Semisymmetric graphs from polytopes

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Abstract

Every finite, self-dual, regular (or chiral) 4-polytope of type $\{3, q, 3\}$ has a trivalent 3-transitive (or 2-transitive) medial layer graph. Here, by dropping self-duality, we obtain a construction for semisymmetric trivalent graphs (which are edge- but not vertex-transitive). In particular, the Gray graph arises as the medial layer graph of a certain universal locally toroidal regular 4-polytope.
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Keywords: Semisymmetric graphs; Abstract regular and chiral polytopes

1. Introduction

The theory of symmetric trivalent graphs and the theory of regular polytopes are each abundant sources of beautiful mathematical ideas. In [22], two of the authors established some general and unexpected connections between the two subjects, building upon a rich variety of examples appearing in the literature (see [4,7,10,11,28,29]). Here we develop these connections a little fur-

ther, with specific focus on semisymmetric graphs. In particular, we reexamine the *Gray graph*, described in [2,3] and [18,24], and here appearing as the medial layer graph of an abstract regular 4-polytope.

We begin with some basic ideas concerning symmetric graphs [1, Chapters 18–19]. Although some of the following results generalize to graphs of higher valency, for brevity we shall *assume outright that $\mathcal{G}$ is a simple, finite, connected trivalent graph* (so that each vertex has valency 3).

By a *t-arc* in $\mathcal{G}$ we mean a list of vertices $[v] = [v_0, v_1, \ldots, v_t]$ such that $\{v_{i-1}, v_i\}$ is an edge for $1 \leqslant i \leqslant t$, but no $v_{i-1} = v_{i+1}$. Tutte has shown that there exists a maximal value of t such that the automorphism group $\text{Aut}(\mathcal{G})$ is transitive on t-arcs. We say that $\mathcal{G}$ is *t-transitive* if $\text{Aut}(\mathcal{G})$ is transitive on t-arcs, but not on $(t+1)$-arcs in $\mathcal{G}$, for some $t \geqslant 1$. Tutte also proved the remarkable result that $t \leqslant 5$ [1, Theorem 18.6]. Any such arc-transitive graph is said to be *symmetric*.

Each fixed t-arc $[v]$ in a t-transitive graph $\mathcal{G}$ has *stabilizer sequence*

$$\text{Aut}\,\mathcal{G} \supset B_t \supset B_{t-1} \supset \cdots \supset B_1 \supset B_0,$$

where the subgroup B_j is the pointwise stabilizer of $\{v_0, \ldots, v_{t-j}\}$. Since $\text{Aut}(\mathcal{G})$ is transitive on r-arcs, for $r \leqslant t$, the subgroup B_j is conjugate to that obtained from any other t-arc. In particular, B_t is the vertex stabilizer, whereas B_0 is the pointwise stabilizer of the whole arc. In fact, $B_0 = \{\epsilon\}$ is trivial [1, Proposition 18.1], so that $\text{Aut}(\mathcal{G})$ acts sharply transitively on t-arcs.

Each t-arc $[v]$ has two *successors*, t-arcs of the form $[v^{(k)}] := [v_1, \ldots, v_t, y_k]$, where v_{t-1}, y_1, y_2 are the vertices adjacent to v_t. The *shunt* τ_k is the (unique) automorphism of $\mathcal{G}$ such that $[v]\tau_k = [v^{(k)}]$. Also let α be the unique automorphism which reverses the basic t-arc $[v]$. Then α has period 2 and $\alpha\tau_1\alpha$ equals either τ_1^{-1} or τ_2^{-1}. We shall say that $\mathcal{G}$ is of *type t^+ or t^-*, respectively. We can now assemble several beautiful results concerning $\text{Aut}(\mathcal{G})$ (see [1, Chapter 18]).

Theorem 1. *Suppose $\mathcal{G}$ is a finite connected t-transitive trivalent graph, with $1 \leqslant t$, and suppose $\mathcal{G}$ has N vertices. Then*:

(a) *For $0 \leqslant j \leqslant t-1$ we have $|B_j| = 2^j$. Also, $|B_t| = 3 \cdot 2^{t-1}$ and*

$$\big|\text{Aut}(\mathcal{G})\big| = 3 \cdot N \cdot 2^{t-1}.$$

(b) *The stabilizers B_j are determined up to isomorphism by t*:

t	B_1	B_2	B_3	B_4	B_5
1	$\mathbb{Z}_3$				
2	$\mathbb{Z}_2$	$\mathbb{S}_3$			
3	$\mathbb{Z}_2$	$(\mathbb{Z}_2)^2$	$\mathbb{D}_{12}$		
4	$\mathbb{Z}_2$	$(\mathbb{Z}_2)^2$	$\mathbb{D}_8$	$\mathbb{S}_4$	
5	$\mathbb{Z}_2$	$(\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^3$	$\mathbb{D}_8 \times \mathbb{Z}_2$	$\mathbb{S}_4 \times \mathbb{Z}_2$

(c) *$\mathcal{G}$ is one of 7 types*: $1^-, 2^+, 2^-, 3^+, 4^+, 4^-$ *or* 5^+.

(Here $\mathbb{Z}_k$ is the cyclic group of order k, $\mathbb{D}_{2k}$ is the dihedral group of order $2k$, $\mathbb{S}_k$ is the symmetric group of degree k.)

Useful lists of symmetric trivalent graphs appear in [4,7]. We refer to [22] for a description of several interesting examples.

We now briefly describe some key properties of abstract regular and chiral polytopes, referring again to [22] for a short discussion, and to [23,25,26] for details. An (*abstract*) *n-polytope* $\mathcal{P}$ is

a partially ordered set with a strictly monotone rank function having range $\{-1, 0, \ldots, n\}$. An element $F \in \mathcal{P}$ with $\mathrm{rank}(F) = j$ is called a *j-face*; typically F_j will indicate a *j*-face; and $\mathcal{P}$ has a unique least face F_{-1} and unique greatest face F_n. Each maximal chain or *flag* in $\mathcal{P}$ must contain $n + 2$ faces. Next, $\mathcal{P}$ must satisfy a *homogeneity property*: whenever $F < G$ with $\mathrm{rank}(F) = j - 1$ and $\mathrm{rank}(G) = j + 1$, there are exactly two *j*-faces H with $F < H < G$, just as happens for convex *n*-polytopes. It follows that for $0 \leqslant j \leqslant n - 1$ and any flag Φ, there exists a unique *adjacent* flag Φ^j, differing from Φ in just the rank *j* face. With this notion of adjacency the flags of $\mathcal{P}$ form a *flag graph* (not to be confused with the medial layer graphs appearing below). The final defining property of $\mathcal{P}$ is that it should be *strongly flag-connected*. This means that the flag graph for each section is connected. Whenever $F \leqslant G$ are faces of ranks $j \leqslant k$ in $\mathcal{P}$, the *section* $G/F := \{H \in \mathcal{P} \mid F \leqslant H \leqslant G\}$ is thus in its own right a $(k - j - 1)$-polytope.

Since our main concern is with 4-polytopes, we now tailor our discussion to that case. A (rank 4) polytope $\mathcal{P}$ is *equivelar* of type $\{p_1, p_2, p_3\}$ if, for $j = 1, 2, 3$, whenever F and G are incident faces of $\mathcal{P}$ with $\mathrm{rank}(F) = j - 2$ and $\mathrm{rank}(G) = j + 1$, then the rank 2 section G/F has the structure of a p_j-gon (independent of choice of $F < G$). Thus, each 2-face (polygon) of $\mathcal{P}$ is isomorphic to a p_1-gon, and there are p_3 of these arranged around each 1-face (edge) of $\mathcal{P}$; and in every 3-face (facet) of $\mathcal{P}$, each 0-face is surrounded by an alternating cycle of p_2 edges and p_2 polygons.

The automorphism group $\mathrm{Aut}(\mathcal{P})$ consists of all order preserving bijections on $\mathcal{P}$. If $\mathcal{P}$ also admits a duality (order reversing bijection), then $\mathcal{P}$ is said to be *self-dual*; clearly $\mathrm{Aut}(\mathcal{P})$ then has index 2 in the group $D(\mathcal{P})$ of all automorphisms and dualities. (Note that $D(\mathcal{P}) = \mathrm{Aut}(\mathcal{P})$ when $\mathcal{P}$ is not self-dual.) If $\mathcal{P}$ is self-dual and equivelar, then it has type $\{p_1, p_2, p_1\}$.

Definition 1. Let $\mathcal{P}$ be a 4-polytope. The associated *medial layer graph* $\mathcal{G}(\mathcal{P})$, or briefly $\mathcal{G}$, is the simple graph whose vertex set is comprised of all 1-faces and 2-faces in $\mathcal{P}$, two such taken to be adjacent when incident in $\mathcal{P}$.

Remarks. Any medial layer graph $\mathcal{G}$ is easily seen to be bipartite and connected. Note that the more desirable phrase 'medial graph' already has a somewhat different meaning in the literature on topological graph theory.

To further focus our investigations, we *henceforth assume that $\mathcal{P}$ is equivelar of type $\{3, q, 3\}$*, where the integer $q \geqslant 2$. Thus $\mathcal{G}$ is trivalent, with vertices of two types occurring alternately along cycles of length $2q$. We say that a *t*-arc in $\mathcal{G}$ is of *type* 1 (respectively *type* 2) if its initial vertex is a 1-face (respectively 2-face) of $\mathcal{P}$. The fact that certain polygonal sections of $\mathcal{P}$ are triangular immediately implies that the action of $D(\mathcal{P})$ on $\mathcal{G}$ is faithful, so that we may regard $D(\mathcal{P})$, or $\mathrm{Aut}(\mathcal{P})$, as a subgroup of $\mathrm{Aut}(\mathcal{G})$ (see [22, §2]).

In Fig. 1 we show a fragment of a polytope $\mathcal{P}$ of type $\{3, q, 3\}$. The vertices of $\mathcal{G}$ are here represented as black and white discs, and the edges of $\mathcal{G}$ are indicated by heavy lines.

Since we shall soon assume that $\mathcal{P}$ is quite symmetric, it is useful now to fix a *base flag*

$$\Phi = \{F_{-1}, F_0, F_1, F_2, F_3, F_4\}$$

in $\mathcal{P}$. Given this, it is convenient to define $v_1 := F_1$, $v_2 := F_2$, and in general let $v_0 = v_{2q}, v_1, v_2, \ldots, v_{2q-1} = v_{-1}$ denote alternate edges and polygons in the rank 2 section F_3/F_0 of $\mathcal{P}$. Thus each v_j is adjacent in $\mathcal{G}$ to $v_{j\pm1}$, taking subscripts mod $2q$. We also let w_j be the third vertex adjacent to v_j in $\mathcal{G}$, as indicated in Fig. 1.
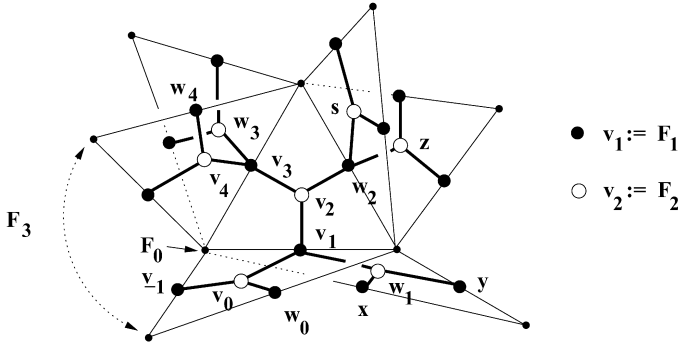
Fig. 1. A fragment of a polytope of type $\{3, q, 3\}$.

We turn now to two significant classes of highly symmetric polytopes. First we recall that $\mathcal{P}$ is *regular* when $\mathrm{Aut}(\mathcal{P})$ acts transitively on the flags of $\mathcal{P}$. Assuming still that $n = 4$, we observe that for $0 \leqslant j \leqslant 3$, there exists a (unique) automorphism ρ_j mapping the base flag Φ to the adjacent flag Φ^j. Then $\mathrm{Aut}(\mathcal{P})$ is generated by the involutions $\rho_0, \rho_1, \rho_2, \rho_3$, which satisfy at least the relations

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = (\rho_1\rho_3)^2 = \epsilon,$$
$$(\rho_0\rho_1)^{p_1} = (\rho_1\rho_2)^{p_2} = (\rho_2\rho_3)^{p_3} = \epsilon, \tag{1}$$

with $2 \leqslant p_1, p_2, p_3 \leqslant \infty$. Indeed, $\mathcal{P}$ is equivelar of type $\{p_1, p_2, p_3\}$. (As before we will actually have $p_1 = p_3 = 3$ and $p_2 = q$ in our applications.)

Furthermore, an *intersection condition* on standard subgroups holds:

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle \tag{2}$$

for all $I, J \subseteq \{0, 1, 2, 3\}$. In short, $\mathrm{Aut}(\mathcal{P})$ is a very particular quotient of a Coxeter group with string diagram.

Conversely, suppose that $\Gamma = \langle \rho_0, \ldots, \rho_3 \rangle$ is a *string C-group*, namely any group generated by specified involutions satisfying (1) and (2). Then one may construct a regular 4-polytope $\mathcal{P} = \mathcal{P}(\Gamma)$, of type $\{p_1, p_2, p_3\}$, with $\mathrm{Aut}(\mathcal{P}) = \Gamma$. We refer to [22, Definition 2] or [23, Theorems 2E11 and 2E12] for details of the construction. Note also that $\mathcal{P}$ is self-dual if and only if $\mathrm{Aut}(\mathcal{P})$ admits an involutory group automorphism δ such that $\delta\rho_j\delta = \rho_{3-j}$ for $j = 0, 1, 2, 3$. Such a polytope $\mathcal{P}$ admits a *polarity* (i.e. involutory duality) which reverses the basic flag Φ. Thus $D(\mathcal{P}) \simeq \mathrm{Aut}(\mathcal{P}) \rtimes \mathbb{Z}_2$ (see [23, 2B17 and 2E12]).

The upshot of Theorem 2 in [22] is that $\mathcal{G}(\mathcal{P})$ is 3-transitive when $\mathcal{P}$ is finite, regular and self-dual of type $\{3, q, 3\}$.

For any regular polytope $\mathcal{P}$, the *rotations* $\sigma_j := \rho_{j-1}\rho_j$ generate a subgroup $\mathrm{Aut}(\mathcal{P})^+$ having index 1 or 2 in $\mathrm{Aut}(\mathcal{P})$. In the latter case, $\mathcal{P}$ is said to be *directly regular*, and certain properties of the σ_j lead, in a natural way, to a parallel theory of *chiral* polytopes (see [25,26] for details).

A polytope $\mathcal{P}$ of rank $n \geqslant 3$ is said to be *chiral* if it is not regular, but there do exist automorphisms $\sigma_1, \ldots, \sigma_{n-1}$ such that σ_j fixes all faces in $\Phi \setminus \{F_{j-1}, F_j\}$ and cyclically permutes consecutive j-faces of $\mathcal{P}$ in the rank 2 section F_{j+1}/F_{j-2} of $\mathcal{P}$. The automorphism group of $\mathcal{P}$ now has two flag orbits, with adjacent flags always in different orbits. Again taking $n = 4$, it is even possible in the chiral case to choose automorphisms $\sigma_1, \sigma_2, \sigma_3$ which generate $\mathrm{Aut}(\mathcal{P})$ and satisfy at least the relations

$$\sigma_1^{p_1} = \sigma_2^{p_2} = \sigma_3^{p_3} = \epsilon,$$
$$(\sigma_1\sigma_2)^2 = (\sigma_2\sigma_3)^2 = (\sigma_1\sigma_2\sigma_3)^2 = \epsilon, \tag{3}$$

for some $2 \leqslant p_1, p_2, p_3 \leqslant \infty$. Once more $\mathcal{P}$ is equivelar of type $\{p_1, p_2, p_3\}$. Here too the specified generators satisfy a revised intersection condition:

$$\langle \sigma_1 \rangle \cap \langle \sigma_2 \rangle = \{\epsilon\} = \langle \sigma_2 \rangle \cap \langle \sigma_3 \rangle,$$
$$\langle \sigma_1, \sigma_2 \rangle \cap \langle \sigma_2, \sigma_3 \rangle = \langle \sigma_2 \rangle. \tag{4}$$

Conversely, if a group $\Lambda = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ satisfies (3) and (4), then there exists a chiral or directly regular 4-polytope $\mathcal{P} = \mathcal{P}(\Lambda)$ of type $\{p_1, p_2, p_3\}$. We refer to [22] or [25, Theorem 1] for the details of the construction. The directly regular case occurs if and only if Λ admits an involutory automorphism ρ such that $(\sigma_1)\rho = \sigma_1^{-1}$, $(\sigma_2)\rho = \sigma_1^2\sigma_2$ and $(\sigma_3)\rho = \sigma_3$.

A chiral polytope $\mathcal{P}$ can be self-dual in two subtly different ways (see [16] or [26, §3]). $\mathcal{P}$ is *properly* self-dual if it admits a polarity δ which reverses the base flag Φ and so preserves the two flag orbits. In $D(\mathcal{P})$ we then have $\delta^2 = \epsilon$ and $\delta\sigma_j\delta = \sigma_{4-j}^{-1}$, for $j = 1, 2, 3$. In contrast, $\mathcal{P}$ is *improperly* self-dual if there exists a duality δ which exchanges the two flag orbits. In fact, we can choose δ so that $\delta^2 = \sigma_1\sigma_2\sigma_3$ (so δ has period 4); and $\delta^{-1}\sigma_1\delta = \sigma_3^{-1}$, $\delta^{-1}\sigma_2\delta = \sigma_1\sigma_2\sigma_1^{-1}$, $\delta^{-1}\sigma_3\delta = \sigma_1$.

In Theorem 5 of [22] we find that $\mathcal{G}$ is 2-transitive when $\mathcal{P}$ is finite, chiral and self-dual of type $\{3, q, 3\}$; more specifically, $\mathcal{G}$ is then of type 2^+ (respectively 2^-) if and only if $\mathcal{P}$ is properly (respectively improperly) self-dual.

In the above results, the self-duality of the polytope $\mathcal{P}$ serves as a natural guarantee that the medial layer graph $\mathcal{G}$ be vertex-transitive. Now ignoring duality, it is quite clear from the symmetry of $\mathcal{P}$ that $\mathrm{Aut}(\mathcal{G})$ is transitive on the edges of $\mathcal{G}$, and separately, at least, is also transitive on t-arcs of types 1 or 2, for some $t \geqslant 2$. We thus ask whether $\mathrm{Aut}(\mathcal{G})$ can be transitive on all such t-arcs, thereby making $\mathcal{G}$ *symmetric*, even when $\mathcal{P}$ is *not* self-dual. In fact, we shall see that this cannot happen, and so we make the following

Definition 2. A finite regular graph $\mathcal{G}$ is *semisymmetric* if $\mathrm{Aut}(\mathcal{G})$ acts transitively on the edges of $\mathcal{G}$ but not transitively on the vertices of $\mathcal{G}$.

Remarks. To be quite clear about terminology, we recall that a 'regular' graph has all vertices of some fixed degree k. Semisymmetric graphs are a little elusive and hence of considerable interest. The so-called *Gray graph* is the earliest known example of a trivalent semisymmetric graph; see [1], [2,3], or [18,24] for neat descriptions, and [8] for another interesting 'small' example. A census of such graphs, with at most 768 vertices, appears in [9].

It is easy to check that a connected, semisymmetric graph $\mathcal{G}$ is bipartite, say with vertices of types 1 and 2. In analogy to the symmetric case, we define $\mathcal{G}$ to be (t_1, t_2)-*semitransitive* if, for $j = 1, 2$, $\mathrm{Aut}(\mathcal{G})$ is transitive on t_j-arcs emanating from vertices of type j (but of course not transitive on longer such arcs). In brief, we say then that $\mathcal{G}$ is *ss of type* (t_1, t_2). The theory of such graphs seems to be largely uncharted, although it was proved in [27] that each $t_j \leqslant 7$. A further generalization is the notion of a *locally s-arc transitive graph*; see [13] for a detailed survey, or [14,15] for more specific investigations. We note that the 's-arc transitivity' discussed in the papers just cited has a more general meaning than that employed here.

In the next section we develop some machinery for manufacturing semisymmetric trivalent graphs from non-self-dual regular or chiral 4-polytopes of type $\{3, q, 3\}$.

2. Vertex-transitive medial layer graphs

We begin by letting $\mathcal{P}$ be a regular polytope of type $\{3, q, 3\}$, with medial layer graph $\mathcal{G}$. Theorem 2 below characterizes the case in which $\mathrm{Aut}(\mathcal{G})$ is vertex-transitive.

Theorem 2. *Suppose that $\mathcal{P}$ is a finite regular 4-polytope of type $\{3, q, 3\}$ with medial layer graph $\mathcal{G}$. Then if $\mathcal{G}$ is vertex-transitive, $\mathcal{G}$ must actually be 3-transitive and $\mathcal{P}$ must be self-dual.*

Proof. Suppose that $\mathcal{G}$ is vertex-transitive. Then $\mathcal{G}$ must be transitive on 3-arcs. In fact, $\mathrm{Aut}(\mathcal{G})$ is already known to be transitive on the 3-arcs of each type, and any element of $\mathrm{Aut}(\mathcal{G})$ which maps a vertex x of $\mathcal{G}$ to a vertex s of different type must necessarily also map a 3-arc with initial vertex x to a 3-arc of the other type, with initial vertex s.

Next we show that $\mathrm{Aut}(\mathcal{G})$ is actually sharply transitive on 3-arcs, that is, $\mathcal{G}$ is 3-transitive. We need to exclude the possibility that $\mathcal{G}$ is t-transitive for $t = 4$ or 5. In the notation of the previous section, we now have $\mathrm{Aut}(\mathcal{P}) = \langle \rho_0, \ldots, \rho_3 \rangle$. It is also useful to specify a few more vertices in Fig. 1: let $x := (v_{-1})\rho_3$ and y to be the two other vertices adjacent to w_1, and likewise let $s := (v_4)\rho_0$ and z be the two others adjacent to w_2.

The case $t = 4$ can be ruled out as in [22, Theorem 2], using the fact that the stabilizer B_4 of a vertex in a finite connected 4-transitive trivalent graph must be isomorphic to $\mathbb{S}_4$. In fact, the element $\eta := \rho_0 \rho_2 \rho_3$ in $\mathrm{Aut}(\mathcal{P})$ is an automorphism of $\mathcal{G}$ that stabilizes the vertex $v_1 = F_1$ of $\mathcal{G}$ and has order 6; and it permutes the vertices at distance 2 from v_1 in the 6-cycle $(x w_0 v_3 y v_{-1} w_2)$. However, $\mathbb{S}_4$ does not contain an element of order 6.

The elimination of the case $t = 5$ is more elaborate. When $t = 5$, the stabilizer $B_5(v_1)$ of the vertex $v_1 = F_1$ of $\mathcal{G}$ in $\mathrm{Aut}(\mathcal{G})$ must be isomorphic to $\mathbb{S}_4 \times \mathbb{Z}_2$. However, the stabilizer of v_1 in $\mathrm{Aut}(\mathcal{P})$ is just the subgroup $\langle \rho_0, \rho_2, \rho_3 \rangle \cong \mathbb{S}_3 \times \mathbb{Z}_2$. We claim that ρ_0 is the central element of $B_5(v_1)$, determining the factor $\mathbb{Z}_2$. In fact, viewing $\mathbb{S}_4 \times \mathbb{Z}_2$ as the symmetry group $[4, 3]$ of the 3-cube $\{4, 3\}$, we observe that its only subgroups of type $\mathbb{S}_3$ are those that fix a vertex of the cube, and that the central inversion is the only non-trivial element in $[4, 3]$ that commutes with a subgroup of this kind. Hence ρ_0, which determines the factor $\mathbb{Z}_2$ in $\mathbb{S}_3 \times \mathbb{Z}_2$, is the central element of $B_5(v_1)$.

Since the vertex-stabilizers in $\mathrm{Aut}(\mathcal{G})$ are all conjugate, we similarly obtain that ρ_1, ρ_2 and ρ_3 are the central elements in the stabilizers of the vertices w_2, w_1 and $v_2 = F_2$, respectively, denoted by $B_5(w_2)$, $B_5(w_1)$ and $B_5(v_2)$ (see Fig. 1). Now consider an element δ in $\mathrm{Aut}(\mathcal{G})$ which maps the 3-arc $[w_1, v_1, v_2, w_2]$ to the reversed 3-arc $[w_2, v_2, v_1, w_1]$. Since the ρ_j's are distinguished as central elements of their respective vertex-stabilizers, we must therefore have $\delta^{-1} \rho_j \delta = \rho_{3-j}$ for $j = 0, 1, 2, 3$. Suppose for a moment that δ is an involution. Then it follows that conjugation by δ in $\mathrm{Aut}(\mathcal{G})$ induces an involutory group automorphism of $\mathrm{Aut}(\mathcal{P})$, so that necessarily $\mathcal{P}$ is self-dual (see [23, 2E12]), contrary to our assumption that $t = 5$. (Recall from [22, Theorem 2] that the medial layer graph of $\mathcal{P}$ must be 3-transitive if $\mathcal{P}$ is self-dual.)

It remains to prove that we may take δ to be an involution. First observe that δ^2 belongs to the pointwise stabilizer of the 3-arc $[w_1, v_1, v_2, w_2]$, which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ (Theorem 1(b)). In all cases, $\delta^4 = \epsilon$ and $x\delta^2 = x$ or y. If $x\delta^2 = x$, then $(x\delta)\delta^2 = x\delta$, so that δ^2 fixes the 5-arc $[x, w_1, v_1, v_2, w_2, x\delta]$. Then $\delta^2 = \epsilon$ as desired. Otherwise, $x\delta^2 = y$ and $y\delta^2 = x$. Consider the unique automorphism γ of $\mathcal{G}$ which fixes the 4-arc $[x, w_1, v_1, v_2, w_2]$ pointwise but interchanges s and z (i.e. $x\delta$ and $y\delta$). Then $\gamma\delta$ reverses the 3-arc $[w_1, v_1, v_2, w_2]$, so we may replace δ by $\gamma\delta$. But

$$x(\gamma\delta)^2 = x\delta\gamma\delta = y\delta^2 = x,$$

so now $\gamma\delta$ is the desired involution.

It follows that $\mathcal{G}$ is 3-transitive. Now we apply the methods of [22, §4]. In particular, associated with $\mathcal{G}$ is a certain subgroup Γ of $\mathrm{Aut}(\mathcal{G})$ with a canonically defined set of four involutory generators (see [22, Definition 3]), and this subgroup Γ is the automorphism group of a certain self-dual ranked partially ordered set (see [22, Theorem 3]). In the present context we can actually identify the generators of Γ with the generators ρ_j for $\mathrm{Aut}(\mathcal{P})$ (and hence Γ with $\mathrm{Aut}(\mathcal{P})$), and then also the new partially ordered set with $\mathcal{P}$ itself. Thus $\mathcal{P}$ is self-dual. This completes the proof.   □

The situation for chiral polytopes is quite similar. We give fewer details in the proof, which relies more closely on ideas used in establishing [22, Theorem 5].

Theorem 3. *Suppose that $\mathcal{P}$ is a finite chiral 4-polytope of type $\{3, q, 3\}$ with medial layer graph $\mathcal{G}$. Then if $\mathcal{G}$ is vertex-transitive, $\mathcal{G}$ must actually be 2-transitive and $\mathcal{P}$ must be self-dual.*

Proof. Let $\mathcal{G}$ be vertex-transitive. First observe that $\mathcal{G}$ is transitive on 2-arcs. In fact, $\mathrm{Aut}(\mathcal{P})$ (and hence $\mathrm{Aut}(\mathcal{G})$) is transitive on the 2-arcs of each type, and the vertex-transitivity allows us again to swap the two kinds of 2-arcs. It follows that $\mathcal{G}$ is t-transitive for $t = 2, 3, 4$ or 5. We must establish that $t = 2$. We now have $\mathrm{Aut}(\mathcal{P}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$.

Suppose first that $t = 3$. We apply the methods of [22, §4] to prove that $\mathcal{P}$ must actually be regular, not chiral. In fact, because $\mathcal{G}$ is 3-transitive, we again have a subgroup Γ of $\mathrm{Aut}(\mathcal{G})$ with canonically defined generators $\rho_0, \rho_1, \rho_2, \rho_3$ (see [22, Definition 3]). Consulting [22, Lemma 1] and its proof we find that the products $\rho_0\rho_1, \rho_1\rho_2, \rho_2\rho_3$ can be identified with the generators $\sigma_1, \sigma_2, \sigma_3$ of $\Gamma(\mathcal{P})$ acting on $\mathcal{G}$. (It is crucial here that $\mathcal{G}$ be 3-transitive.) Moreover, the self-dual "regular" ranked poset (with a flag-transitive action) associated with Γ as in [22, Theorem 3] is actually isomorphic to $\mathcal{P}$. In fact, this poset can be defined completely in terms of the generators $\rho_0\rho_1, \rho_1\rho_2, \rho_2\rho_3$ of the "rotation subgroup" Γ^+ of Γ (see [25, p. 510]), that is, in terms of the generators $\sigma_1, \sigma_2, \sigma_3$ of $\mathrm{Aut}(\mathcal{P})$. However, the poset associated with $\sigma_1, \sigma_2, \sigma_3$ is just $\mathcal{P}$ itself. Hence $\mathcal{P}$ must be regular. It follows that we cannot have $t = 3$.

To rule out the cases $t = 4, 5$ we mimic part of the proof of [22, Theorem 5], which utilized certain universal relations satisfied by generators of $\mathrm{Aut}(\mathcal{G})$, as described in [6, §1]. In each case it is impossible to achieve $(\sigma_2\sigma_3)^2 = \epsilon$, given the other relations in (3). (We note that Theorem 5 of [22] has almost the same hypotheses as here, except that $\mathcal{P}$ is there assumed to be self-dual; but this self-duality is used only to guarantee that the medial layer graph $\mathcal{G}$ be vertex-transitive.)

Thus we must have $t = 2$. Now, as in the proof of [22, Theorem 6], the sharp transitivity of $\mathrm{Aut}(\mathcal{G})$ on 2-arcs enables a definition of a duality on $\mathcal{P}$, whether $\mathcal{G}$ is of type 2^+ or 2^-: see [22, Eqs. (7) and (13)].   □

3. Graphs from polytopes of type $\{3, q, 3\}$

There is a wealth of finite trivalent semisymmetric graphs that are medial layer graphs of regular or chiral polytopes $\mathcal{P}$ of type $\{3, 6, 3\}$. Necessarily, by Theorems 2 and 3, $\mathcal{P}$ must not be self-dual. However, before exploring such polytopes we must first review some key constructions.

For any pair $\mathbf{s} = (s, t)$ of integers satisfying $s^2 + st + t^2 > 1$, the *toroidal map* $\{3, 6\}_\mathbf{s}$ has the structure of a finite 3-polytope (or polyhedron), usually chiral, but regular just when $st(s - t) = 0$. Referring to [23, 1D], we merely note here that $\{3, 6\}_\mathbf{s}$ is obtained from the regular triangular tessellation $\{3, 6\}$ of the Euclidean plane by factoring out a suitable subgroup of the group of translation symmetries. Taking $v = s^2 + st + t^2$, we find that $\{3, 6\}_\mathbf{s}$ has v vertices, $3v$ edges,

$2v$ triangular facets and a rotation group $\langle \sigma_1, \sigma_2 \rangle$ of order $6v$. The toroidal map $\{6, 3\}_{\mathbf{s}}$ can be constructed similarly and is dual to $\{3, 6\}_{\mathbf{s}}$, both as a map on a compact surface and as an abstract polyhedron.

In any regular (or chiral) n-polytope $\mathcal{P}$, all facets are isomorphic to a particular $(n - 1)$-polytope, say $\mathcal{M}$; likewise each *vertex-figure* $\mathcal{N}$ (maximal section over a vertex in $\mathcal{P}$) is isomorphic to one $(n - 1)$-polytope $\mathcal{N}$. Conversely, given regular $(n - 1)$-polytopes $\mathcal{M}, \mathcal{N}$, there may or may not exist a regular n-polytope $\mathcal{P}$ with facets $\mathcal{M}$ and vertex-figures $\mathcal{N}$; but if one such polytope exists, then there is a *universal polytope* of this type, denoted

$$\{\mathcal{M}, \mathcal{N}\},$$

and from which all others are obtained by identifications [23, 4A]. Somewhat more intricate results like this hold for chiral polytopes [25,26].

3.1. Medial layer graphs of finite universal polytopes

Rephrasing the introductory remarks above, we observe that every (finite) regular polytope $\mathcal{P}$ of type $\{3, 6, 3\}$ has certain facets $\{3, 6\}_{\mathbf{s}}$ and vertex-figures $\{6, 3\}_{\mathbf{t}}$, with $\mathbf{s} = (s^k, 0^{2-k})$, $\mathbf{t} = (t^l, 0^{2-l})$; here, $s \geqslant 2$ if $k = 1$ and $s \geqslant 1$ if $k = 2$; likewise $t \geqslant 2$ if $l = 1$ and $t \geqslant 1$ if $l = 2$. In particular, $\mathcal{P}$ is a quotient of the (generally infinite) universal regular 4-polytope

$$\mathcal{P}_{\mathbf{s},\mathbf{t}} := \left\{ \{3, 6\}_{\mathbf{s}}, \{6, 3\}_{\mathbf{t}} \right\}.$$

(See [23, Section 11E] for details. In some cases, the only available construction for $\mathcal{P}_{\mathbf{s},\mathbf{t}}$ is via the corresponding string C-group, which in turn is naturally defined by a presentation encoding the local structure of the polytope. We can expect no simple expression for the order of the group.)

For certain small parameter values these universal polytopes are known to be finite; however, the finite polytopes $\mathcal{P}_{\mathbf{s},\mathbf{t}}$ have not yet been completely enumerated. Clearly, if $\mathbf{s} \neq \mathbf{t}$, then $\mathcal{P}$ cannot be self-dual and hence its medial layer graph is semisymmetric.

We list in Table 1 data for the medial layer graphs $\mathcal{G}_{\mathbf{s},\mathbf{t}}$ of those universal polytopes $\mathcal{P}_{\mathbf{s},\mathbf{t}}$ which are known to be finite; in the last column we use 'ss-(t_1, t_2)' or '3^+,' respectively, to indicate that $\mathcal{G}_{\mathbf{s},\mathbf{t}}$ is semisymmetric of type (t_1, t_2) or 3-transitive. (The type of the last semisymmetric graph with 40,320 vertices seems to be beyond brute force calculation in GAP [12], for example.) Recall that N is the number of vertices.

When $\mathbf{s} = \mathbf{t}$, the universal polytope $\mathcal{P}_{\mathbf{s},\mathbf{t}}$ is self-dual and generally has many self-dual quotients. For example, when the standard representations of the crystallographic Coxeter groups $[3, 6, 3]$ and $[3, \infty, 3]$ are reduced modulo an odd prime p, we obtain interesting self-dual (in one case, non-self-dual) regular polytopes of types $\{3, 6, 3\}$ or $\{3, p, 3\}$, respectively, with automorphism groups isomorphic to finite reflection groups over the finite field $\mathbb{Z}_p$ (see [20, (28), (31)]).

Table 1
The medial layer graphs of the known finite polytopes $\mathcal{P}_{\mathbf{s},\mathbf{t}}$

s	**t**	N	Transitivity type
$(1, 1)$	$(1, 1)$	18	3^+
$(1, 1)$	$(3, 0)$	54	ss-$(4, 3)$
$(2, 0)$	$(2, 0)$	40	3^+
$(2, 0)$	$(2, 2)$	120	ss-$(3, 3)$
$(3, 0)$	$(3, 0)$	486	3^+
$(3, 0)$	$(2, 2)$	6912	ss-$(3, 3)$
$(3, 0)$	$(4, 0)$	40,320	ss-$(?, ?)$

The exception occurs for [3, 6, 3] with $p = 3$, yielding the non-self-dual polytope $\mathcal{P}_{(1,1),(3,0)}$, whose medial layer graph is the Gray graph (see Section 3.4). All other polytopes obtained by this construction have finite trivalent symmetric graphs as medial layer graphs. In particular, when $p > 3$, the polytopes obtained from [3, 6, 3] have facets $\{3, 6\}_{(p,0)}$ and vertex-figures $\{6, 3\}_{(p,0)}$ and hence are quotients of $\mathcal{P}_{(p,0),(p,0)}$.

3.2. Non-constructive methods

Even if the universal polytope $\mathcal{P}_{\mathbf{s},\mathbf{t}}$ of Section 3.1 is not finite, we often can still establish the existence of semisymmetric medial layer graphs through non-constructive methods by appealing to [23, Theorem 4C4]. Recall that a group Γ is *residually finite* if, for each finite subset of $\Gamma \setminus \{\epsilon\}$, there exists a homomorphism of Γ onto a finite group such that no element of the subset is mapped to the identity element.

Suppose that Q is an infinite regular 4-polytope with facets $\{3, 6\}_{\mathbf{s}}$ and vertex-figures $\{6, 3\}_{\mathbf{t}}$, whose group $\Gamma(Q)$ is residually finite. Then [23, Theorem 4C4], applied with $\mathcal{P}_1 = \{3, 6\}_{\mathbf{s}}$ and $\mathcal{P}_2 = \{6, 3\}_{\mathbf{t}}$, says that there are infinitely many finite regular 4-polytopes with facets $\{3, 6\}_{\mathbf{s}}$ and vertex-figures $\{6, 3\}_{\mathbf{t}}$, which are quotients of Q. When $\mathbf{s} \neq \mathbf{t}$, these polytopes yield trivalent semisymmetric graphs.

Such polytopes Q are known to exist at least for certain parameter values, including $\mathbf{s} = (s, s)$ and $\mathbf{t} = (s, 0)$ or $(3s, 0)$, with $s \geqslant 2$ (but excluding the pair $\mathbf{s} = (2, 2)$ and $\mathbf{t} = (2, 0)$). In fact, inspection of the methods employed in the proof of [23, Theorem 11E5] reveals the existence of certain infinite regular 4-polytopes Q with facets $\{3, 6\}_{\mathbf{s}}$ and vertex-figures $\{6, 3\}_{\mathbf{t}}$, whose group $\Gamma(Q)$ is a semi-direct product of an infinite, finitely generated, 4-dimensional complex linear group by a small group ($\mathbb{S}_3$, in fact); then $\Gamma(Q)$ itself also is a complex linear group, in a space of dimension larger than 4 (see [23, pp. 415–416]). By a theorem of Malcev [17], every finitely generated linear group is residually finite. Thus $\Gamma(Q)$ is residually finite.

In summary, we obtain the following

Theorem 4. *Let $s \geqslant 2$, and let $\mathbf{s} := (s, s)$ and $\mathbf{t} := (s, 0)$ or $(3s, 0)$, but excluding the pair $\mathbf{s} = (2, 2)$ and $\mathbf{t} = (2, 0)$. Then there are infinitely many finite trivalent semisymmetric graphs which are medial layer graphs of finite regular polytopes with facets $\{3, 6\}_{\mathbf{s}}$ and vertex-figures $\{6, 3\}_{\mathbf{t}}$.*

As a final application of these methods, we mention a similar such theorem for symmetric graphs.

Theorem 5. *For each $q \geqslant 5$, there are infinitely many finite, trivalent symmetric (indeed 3-transitive) graphs which are medial layer graphs of finite self-dual regular polytopes of type $\{3, q, 3\}$.*

Proof. The Coxeter group $[3, q, 3]$ is the automorphism group of the self-dual universal regular polytope $\mathcal{P} := \{3, q, 3\}$. In particular, $D(\mathcal{P}) \cong [3, q, 3] \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ is generated by the polarity δ that fixes the base flag of $\mathcal{P}$ (δ corresponds to the symmetry of the string Coxeter diagram). Hence $D(\mathcal{P})$ is residually finite, since $[3, q, 3]$ is residually finite. Now adapt the proofs of [23, Theorem 4C4] and [23, Corollary 4C5], applying Malcev's theorem to $D(\mathcal{P})$ in place of Aut($\mathcal{P}$), and requiring that δ does not become trivial under the homomorphisms onto finite groups. Then the latter guarantees the self-duality of the resulting quotients of $\mathcal{P}$; hence their medial layer graphs are symmetric. $\quad\square$

3.3. Polytopes and graphs from the Eisenstein integers

Next we consider from [21, §6] a family of regular or chiral polytopes $\mathcal{Q}_m^A$, again of type $\{3, 6, 3\}$. Here, the parameter m is chosen from $\mathbb{D} := \mathbb{Z}[\omega]$, the domain of Eisenstein integers. (Recall that $\omega = e^{2\pi i/3}$ is a primitive cube root of unity.) The construction begins with a certain group H_m of 2×2 matrices over the residue class ring $\mathbb{D}_m := \mathbb{D}/(m)$; and then any subgroup A of the unit group of $\mathbb{D}_m$, with $-1 \in A$, is said to be *admissible*. Without going into many details, we note simply that the rotation group $H_m^A = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ for $\mathcal{Q}_m^A$ is obtained from the matrix group by factoring out the subgroup consisting of scalar multiples of the identity, with scalars from A. Thus $\mathcal{Q}_m^A$ is finite when $m \neq 0$. On the other hand, $H_0^{\pm 1}$ is the infinite rotation group for $\mathcal{Q}_0^{\pm 1}$, which is isomorphic to the regular honeycomb $\{3, 6, 3\}$ of hyperbolic space $\mathbb{H}^3$.

If the Eisenstein prime $1 - \omega$ does not divide m, the polytope will be self-dual. Interesting as it is, we leave this case behind (see [22]). Suppose therefore that

$$m = (1 - \omega)^e d,$$

where $e \geqslant 1$ and $d \in \mathbb{D} \setminus \{0\}$. To avoid degeneracies, we also assume that d is a non-unit if $e = 1$. It follows from [21, Theorem 6.1] that $\mathcal{Q}_m^A$ is a finite quotient of the universal polytope

$$\left\{ \{3, 6\}_{(c,b)}, \{6, 3\}_{\left(\frac{c-b}{3}, \frac{c+2b}{3}\right)} \right\},$$

where $m = c - b\omega$, for certain $b, c \in \mathbb{Z}$. Since the facets of $\mathcal{Q}_m^A$ are clearly not dual to its vertex-figures, $\mathcal{Q}_m^A$ itself cannot be self-dual. Furthermore, $\mathcal{Q}_m^A$ is regular if $m \mid \bar{m}$ and $A = \bar{A}$ (i.e. the scalar subgroup is invariant under complex conjugation); otherwise, $\mathcal{Q}_m^A$ is chiral. Consequently, by Theorems 2 and 3 above, we obtain a trivalent, semisymmetric medial layer graph $\mathcal{G}_m^A$ with

$$N = 2 \left[\frac{(m\bar{m})^3}{12 \cdot |A|} \prod_{\pi \mid m} \left(1 - (\pi \bar{\pi})^{-2} \right) \right]$$

vertices. (The product here is over all *non-associated* prime divisors π of m.) We can summarize our construction in the following

Theorem 6. *Suppose the Eisenstein integer m satisfies $m\bar{m} = 3k$, for some rational integer $k > 1$; and let A be any admissible group of scalars. Then $\mathcal{G}_m^A$ is a finite, trivalent semisymmetric graph.*

Remarks. When $m = 3 = (1 - \omega)^2(-\omega^2)$ and $A = \{\pm 1\}$, we get the dual of the universal polytope $\{\{3, 6\}_{(1,1)}, \{6, 3\}_{(3,0)}\}$ mentioned in Section 3.1 above. The medial layer graph $\mathcal{G}_3^{\pm 1}$ is the Gray graph, which we examine more closely below. (For easier reading we omit the brackets from $\{\pm 1\}$.) Similarly, for $m = 2(1 - \omega)$ we find that $\mathcal{Q}_{2-2\omega}^{\pm 1}$ is the dual of the universal polytope $\{\{3, 6\}_{(2,0)}, \{6, 3\}_{(2,2)}\}$ described in Section 3.1. The medial layer graph has 120 vertices. According to the census in [9], we have thus described the unique trivalent semisymmetric graphs with these orders.

We note that $\mathcal{Q}_m^A$ itself is not usually the universal polytope for the specified toroidal facets and vertex-figures. Certainly we get a proper quotient of the universal cover when $|A| > 2$, which is possible when m has distinct prime divisors. In any case, the scalar group A has order 2^a and depends in an intricate way on the prime factorization of m in $\mathbb{D}$; see [21, pp. 105–106].

3.4. The Gray graph

The Gray graph $\mathcal{C}$ is the smallest trivalent, semisymmetric graph (see [9]). Following [3], we define $\mathcal{C}$ to be the (bipartite) incidence graph of cubelets and columns in a $3 \times 3 \times 3$ cube. Thus vertices of the first type are the 27 cubelets; and vertices of the second type are the $9 + 9 + 9$ columns of 3 cubelets parallel to edges of the cube. It is not hard to check that $|\text{Aut}(\mathcal{C})| = 1296$ [3, Theorem 1.1]. Recent work has concerned various interesting features of the graph [18,19]; and here, of course, we construct it in a new way.

Before confirming that $\mathcal{G}_3^{\pm 1}$ *really is* isomorphic to $\mathcal{C}$, we develop a more concrete geometric description. First of all, using GAP it is easy to check that $|\text{Aut}(\mathcal{G}_3^{\pm 1})| = 1296$. Somewhat unexpectedly this is 4 times the order of $\text{Aut}(\mathcal{Q}_3^{\pm 1})$. This discrepancy hints that we might examine a related embedding of the honeycomb $\{3, 6, 3\}$ into a different hyperbolic honeycomb $\{3, 3, 6\}$.

Let us pick an arbitrary tetrahedral tile T of $\{3, 3, 6\}$ and denote its center by F_3. In Fig. 2, F_0 is a vertex of T (and is an ideal point on the sphere at infinity); let F_1 be the center of an edge of T through F_0 and F_2 the center of a triangle of T with that edge. The points F_i ($i = 0, 1, 2, 3$) are the vertices of a fundamental region for the hyperbolic Coxeter group $[3, 3, 6]$. Thus, taking ρ_i to be the reflection in the face opposite F_i in this fundamental region, we have $[3, 3, 6] = \langle \rho_0, \rho_1, \rho_2, \rho_3 \rangle$.

Now let E_3 be the vertex of the tetrahedron T which does not belong to the triangle centered at F_2. Then F_0, F_1, F_2 and E_3 are vertices of a new fundamental region for the hyperbolic Coxeter group $[3, 6, 3]$, appearing here as the subgroup of $[3, 3, 6]$ generated by

$$\rho_0, \ \rho_1, \ \rho_2' := \rho_2 \rho_3 \rho_2 \ \text{and} \ \rho_3.$$

In fact, $[3, 6, 3]$ has index 4 in $[3, 3, 6]$ (see [23, 11G]).

Through the edge containing F_1 there are six triangles of $\{3, 3, 6\}$, but only three of them belong to the honeycomb $\{3, 6, 3\}$, as we indicate in Fig. 3. Hence, the vertices of the medial
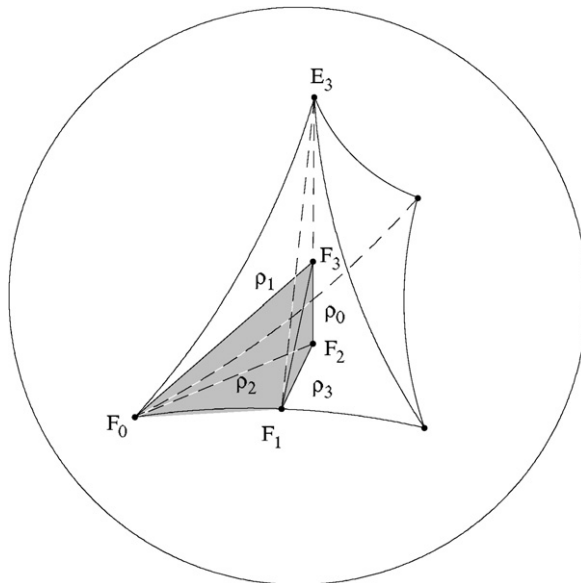

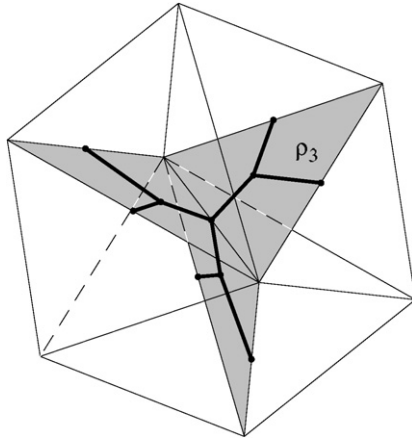Fig. 2. A tetrahedral tile in the hyperbolic honeycomb $\{3, 3, 6\}$.

Fig. 3. The medial layer graph of $\{3, 6, 3\}$ inscribed in $\{3, 3, 6\}$.
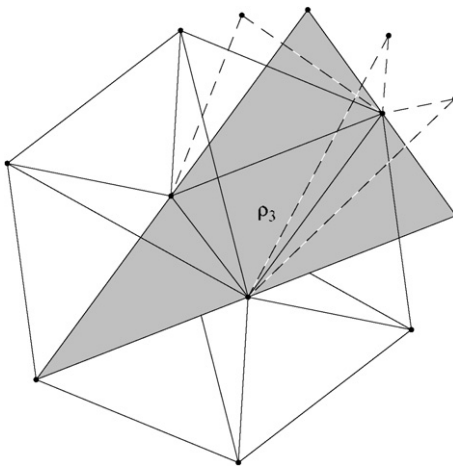

Fig. 4. The tessellation $\{3, \infty\}$ cutting through $\{3, 3, 6\}$.

layer graph of $\{3, 6, 3\}$ are comprised of just 'half' the edges and 'one quarter' of the triangles of $\{3, 3, 6\}$.

Now a typical 2-face of $\{3, 6, 3\}$ is the ideal triangle $\{3\}$ with vertex F_0, edge F_1 and center F_2 in the hyperbolic plane p which serves as the mirror for the reflection ρ_3. But p is perpendicular to the mirrors for reflections ρ_0, ρ_1 and $\rho := \rho_2' \rho_2 \rho_2' = \rho_2 (\rho_3 \rho_2)^2$ in $[3, 3, 6]$. Since the latter two mirrors are parallel at F_0, we see that $\langle \rho_0, \rho_1, \rho \rangle \simeq [3, \infty]$. Hence the honeycomb $\{3, 3, 6\}$ cuts the plane p into the triangles of the regular tessellation $\{3, \infty\}$ (see Fig. 4).

At this point we recall from Section 3.3 that the polytopes $\mathcal{Q}_m^A$ ultimately arise from a modular representation of the rotation subgroup in the reflection group $[3, 6, 3]$ for $\{3, 6, 3\}$. A parallel construction based on the group $[3, 3, 6]$ is described in [21, §4], thereby yielding a family of polytopes $\mathcal{P}_m^A$ of type $\{\{3, 3\}, \{3, 6\}_{(b,c)}\}$. When $1 - \omega$ divides m, the rotation group H_m^A for
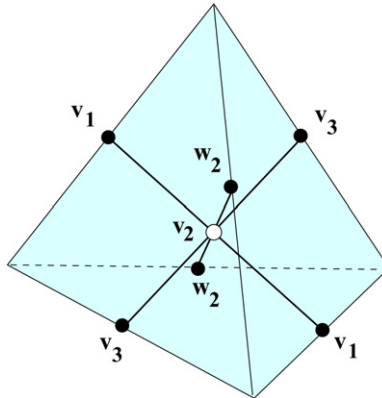
Fig. 5. The graph constructed from the 'medial tetrahedra.'

$\mathcal{Q}_m^A$ still has index 4 in the rotation group G_m^A for $\mathcal{P}_m^A$. In particular, when $m = 3$ we obtain the universal polytope

$$\mathcal{P}_3^{\pm 1} = \big\{ \{3, 3\}, \{3, 6\}_{(3,0)} \big\},$$

whose automorphism group has order 1296 [23, 11B5]. (Note that the rotation group for $\mathcal{Q}_3^{\pm 1}$ has order just 324.) To see that the larger group of order 1296 is the automorphism group for the Gray graph, we must consider how reduction modulo $m = 3$ affects the picture in $\mathbb{H}^3$.

In identifying vertices of $\{3, 3, 6\}$ to obtain the universal polytope $\mathcal{P}_3^{\pm 1}$, we note that $\rho_1 \rho = \rho_1 \rho_2 (\rho_3 \rho_2)^2$ has order 3, so that the tessellation $\{3, \infty\}$ in the plane p collapses to a 'medial tetrahedron' $\{3, 3\}$. At the same time, the 'inscribed' honeycomb $\{3, 6, 3\}$ collapses to $\mathcal{Q}_3^{\pm 1}$ and its medial layer graph to $\mathcal{G}_3^{\pm 1}$.

Now working in $\mathcal{P}_3^{\pm 1}$, we define a graph $\mathcal{M}$ whose vertex set consists of all $27 = 1296/(2 \cdot 24)$ of the medial tetrahedra from $\mathcal{P}_3^{\pm 1}$, together with all $27 = (27 \cdot 3)/3$ pairs of opposite edges from such tetrahedra. A vertex representing a tetrahedron is adjacent to a vertex representing a pair of edges whenever the tetrahedron contains the edges (see Fig. 5). Evidently, $\mathcal{G}_3^{\pm 1}$ is isomorphic to $\mathcal{M}$ and so inherits all 1296 automorphisms of $\mathcal{P}_3^{\pm 1}$. In other words,

$$\text{Aut}\big(\mathcal{G}_3^{\pm 1}\big) \cong \text{Aut}\big(\mathcal{P}_3^{\pm 1}\big).$$

(This group appears as $[112]^3 \rtimes \mathbb{Z}_2$ in [23, 11B5].)

Now it is a simple matter to identify $\mathcal{G}_3^{\pm 1} \cong \mathcal{M}$ with the Gray graph $\mathcal{C}$: take the medial tetrahedra to be the columns in the $3 \times 3 \times 3$ cube, and pairs of opposite edges to be the cubelets themselves. But each edge of $\{3, 6, 3\}$ belongs to three facets $\{3, 6\}$; so after reducing modulo $m = 3$, each of our 27 pairs of edges must lie on three distinct medial tetrahedra.

We note finally that it is not at all clear, for general moduli $m = (1 - \omega)^e d$, just how large $\text{Aut}(\mathcal{G}_m^A)$ is relative to its subgroup $\text{Aut}(\mathcal{Q}_m^A)$.

For example, when $m = 2(1 - \omega)$, $\mathcal{Q}_{2(1-\omega)}^{\pm 1}$ is a regular polytope whose full reflection group of order 720 is isomorphic to $\text{Aut}(\mathcal{G}_{2(1-\omega)}^{\pm 1})$. A similar isomorphism holds for the chiral polytope obtained when $m = (1 - \omega)(1 + 3\omega)$. But when $m = 3(1 - \omega)$, the reflection group $\text{Aut}(\mathcal{Q}_{3(1-\omega)}^{\pm 1})$ once more has index 4 in the $\text{Aut}(\mathcal{G}_{3(1-\omega)}^{\pm 1})$, whose order is 34,992.

Based on this flimsy evidence, we conjecture that the index is always 4 whenever $m = (1 - \omega)^e$, for $e \geqslant 2$.

Some history and words of thanks

At this point we happily thank Izak Bouwer for several comments concerning the provenance of the Gray graph. In 1968, Izak gave the first published description [2]. A year later, in private correspondence with him, Dr. Marion C. Gray (1902–?) wrote that she had encountered the graph while investigating 'completely symmetric networks.' (This happened about 1932, early in her career at Bell Labs.) In fact, with some uncertainty, Dr. Gray even attributed the graph to R.D. Carmichael. Perhaps the configurations described in [5] were an inspiration.

It is also a pleasure to thank the referees for several suggestions and for pointing out related material in [13,27].

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