$

Consequently, $J \equiv B''$ and therefore

$$\sigma(J^{p-1}) \setminus \{0\} \equiv \sigma(B''^{p-1}) \setminus \{0\} \equiv \sigma(B''^{p-1}_1) \setminus \{0\} \equiv \sigma(B^p) \setminus \{0\}. \tag{3.6}$$

Based on the result just obtained we can readily prove the following statement.

Theorem 3.1. *Under the notation of Section 2 let $A \in \mathbb{C}^{n,n}$ be a nonsingular block p-cyclic consistently ordered matrix of the form (3.1) with $\rho(B) < 1$. Then for the spectral radii of the block Jacobi and Gauss–Seidel iteration matrices B, B'', B''_1, H, H'' and H''_1 the following relationships hold*:

$$\rho(B''_1) \equiv \rho(B'') = \rho^{\frac{p}{p-1}}(B) < \rho(B) < 1 \tag{3.7}$$

and

$$\rho(H''_1) \equiv \rho(H'') = \rho(H) = \rho^p(B) < 1. \tag{3.8}$$

Proof. The proof of (3.7) is an immediate consequence of (3.6). The validity of (3.8) is because the spectrum of the Gauss–Seidel matrix of a p-cyclic consistently ordered matrix is the union of 0 and of the pth powers of the eigenvalues of the corresponding Jacobi matrix. □

We conclude this section by stating a theorem concerning the best of the optimal block SOR methods associated with a block p-cyclic consistently ordered matrix A of the form (3.1) and its preconditioned one $\tilde{A}$ (or $\tilde{A}_1$) when $\sigma(B^p) \subset \mathbb{R}$.

Theorem 3.2. *Let B be the block Jacobi matrix* (3.2) *associated with the block p-cyclic consistently ordered matrix A in* (3.1), *$p \geqslant 3$, and let $\sigma(B^p) \subset [-\alpha^p, \beta^p]$ with $-\alpha^p$, $\beta^p \in \sigma(B^p)$, where $0 \leqslant \beta < 1$ and $0 \leqslant \alpha < \infty$. Consider $\tilde{A}$ partitioned in a block $(p-1)$-cyclic form consistent with the partitioning of A in* (3.5). *Denote by ω_k and $\rho(\mathscr{L}_{\omega_k})$, $k = p, p-1$, the real optimal SOR factor and the optimal spectral radius associated with A ($k = p$) and $\tilde{A}$ or $\tilde{A}_1$ ($k = p-1$), respectively, and by $\rho(\mathscr{L}_\omega(A))$, $\rho(\mathscr{L}_\omega(\tilde{A}))$ the spectral radii of the SOR matrices for A and $\tilde{A}$. Then there hold*:

If

$$\frac{\alpha}{\beta} \in \left(\left(\frac{p-3}{p-1} \right)^{(p-1)/p}, \frac{p-2}{p} \right),$$

then for any $\beta \in (0,1)$ *there exists a unique value of* α,

$$\alpha_{p-1,p} := \alpha(\beta) \in \left(\left(\frac{p-3}{p-1}\right)^{(p-1)/p} \beta, \frac{p-2}{p}\beta \right)$$

which is given by the expression

$$\alpha_{p-1,p} = \left(\frac{2\rho^{\frac{1}{p-1}} - (1+\rho)\beta^{\frac{p}{p-1}}}{1-\rho} \right)^{(p-1)/p},$$

where ρ *is the unique root in* $(0,1)$ *of the equation*

$$\beta^p (p-1+\rho)^p - p^p \rho = 0. \tag{3.9}$$

The root ρ *is such that*

$$\rho(\mathscr{L}_{\omega_{p-1}}) < \rho(\mathscr{L}_{\omega_p}) < 1 \quad \text{for} \quad \left(\frac{p-3}{p-1}\right)^{(p-1)/p} \beta < \alpha < \alpha_{p-1,p}, \tag{3.10}$$

$$\rho(\mathscr{L}_{\omega_{p-1}}) = \rho(\mathscr{L}_{\omega_p}) < 1 \quad \text{for} \quad \alpha = \alpha_{p-1,p}, \tag{3.11}$$

$$\rho(\mathscr{L}_{\omega_p}) < \rho(\mathscr{L}_{\omega_{p-1}}) < 1 \quad \text{for} \quad \alpha_{p-1,p} < \alpha < \left(\frac{p-2}{p}\right)\beta. \tag{3.12}$$

Note: *The theorem presents only one out of eight*(!) *possible cases. In each of them it is determined which of the optimal* p*-cyclic and* $(p-1)$*-cyclic repartitioning is the best to use for SOR when the ratio* α/β *runs over the set of real numbers. Specifically, the following intervals are considered*:

$$\text{(i)} \quad \left[0, \left(\frac{p-3}{p-1}\right)^{(p-1)/p}\right], \quad \text{(ii)} \quad \left(\left(\frac{p-3}{p-1}\right)^{(p-1)/p}, \frac{p-2}{p}\right),$$

$$\text{(iii)} \quad \left(\frac{p-2}{p}, 1\right), \quad \text{(iv)} \quad [1,1],$$

$$\text{(v)} \quad \left(1, \frac{p}{p-2}\right), \quad \text{(vi)} \quad \left(\frac{p}{p-2}, \frac{p}{\beta(p-2)}\right),$$

$$\text{(vii)} \quad \left(\frac{p}{\beta(p-2)}, \frac{1}{\beta}\left(\frac{p-1}{p-3}\right)^{(p-1)/p}\right), \quad \text{(viii)} \quad \left[\frac{1}{\beta}\left(\frac{p-1}{p-3}\right)^{(p-1)/p}, \infty\right]. \tag{3.13}$$

Proof. Only a sketch of the proof will be given, since the main line of reasoning is analogous to that found in [7]. It is known that in a p-cyclic consistently ordered case, like the one we are working on, when $\alpha \geqslant p/(p-2)$ there is no $\omega \in \mathbb{R}$ for which the associated SOR method converges. Having in mind the point just made, then by means of the formulas given in Theorem 2.1 and Table 1 of [7] (of the *Best Cyclic Repartitioning for Optimal SOR*) one can determine the value of $k = 2(1)p$ that gives the best (repartitioned) optimal SOR. Having determined the specific value of k one can find the optimal SOR parameter by means of the formulas (2.20)–(2.22) of [7]. In our case, however, we must restrict to the values of $k = p-1$ and p. So, we have to appropriately exploit the results in [7]. For this we observe that because of (3.6), it will be $\alpha_{p-1}^{p-1} = \alpha^p$ and $\beta_{p-1}^{p-1} = \beta^p$, where $\alpha_{p-1}, \beta_{p-1}$ play the roles of α, β for the block $(p-1)$-cyclic consistently ordered matrix $\tilde{A}$ or $\tilde{A}_1$. However, for these matrices convergence of the associated block SOR will take place for values of $\alpha_{p-1} \in [0, (p-1)/(p-3))$ while for the original p-cyclic SOR the corresponding interval for α will be $[0, p/(p-2))$. The latter interval for α is contained in the former one for α which is then $[0, ((p-1)/(p-3))^{(p-1)/p})$ since as can be proved the function $((x-2)/x)^x$ (resp. $(x/(x-2))^x$), $x \in [2, \infty)$, strictly increases (resp. decreases) with x. So, the $(p-1)$-cyclic case for $\tilde{A}$ can be advantageous over the p-cyclic case for A. From this point on the statement of our theorem describes more specifically one of the eight possible situations that can arise and gives for it (case (vi) of (3.13)) the (optimal) convergence results that can be obtained. (Note: *A complete list of the results for the other seven cases in* (3.13) *is given in* [2].) □

4. Comments and discussion on the singular case

(I) Let $A \in \mathbb{R}^{n,n}$, in (1.3), be a *singular* irreducible M-matrix. For such an A, each principal submatrix, except itself, is a nonsingular M-matrix [3]. By Lemma 2.1 and Theorem 2.1, $\tilde{A}_1$ is irreducible (unless $p=2$ and $n_1 = n-1$, when $\tilde{A}_1 = O \in \mathbb{R}^{1,1}$) and both $\tilde{A}$ and $\tilde{A}_1$ are singular M-matrices. All the splittings in Sections 2.2 and 2.3 that were M-splittings still are and so all the corresponding iteration matrices there, e.g., B, H, $\mathscr{L}_\omega$, $\omega \in (0,1]$, are well defined and have spectral radii 1. As is known [3] for the (semi)converge of a linear first order iterative scheme, with iteration matrix satisfying $\rho(T)=1$, for any initial guess $x_0 \in \mathbb{R}^n$, provided that $b \in \mathrm{range}(A)$, $\sigma(T)$ must satisfy the three conditions below (see [3]). In such a case a factor $\gamma(T)$, which is equal to the modulus of the second largest in modulus eigenvalue of T, plays the role of the spectral radius.

(i) $\rho(T)=1$.
(ii) If $\lambda \in \sigma(T)$ with $|\lambda|=1$, then $\lambda=1$.
(iii) $\mathrm{index}(I-T)=1$, that is in the Jordan canonical form of T all eigenvalues of modulus 1 are associated with 1×1 Jordan blocks.

All iteration matrices arising from M-splittings satisfy condition (i) and in view of the irreducibility of A, and of $\tilde{A}_1$, the iteration matrices associated with them satisfy condition (iii). However, when the splitting is a non-parametric one, condition (ii) cannot be always satisfied. E.g., for A being also p-cyclic consistently ordered its Jacobi matrix has, besides 1, as eigenvalues the numbers $\exp(i(2\pi k/p)$, $k = 1(1)p-1$, of modulus 1. So, stronger conditions are necessary or parametric iterative schemes based on the given one such as "Extrapolated Schemes", with parameter $\omega \in (0,1)$ (see, e.g., [22] and also [9] for more general cases), or SOR Schemes, see below, can produce semiconvergent schemes. We also note that even if all three conditions (i)–(iii) are met for the iteration matrices of Theorems 2.2 and 2.4 it is not clear whether their semiconvergence factors $\gamma(\cdot)$ will satisfy relationships analogous to those in (2.21)–(2.24) and in (2.32)–(2.36). In this direction some recent results due to Marek and Szyld that can be found in [16,17] have contributed a lot.

(II) The case where some of the results obtained are carried over to the singular case is when $A \in \mathbb{C}^{n,n}$ is block p-cyclic consistently ordered of the form (3.1). Suppose then that $B = I - D^{-1}A$ satisfies condition (iii). This condition, $\mathrm{index}(I-B)=1$, implies $\mathrm{index}(I-\mathscr{L}_\omega)=1$, $\forall \omega \neq \{0, p/(p-1)\}$

(see [9, Theorem 3.1]). Hence by virtue of [11, Theorem 3.1], Theorem 3.2 holds, provided that in its assumptions $\sigma(B^p) \subset [-\alpha^p, \beta^p] \cup \{1\}$ will replace $\sigma(B^p) \subset [-\alpha^p, \beta^p]$ and therefore Theorem 3.2 will be valid except that $\gamma(\cdot)$'s will replace the corresponding $\rho(\cdot)$'s.

A direct application to the previous results in the present section is the determination of the stationary probability distribution vector in the Markov Chain Analysis where the coefficient matrix A is a singular M-matrix with zero column sums.

If A is irreducible (ergodic chain) Theorems 2.2 and 2.4 hold in the way explained above.

If A is, in addition, of the form (3.1) (periodic chain of period p) and $B = I - D^{-1}A$ is irreducible, then Theorem 3.2 holds as was explained. Since $\rho(B) = 1$, α in Theorem 3.2 must satisfy $\alpha < 1$. This restriction modifies slightly the conclusions of the theorem. Specifically: The three cases (vi), (vii) and (viii) are expressed as one under the main assumption $p/(p-2) < \alpha/\beta \leqslant \infty$, where "$=\infty$" means $\beta = 0$ and $\alpha > 0$. Then, $\beta_{p-1,p} \in [0, [(p-2)/p]\alpha]$, with $\beta_{p-1,p} = 0$ corresponding to $\alpha/\beta = \infty$. The conclusions are those of case (vi) of [2].

5. Numerical examples

Example 1. The matrix below (without the partitioning is found in [8]) is obviously an irreducible Z-matrix and since $\rho(I - A) \approx 0.9807 < 1$, A is also a nonsingular M-matrix,

$$A = \left[\begin{array}{cc|c|cc} 1 & -0.2 & -0.1 & -0.4 & -0.2 \\ -0.2 & 1 & -0.3 & -0.1 & -0.6 \\ \hline -0.3 & -0.2 & 1 & -0.1 & -0.6 \\ \hline -0.1 & -0.1 & -0.1 & 1 & -0.01 \\ -0.2 & -0.3 & -0.4 & -0.3 & 1 \end{array}\right]. \tag{5.1}$$

Therefore, Theorems 2.1–2.4 apply. Indeed, preserving the notation in these statements it is:

$$\begin{aligned} &\rho(B^{(0)}) = 0.9807, \quad \rho(H^{(0)}) = 0.9611, \quad \rho(\mathscr{L}^{(0)}_{0.75}) = 0.9768, \\ &\rho(B) = 0.9785, \quad \rho(H) = 0.9570, \quad \rho(\mathscr{L}_{0.75}) = 0.9743, \\ &\rho(B'^{(n_1)}) = 0.9778, \quad \rho(H'^{(n_1)}) = 0.9565, \quad \rho(\mathscr{L}'^{(n_1)}_{0.75}) = 0.9749, \\ &\rho(B''^{(n_1)}) = 0.9770, \quad \rho(H''^{(n_1)}) = 0.9531, \quad \rho(\mathscr{L}''^{(n_1)}_{0.75}) = 0.9734, \\ &\rho(B') = 0.9684, \quad \rho(H') = 0.9505, \quad \rho(\mathscr{L}'_{0.75}) = 0.9676, \\ &\rho(B'') = 0.9577, \quad \rho(H'') = 0.9172, \quad \rho(\mathscr{L}''_{0.75}) = 0.9502, \end{aligned} \tag{5.2}$$

where the matrices in the first two rows refer to the "point" and "block" iteration matrices associated with A, the matrices of the third and fourth rows refer to the "point" iteration matrices of $\tilde{A}$ and the matrices of the last two rows refer to the "block" ones of $\tilde{A}$. It is checked that all the relationships (strict inequalities) of Theorems 2.1–2.4 are verified.

For the above example we determined the "point" and "block" Jacobi and Gauss–Seidel iteration matrices after applying the "point" and "block" preconditioners of the Gunawardena et al. [8]. For the "point" preconditioner we used the known one

$$\begin{bmatrix} 1 & -a_{12} & 0 & 0 & \dots & 0 \\ 0 & 1 & -a_{23} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \vdots \\ 0 & \dots & \dots & \dots & 1 & -a_{n-1,n} \\ 0 & \dots & \dots & \dots & 0 & 1 \end{bmatrix}, \tag{5.3}$$

while for the "block" the one below

$$\begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} & 0 & 0 & \dots & 0 \\ 0 & A_{22}^{-1} & -A_{22}^{-1}A_{23}A_{33}^{-1} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \vdots \\ 0 & \dots & \dots & \dots & A_{p-1,p-1}^{-1} & -A_{p-1,p-1}^{-1}A_{p-1,p}A_{pp}^{-1} \\ 0 & \dots & \dots & \dots & 0 & A_{pp}^{-1} \end{bmatrix}. \tag{5.4}$$

The corresponding spectral radii were found to be

$$\begin{aligned} &\rho(B'^{(n_1)}_G) = 0.9784, \quad \rho(H'^{(n_1)}_G) = 0.9611, \\ &\rho(B'_G) = 0.9729, \quad \rho(H'_G) = 0.9570, \end{aligned} \tag{5.5}$$

where the superfix (n_1) refers to the "point" case and the absence of it to the "block" one. It seems that the corresponding "point" and "block" preconditioners of Milaszewicz's type give better results in all cases. However, as was stated in Section 1, no final conclusion regarding the relative effectiveness of the two types of preconditioners should be drawn before a complete theoretical analysis has taken place.

Example 2. We consider the following singular block three-cyclic consistently ordered matrix

$$A = \begin{bmatrix} I_3 & O & -C \\ -C & I_3 & O \\ O & -C & I_3 \end{bmatrix}, \quad \text{with } C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha\beta & \alpha-\beta+\alpha\beta & 1-\alpha+\beta \end{bmatrix}, \tag{5.6}$$

$I_3 \in \mathbb{R}^{3,3}$ the identity matrix and $-\alpha \leqslant 0 \leqslant \beta < 1,\ \alpha \neq \beta$. It is $B = \mathrm{diag}(I_3, I_3, I_3) - A$ hence $B^3 = \mathrm{diag}(C^3, C^3, C^3)$, with $\sigma(C^3) = \{1, -\alpha^3, \beta^3\}$, $-\alpha^3 \leqslant 0 \leqslant \beta^3 < 1,\ \alpha \neq \beta$. Therefore,

$$\sigma(B) = \left\{ \exp\left(\mathrm{i}\,\frac{2k\pi}{3}\right), \quad \beta \exp\left(\mathrm{i}\,\frac{2k\pi}{3}\right), -\alpha \exp\left(\mathrm{i}\,\frac{2k\pi}{3}\right) \right\}, \quad k = 0, 1, 2,$$

the eigenvalue 1 is simple and $\mathrm{index}(I - B) = 1$. Consequently, all the assumptions of Theorem 3.2, in the singular case, are satisfied. So, depending on the value of the ratio α/β the best of the two optimal SORs associated with A and $\tilde{A}_1$, if they exist, will be associated with either the original block three-cyclic or, after the preconditioning takes place, with the block two-cyclic one. It is noted that for the optimal three-cyclic SOR to exist, $\alpha < 3$ must hold, while the two-cyclic one exists for all $\alpha \in [0, \infty)$!

Example 3. We consider the convection–diffusion equation in the unit square under Dirichlet boundary conditions

$$-u_{xx} - u_{yy} + cu_x = f(x,y) \quad \text{in } \Omega := (0,1)\times(0,1), \quad u = g(x,y) \quad \text{on } \partial\Omega, \tag{5.7}$$

where c is a positive constant. A uniform discretization is imposed on $\Omega \cup \partial\Omega$ with n internal nodes in each coordinate direction. The n^2 internal nodes, at which the approximate values of the unknown function are sought, are ordered in the natural ordering, i.e., from left to right and from bottom to top. The classical five-point central difference scheme is adopted for the discretization. (Note: $c \leqslant 2(n+1)$ is a sufficient and necessary condition for the coefficient matrix A to be a nonsingular M-matrix.)

A number of numerical examples were run on a computer using Matlab 6.5 (with double precision arithmetic) with various values of c and n. The conclusions at which we arrived were in almost all the cases pretty much the same. In the table below we present a simple case, that can be readily checked by the interested reader, so $c = 0$ is selected and the PDE considered becomes the Poisson equation. The functions f and g in (5.7) are selected such that the theoretical solution to the PDE problem is $u(x,y) = e^{x+y}\sin(\pi x/2)\sin(\pi y/2)$. To make fair comparisons we used $\max_i |(x_i^{(m+1)} - x_i^{(m)})/x_i^{(m+1)}| \leqslant \varepsilon = 0.5 \times 10^{-12}$, $i = 1(1)n^2$, as a stopping criterion, with $x_i^{(m)}$ denoting the ith component of the mth approximation to the actual solution vector of the linear system yielded from the discretization. The block partitioning considered was the one suggested by the discretization used and the theory we developed, that is $A_{11} \in \mathbb{R}^{n,n}$ and $A_{22} \in \mathbb{R}^{n(n-1),n(n-1)}$. In the following table, because the PDE considered leads to a real symmetric positive definite linear system, the ICC(0)/CG instead of the ILU(0)/GMRES method was used, where ICC stands for Incomplete Cholesky. The ICC(0)/CG was restarted every n iterations, which n we found fairly good in almost all the examples we run on the computer. It should be said that the simple CG performed worse than the restarted one and that is why we preferred the latter. We also have to add that in all cases treated full exploitation of the presence of zero elements in the structure of the main matrix and its submatrices was taken so that calculations with zero elements were completely avoided.

n		Jacobi	Gauss–Seidel	ICC(0)/CG
	iter	260	140	17
10	CPU time	7.6250	3.750	0.3590
	rel error	1.3832e-011	2.4428e-012	1.0689e-004
	iter	842	454	27
20	CPU time	14.449	7.703	1.532
	rel error	5.0075e-011	1.0818e-11	1.0225e-005
	iter	1682	914	38
30	CPU time	119.44	48.125	4.109
	rel error	1.0992e-010	2.3587e-011	2.4481e-006
	iter	2748	1502	47
40	CPU time	91.625	69.172	7.281
	rel error	1.9429e-010	4.1897e-011	8.7150e-007
	iter	4020	2207	58
50	CPU time	292.08	131.27	12.06
	rel error	3.0014e-010	6.5602e-011	3.8770e-007

In the above table the following items are illustrated: In the first column the number n. In the second one, for each n, and in three consecutive rows, the number of iterations needed to satisfy the criterion imposed (iter), the CPU time consumed in seconds and the actual absolute relative error (rel error) achieved with respect to the theoretical solution at the corresponding nodal point, that is $\max_i |(x_i^{(m+1)} - u_i)/u_i|$, $i = 1(1)n^2$, for each of the three methods Jacobi, Gauss–Seidel, ICC(0)/CG, shown in columns three to five.

As one can note from the various values in the table, in all the cases the block preconditioned Gauss–Seidel performs twice as good as the block preconditioned Jacobi, regarding both iterations needed and CPU time consumed, and this is in accordance with what one would expect due to the block two-cyclic consistently ordered nature of the coefficient matrix of the linear system solved. Although the criterion imposed was satisfied in all the cases by the Jacobi, Gauss–Seidel and ICC(0)/CG methods, the former two methods needed much more time than the latter one to reach it. On the other hand, however, the actual absolute relative error in the former two methods is (very) close to the required one and much better than the corresponding quantity for the latter method. Things seem to improve in favor of the ICC(0)/CG method as the number of subdivisions n increases. From the results and the data presented it is rather clear that for moderate values of n block preconditioned Gauss–Seidel should be preferred instead of ILU(0)/GMRES, while for large values of n the situation may be reversed.

Before we conclude with this example we have to say that in each case examined we tried to increase the number of iterations in the ICC(0)/CG method in order to reach the best *actual* relative error achieved by either of the methods Jacobi or Gauss–Seidel. Although we exhausted the time limits defined by the latter two methods, the result was that the aforementioned actual relative error could not be reached! The results illustrated in the table could *not* be improved further.

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