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# Duality between QCD perturbative series and power corrections

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ABSTRACT

We elaborate on the relation between perturbative and power-like corrections to short-distance sensitive QCD observables. We confront theoretical expectations with explicit perturbative calculations existing in literature. As is expected, the quadratic correction is dual to a long perturbative series and one should use one of them but not both. However, this might be true only for very long perturbative series, with number of terms needed in most cases exceeding the number of terms available. What has not been foreseen, the quartic corrections might also be dual to the perturbative series. If confirmed, this would imply a crucial modification of the dogma. We confront this quadratic correction against existing phenomenology (QCD (spectral) sum rules scales, determinations of light quark masses and of $\alpha_s$ from $\tau$-decay). We find no contradiction and (to some extent) better agreement with the data and with recent lattice calculations.



## 1. Introduction

Because of the asymptotic freedom, predictions for short-distance processes are very simple in QCD and essentially reduce to parton model, or to lowest order perturbation theory. This is true, however, only in the leading order approximation. As far as corrections are concerned, there is a double sum which includes expansion in $\alpha_s(Q^2)$ where $Q^2$ is a generic large mass parameter and powers of $(\Lambda_{\rm QCD}/Q)^k$. Consider for example the best studied case of current correlators which determine QCD sum rules [1] (for a review, see e.g. [2]). Then, one usually assumes the following form of the correlator in the $x$-space:

$$\langle 0|J(x), J(0)|0\rangle \approx C_I\big(\alpha_s(x)\big)I + C_{G^2}\big(\alpha_s(x)\big)G^2(0)x^4 + \cdots, \qquad (1)$$

where $J(x)$ is the hadronic current, $I$ is the unit operator and $G^2$ is the dimension four operator. The coefficient functions $C_{I,G^2}$ are calculable perturbatively as infinite sums in the running coupling.

Moreover, Eq. (1) does not apparently contain quadratic corrections, while such corrections are included in many cases on the phenomenological grounds (see in particular [3–9]). These quadratic corrections and their phenomenological significance will be in fact focus of our attention. Let us remind the reader what is understood by these corrections.

Start with the heavy quark potential at short distances. The Cornell version of this potential (which describes the lattice data very well) is very simple:

$$V_{Q\bar{Q}}(r) \approx -\frac{4}{3}\frac{\alpha_s}{r} + \sigma \cdot r, \qquad (2)$$

where $r$ is the distance, $\alpha_s \equiv g^2/(4\pi)$ is the QCD coupling, $\sigma \approx 0.2\ \text{GeV}^2$ is the string tension. The fit in Eq. (2) works well at all distances. The question is whether such a form of the potential at *short* distances – let it be only approximate – is acceptable theoretically. There are papers which ascertain a positive answer to this question (see, in particular, [3,4]). The observable (heavy-quark potential in our case) is viewed as represented by a short perturbative series (a single *const*/$r$ term in our case) plus a leading power correction (quadratic correction, in our case, $\sigma \cdot r$).

The version used in some other papers (see, in particular, [10]) looks as:

$$\lim_{r\to 0} V_{Q\bar{Q}}(r) \approx \frac{1}{r}\sum_{n=1}^{n=4} a_n \alpha_s^n(r) + (const) + \tilde{\sigma}_n \cdot r, \qquad (3)$$

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where $n = 4$ is the realistic number of perturbative terms calculated explicitly and *const* stands for an infrared renormalon contribution (this could be added to the version in Eq. (2) as well). The last term, proportional to $\tilde{\sigma}_n$ imitates the power correction.

It is quite common [10] to identify the parameters $\sigma$ from Eq. (2) and $\tilde{\sigma}$ from Eq. (3) and compare their numerical values. Our point is that such an identification is not justified[1]:

*There are two dual descriptions: either one uses a short perturbative series and adds the leading quadratic correction by hand, or one uses long perturbative series and then there is no reason to add the quadratic correction.*

Numerically both Eqs. (2) and (3) work well. Chronologically, the papers in the series [3,4] appeared first. At that time, the common belief was that the Voloshin–Leutwyler potential is valid non-perturbatively. This would correspond to a cubic correction in Eq. (2) (or (3)). The papers [3,4] established validity of the unconventional (at that time) quadratic correction. The emphasis in later papers [10] was in fact on finding another interpretation to the already known quadratic power correction.

The problem of mixing between power-like corrections and perturbative series is not new at all. The standard view is that power corrections are related to divergences in perturbative series due to the factorial growth of the expansion coefficients (for review see, e.g., [12]). This viewpoint formulated long time ago still dominates theoretical thinking. In practice, however, no factorial growth of the expansion coefficients has been observed so far. The reason could be that the ability to calculate the expansion coefficients is limited and the series known explicitly are not *long* enough.

Here we come actually to a key point. Because in phenomenological applications, one usually assumes, explicitly or tacitly, that large-order asymptote sets in immediately after the terms known explicitly (see, e.g., [13,14] and references therein).

There exists, however, an example of a long perturbative series which allows to check the current ideas on the expansion coefficients. We have in mind the perturbative calculations of the gluon condensate [15–17]. This example indicates strongly that theorems on the asymptotic behaviour of the expansion coefficients might apply only in case of much longer series than are available in reality. Thus, we argue that, in realistic phenomenological fits, one should keep the quadratic corrections which are absent from the symbolic expansion in Eq. (1).

*Thus, our main point is that the properties of the relatively short perturbative series are different from properties of long perturbative series.*

Another new point is the impact of the dual models. We will argue, basing on the results of [18], that there exists another source of the quartic corrections, which are usually identified with the infrared-sensitive part of the gluon condensate $\langle G^2\rangle$. Namely, the same short-distance contributions which control the quadratic correction taken to second order, produce a *calculable* quartic correction. We confront this insight brought by the dual models with explicit perturbative calculations of papers in Refs. [15–17].

In Section 2, we discuss an argumentation in favor of the duality between the quadratic correction and long perturbative series. In Section 3, we emphasize lessons for the generic structure of perturbative series brought by the explicit calculations of the gluon condensate. In Section 4, we propose a simplified generic version of perturbative series. In Section 5, we summarize lessons brought by the holographic models. In Section 6, we discuss the unexpected duality between the perturbative and quartic power corrections. Section 7 is devoted to phenomenology of particular processes. In Section 8, we present our conclusions.

[1] The conjecture was made first in a conference talk [11]. Present, original paper also includes material submitted to other proceedings by the authors.

## 2. Duality expected (quadratic correction)

### 2.1. Duality between s- and t-channels

Because of the existing confusion in the literature concerning the duality between long perturbative series and quadratic correction, let us start with the notion of the duality itself.

Consider a hadronic reaction $a + b \to c + d$ at relatively low energies. Then the following representation of the amplitude can be reasonable:

$$A(a+b \to c+d) \approx (\text{nearest } s\text{-channel exchange}) + (\text{nearest } t\text{-channel exchange}). \tag{4}$$

Such a phenomenology was popular a few decades ago and turned successful.

Now, imagine that one starts improving Eq. (4) by summing up the $s$-channel exchanges:

$$A(a+b \to c+d) \approx \sum_{n=1}^{N \text{ particles}} (s\text{-channel exchange}) + (t\text{-channel exchange}), \tag{5}$$

where the sum over the $s$-channel resonances is taken.

Then, if $N$ is large enough one would notice that there is no more space for the $t$-channel exchanges. The conclusion could be that there are no $t$-channel particles or that they are decoupled from our hadrons $a$, $b$, $c$, $d$.

As everybody knows, beginning with the celebrated Veneziano's paper [19], such a conclusion would be wrong. Namely, if one uses sums over the resonances, then it is *either* $s$-channel *or* $t$-channel exchanges that are allowed but not both.

Similar things happen in the case of the quadratic corrections to the parton model (of which a linear potential is an example). One uses *either* a short perturbative series and adds a linear term by hand. This is an analogy to the nearest-singularity amplitude in Eq. (4) and corresponds to the form in Eq. (2). *Or* one uses a long perturbative series and then *does not* add by hand the linear term. Since it is already included into the perturbative series, by virtue of the general theorems inherent to the Yang–Mills theories. This is then the version in Eq. (3).

Thus, claiming that the parameter $\tilde{\sigma} \approx 0$ in Eq. (3) contradicts $\sigma \neq 0$ in Eq. (2) is like claiming that summing up the $s$-channel exchanges proves that there are no $t$-channel particles in nature.

### 2.2. Quadratic correction and OPE

The proof [20] that there are no genuine non-perturbative quadratic corrections is simple. Indeed, originally, the quadratic correction was associated with the so-called ultraviolet renormalon which corresponds to the following asymptotic series:

$$f(Q^2)_{\text{UV renorm.}} \sim \sum_{N_{cr}}^{\infty} \left[\alpha_s(Q^2)\right]^k (-1)^k k! (b_0)^k, \tag{6}$$

where $Q^2$ is a generic large mass parameter inherent to the problem and $b_0 \equiv \frac{1}{4\pi}(11 - (2/3)n_f)$ is the first coefficient in the $\beta$-function for $n_f$ flavours. Note that if one treats the expansion in Eq. (6) as an asymptotic series, then its uncertainty is a quadratic correction, $\Lambda^2_{\text{QCD}}/Q^2$. On the other hand, one can sum up the series à la Borel and then there is no uncertainty at all.

The crucial observation is that the factorial growth of the expansion coefficients in Eq. (6) are associated with an integration over very large momenta, $p^2 \gg Q^2$. However, because of the asymptotic freedom, this region of integration should not be a source of uncertainty in QCD. Indeed, by introducing a cut off $a$ and using the coupling $\alpha_s(a)$ normalized in the UV, one eliminates the integration over momenta $p^2 > a^{-2}$ and, therefore, there is no ambiguity of order $1/Q^2$ [20].

From the point of view of the operator product expansion (OPE) in Eq. (1), the quadratic correction we are discussing is hidden in the coefficient function in front of the unit operator, $C_I(\alpha_s(x))$ and, in no way, violates the OPE.

However, the QCD (spectral) sum rules were originally based [1] (for a review, see e.g. [2]) on a simplified assumption that the coefficient functions can be approximated by the first terms, while the effect of the confinement is encoded in the power corrections. It is only *within this terminology* that one might say that the form in Eq. (2) violates the OPE. In a more correct but longer language, what is violated is the assumption that the coefficient functions are approximated by their first terms. More advanced applications of the sum rules are keeping longer and longer perturbative series. Then, the terminology with 'violations of the OPE' due to the quadratic correction becomes obsolete.

Another source of confusion is the observation that, in the Euclidean space, one can ascribe a gauge-invariant meaning to the vacuum matrix element of dimension-two operator $(A^a_\mu)^2$ [21]. The quantity $\langle 0|(A^a_\mu)^2|0\rangle_{\min}$ turns to be of significant interest in many applications. This does not change of course the fact that $\langle 0|(A^a_\mu)^2|0\rangle_{\min}$ does not appear in the operator product expansion (1). None of the papers in Ref. [3] claims either violation of the OPE with 'long' perturbative expansions or appearance of $\langle 0|(A^a_\mu)^2|0\rangle_{\min}$ in the OPE equations. Nevertheless, sometimes one fights just with these would-be-made claims [10].

From the perspective of the perturbative expansion, the most difficult question is: why the quadratic correction could be at all important? Therefore, it is worth emphasizing that the quadratic correction is required by phenomenology, not yet by the theory. For example, on the lattice, one can give a definition of the 'non-perturbative' heavy quark potential (for review see e.g. [22]). Then, this potential is pure linear starting from the smallest distances available:

$$V_{Q\bar{Q}}(r)\big|_{\text{non-pert}} \approx \sigma \cdot r. \tag{7}$$

Moreover, this non-perturbative contribution encodes confinement as well. Thus, it is strongly suggested by the phenomenology that the effects of the confinement are encoded in the quadratic correction which is not explicit in the general OPE in Eq. (1).[2] As we argue in Section 5, a natural framework for the quadratic correction is provided by the stringy, or holographic formulation of QCD (for a review see, e.g., [23]).

## 3. Lessons from PT calculation of $\langle \alpha_s GG\rangle$

The best check of this logic is provided by the beautiful results for perturbative calculation of the gluon condensate on the lattice (the most advanced calculations are due to Rakow et al. [15–17]): More precisely, the results refer to perturbative evaluations of the quantity:

$$a^4 \frac{\pi}{12N_c}\left[\frac{-b_0 g^3}{\beta(g)}\right]\langle \alpha_s GG\rangle = 1 + \sum_{n=1}^{N} p_n g^{2n} + \Delta_N, \tag{8}$$

where $a$ is the lattice spacing and $\alpha_s(a)$ is the running coupling normalized at the ultraviolet cut off, $p_n$ are the expansion coefficients which are calculated explicitly up to $n = N$. Finally the difference $\Delta_N$ is known numerically since the total value of the l.h.s. of Eq. (8) coincides with the plaquette action and is known to a very good precision. Moreover, the difference is fit to power-like corrections:

$$\Delta_N = b_2^N(\Lambda_{\text{QCD}} \cdot a)^2 + b_4^N(\Lambda_{\text{QCD}} \cdot a)^4, \tag{9}$$

where the coefficients $b_{2,4}$ are fitting parameters which depend on the number of perturbative terms calculated explicitly.

Explicit results [15–17] demonstrate that, indeed, the power corrections in Eq. (8) depend strongly on the number $N$ of perturbative terms taken into account explicitly. Namely, up to $N \approx 10$ the power corrections are dominated by a quadratic term:

$$\Delta_N \approx b_2^N \cdot (a \cdot \Lambda_{\text{QCD}})^2 \quad \text{for } N \leqslant 10.$$

That is, the coefficient $b_4^N$ is consistent with zero for such $N$. However, the numerical value of the coefficient $b_2^N$ in front of the power correction diminishes with increasing $N$. Thus, perturbative corrections 'eat up' the power correction. In more refined terminology, the perturbative terms are dual to the leading power correction.

At $N > 10$, a quartic correction emerges as a result of subtracting the perturbative contributions from the total matrix element $\langle \alpha_s G^2\rangle$:

$$\Delta_N \approx const \cdot (a \cdot \Lambda_{\text{QCD}})^4 \quad \text{for } N \geqslant 10. \tag{10}$$

And, finally, at about $N \approx 16$ one restores the value of the quartic correction which is [15,16]:

$$\langle \alpha_s G^2\rangle_{\text{pert}} \approx 0.12\ \text{GeV}^4, \tag{11}$$

with a large error, but the result is comparable in magnitude with the standard gluon condensate entering the QCD (spectral) sum rules. Another remarkable finding [15,16] is that perturbative coefficients $p_n$ entering (8) are well approximated by a simple geometric series:

$$r_n \equiv \frac{p_n}{p_{n+1}} = u\left(1 - \frac{1+q}{n+s}\right), \tag{12}$$

where the fitting parameters $u = 0.961(9)$, $q = 0.99(7)$, $s = 0.44(10)$. The perturbative series with such coefficients is convergent for

$$|g^2| < |u|^{-1}.$$

This simple geometrical series fits explicit calculations of the PT coefficients at least for the first 16 terms. Extending $n \to \infty$ ($n \geqslant 50$), the geometric series reproduces the full answer to the accuracy better than $10^{-3}$, which is a remarkable result.

## 4. Geometric growth of the PT coefficients?

Physics-wise, one can say that the series found in [15–17] is determined by the singularity due to the crossover from strong to weak coupling. This is true in pure gluonic channel. This could also be true with account of quarks. Then, we would have, in different channels, geometric series, with approximately the same range of convergence. To see whether a such hypothesis can be ruled out, we compile below the calculated expressions of the Adler-like function in the Euclidian region[3] for different channels.

[2] A phenomenologically successful fit to the power corrections is provided by the 'short-distance' gluon mass (see [4–6,8,9]). However, the very notion of the short-distance gluon mass can be introduced only in the Born approximation and only for a certain class of processes [3,4].

[3] One can notice that the PT corrections in the theory of $\tau$-decay: $\delta^{(0)} = a_s + 5.202a_s^2 + 26.366a_s^3 + 127.079a_s^4$ [24–27] indicates a geometric growth, but the

In the vector channel with massless quarks, it reads [27,29,30]:

$$-Q^2\frac{d}{dQ^2}\Pi_V\left(Q^2\right) = \frac{N_c}{12\pi^2}\left[1+a_s+1.64a_s^2+6.31a_s^3+49.25a_s^4\right]. \tag{13}$$

The perturbative corrections to this expression due to the strange quark mass for the neutral (resp. charged) vector current, read [24, 31]:

$$Q^2\Pi^{D=2}_{\bar ss} = -\frac{6\bar m_s^2}{4\pi^2}\left[1+2.67a_s+24.14a_s^2+250a_s^3\right],$$
$$Q^2\Pi^{D=2}_{\bar us} = -\frac{3\bar m_s^2}{4\pi^2}\left[1+2.33a_s+19.58a_s^2+202a_s^3\right]. \tag{14}$$

The difference from $\alpha_s^2$ is due to the light by light scattering diagram contributing in the neutral vector two-point correlator.

For the pseudoscalar channel, the QCD expression of the Adler-like function reads for $n_f=3$ [32–34]:

$$-Q^2\frac{d}{dQ^2}\Pi_5\left(Q^2\right) = \frac{N_c}{8\pi^2}\left[1+5.67a_s+45.85a_s^2+465.8a_s^3+5589a_s^4\right]. \tag{15}$$

Similarly, one can also present the PT expressions of moments in deep-inelastic scatterings. The ones of the well-known Ellis–Jaffe[4] for polarized electroproduction or of the Gross–Llewellyn Smith sum rule for neutrino–nucleon scattering, read, for $n_f=3$ [35,36]:

$$\int_0^1 dx\, g_1^{p(n)} \simeq \left(1-a_s-3.58a_s^2-20.22a_s^3\right)\left[\pm\frac{|g_A|}{12}+\frac{a_8}{36}\right] - \frac{a_0}{9}\left(1-0.33a_s-0.55a_s^2-4.45a_s^3\right),$$
$$\int_0^1 dx\, F_3^{\bar\nu p+\nu p} \simeq 6\left(1-a_s-3.58a_s^2-18.976a_s^3\right), \tag{16}$$

where $a_8$ and $a_0$ are the octet and singlet structure functions. One can notice that, in all the cases, the series found, do not show any factorial growth nor an alternate sign but, are consistent with geometric series, with sizable corrections at small $n$ similar to the case of the gluon condensate.[5] Thus, there is an exciting perspective that all the perturbative series are in fact quite simple in large orders.

## 5. Insight from dual models

### 5.1. Holographic quadratic correction

In the holographic language, one evaluates the same observables, as within the field theoretic formulation of QCD, but in terms of strings living in extra dimensions. There is no direct derivation of the metrics of the extra dimensions in the QCD case. One rather uses phenomenologically motivated assumptions (see, e.g., [18]).

The crucial element is the metrics $z$ in the fifth dimension. Following [18], let us choose the following model:

$$ds^2 = R^2\frac{h(z^2)}{z^2}\left(dx_i^2+dz^2\right), \tag{17}$$

where $R^2$ is a constant whose explicit definition is not important for us here and the function $h(z^2)$ is specified below. Note that, at $z\to 0$, one needs $h(z^2)\to const$ in order to reproduce an approximate conformal symmetry of the Yang–Mills theories (due to the asymptotic freedom).

We would like to define the function $h(z^2)$ in such a way as to ensure confinement at large distances and to reproduce the (leading) quadratic power correction at short distances. The following choice:

$$h\left(z^2\right) = \exp\left(c^2z^2/2\right) \tag{18}$$

satisfies these conditions. Note that, while the condition to reproduce confinement, or the area law for the Wilson line is common to all the holographic models, the condition to reproduce the quadratic correction at short distances assumes that it is this correction which encodes the confinement at short distances. One can demonstrate that, assuming Eq. (18), is equivalent to assuming the Cornell potential for the heavy quarks interaction. The numerical value of the constant $c$ can be fixed in terms of the string tension, $c^2=(0.9\ \text{GeV})^2$.

The simple model in Eqs. (17) and (18) turns to be successful phenomenologically (see, in particular, [37] and references therein).

A crucial advantage of using the holographic language is that it allows for a perfectly gauge invariant way to introduce and parameterize the quadratic correction. Also, the simple expression (18) looks much more 'natural' than the assumption on approximate equality of the long perturbative series and quadratic correction plus short series discussed above. What is lacking, is further applications of the same metrics in Eq. (18) to evaluate quadratic corrections to the parton model in other cases, such as the current correlators.

### 5.2. Holographic quartic correction

Presence of the quadratic correction in the string-based approach is an assumption which allows to model the metric in the fifth dimension. However, once the metric is fixed, one can calculate the full answer for the gluon condensate [38].

The model does not account for the running of the coupling but allows to evaluate power corrections. In particular, it produces the value of the 'physical gluon condensate' of the magnitude:

$$\left\langle\alpha_sG^2\right\rangle_{\text{holographic}} \approx 0.03\ \text{GeV}^4, \tag{19}$$

which is reasonable phenomenologically [1].

What appears even more important is that the dual-model approach provides a new qualitative picture for the power corrections. Namely, in the holographic language $\langle\alpha_sG^2\rangle\sim\Lambda^4_{\text{QCD}}$ appears as a second-order effect in the coefficient $c$ introduced in Eq. (18):

$$\left\langle\alpha_sG^2\right\rangle_{\text{holographic}} \sim c^2 \sim \Lambda^4_{\text{QCD}}. \tag{20}$$

Since the coefficient $c$ (or the quadratic correction in the holographic language) is associated with short distance, the same is true for the gluon-condensate contribution in Eq. (20).

In short, the stringy calculation does not have a counterpart to the infrared-renormalon contribution which is taken for granted in field theoretic approach. This point is worthy to be elaborated.

effects due to the analytic continuation and to the $\beta$-functions induced by the renormalization group equation obscure the exact behaviour of the coefficients. Interpretations of a this fact require more involved analysis (see e.g. [28]).

4 The Bjorken sum rule corresponds to: $\int_0^1 dx\,(g_1^p-g_1^n)$.

5 A geometric growth of the PT coefficient has been assumed in [25] for predicting the $\alpha_s^4$ term of the PT series of the D-function in the V + A channel. This result has been (approximately) confirmed later on by the analytic calculation of [27].

In both cases of field theory and of stringy calculation, one deals with a propagator, of a particle or a string respectively. In both cases, the leading contribution comes from short distances. If the typical size is of order $a$, then, in both cases, $\langle\alpha_s G^2\rangle \sim a^{-4}$. However, the probability for a (virtual) particle to propagate to the distance of order $\Lambda_{\rm QCD}^{-1}$ is power-like suppressed:

$$a^4\langle\alpha_s G^2\rangle_{\rm IR,\ particle} \sim (\Lambda_{\rm QCD}\cdot a)^4, \qquad (21)$$

as revealed by the infrared renormalon (see, e.g., [12]). In the case of strings, the suppression of the infrared region turns to be exponential:

$$a^4\langle\alpha_s G^2\rangle_{\rm IR,\ string} \sim \exp\left(-const/[\Lambda_{\rm QCD}\cdot a]^\gamma\right), \qquad (22)$$

where $\gamma$ is positive. Intuitively, this strong suppression is due to the fact that string corresponds to a collection of particles.

## 6. Duality unexpected: quartic correction

Let us emphasize again that the standard assumption is that the quartic correction in Eq. (10) emerges simultaneously with the factorial divergence in expansion coefficients $a_n$ (see Eq. (8)):

$$\left(\frac{p_{n+1}}{p_n}\right)_{\rm IR\ renormalon} \sim n \quad \text{for } n \gg 1. \qquad (23)$$

This divergence is due to the infrared renormalon (for a review, see [12]).

So far [15–17], one does not run into the problem of the divergence in Eq. (23):

$$\left(\frac{p_{n+1}}{p_n}\right)_{n<15} \sim 1. \qquad (24)$$

It is even more amusing that, with presently available perturbative terms in Eq. (8), one can extract [15–17] the 'genuine' gluon condensate in Eq. (10):

$$a^4\langle\alpha_s G^2\rangle \sim (\Lambda_{\rm QCD}\cdot a)^4, \qquad (25)$$

so that the quartic correction gets disentangled from the infrared renormalon. This observation, if confirmed, is a radical change of dogma.

It is not ruled out that the infrared renormalon still shows up in higher orders of perturbation theory, say, at $n \sim 25$, as discussed in [15,16]. However, its contribution will be in any case smaller than the condensate in Eq. (25) determined from perturbative series which looks like a geometric series and exhibits no factorial growth of the coefficients.

It is amusing that the dual models independently provide a mechanism of generating the quartic correction from short distances [see discussion in Eq. (8)]. The condensate in Eq. (8) is not related to any divergence of the perturbative theory either. Thus, two independent approaches result in similar pictures.

## 7. Phenomenology of $1/Q^2$ corrections

In this section we review results of numerical fits which keep both a few-term perturbative series and a quadratic correction (assuming, therefore that considerably more perturbative terms are needed to apply the duality).

### 7.1. Tachyonic gluon mass squared $\lambda^2$

From the phenomenological point of view, it would be important to relate the quadratic corrections in various channels. A model which turns successful in this respect is the introduction of a tachyonic gluon mass $\lambda^2$ at short distances [3,4]. From the calculational point of view one changes the gluon propagator:

$$D^{ab}_{\mu\nu}(k^2) = \frac{\delta^{ab}\delta_{\mu\nu}}{k^2} \to \frac{\delta^{ab}\delta_{\mu\nu}}{k^2}\left(1+\frac{\lambda^2}{k^2}\right) \qquad (26)$$

and checks that the quadratic correction is associated with large momenta $k^2 \sim Q^2$. To the lowest order the analysis is gauge invariant. The model in Eq. (26) is purely heuristic in nature and can be used only for estimates in conjunction with short perturbative series.

### 7.2. Estimate of $\langle\alpha_s G^2\rangle$ and of $\alpha_s\lambda^2$

One can extract these two parameters by using ratio of exponential QCD (spectral) sum rules in $e^+e^- \to$ hadrons data [5], which is not sensitive to the leading $\alpha_s$ corrections. It is worth mentioning that FESR may not be appropriate for extracting such small quantities, as it requires a cancellation of two large numbers which depend on the high-energy parametrization of the spectral function. This feature is signaled by the large range spanned by the determinations of power corrections using FESR [39] and the discrepancies of the estimated quadratic corrections in [40] and [41], which both also differ from the one using the ratio of exponential Borel/Laplace (LSR) used in [5].

In addition to previous channels, the gluon condensate can be also obtained using a ratio of LSR for the $J/\psi$–$\eta_c$ and $\Upsilon$–$\eta_b$ mass-splittings, which has a minimum sensitivity on the heavy quark mass effects and on the $\alpha_s$ corrections [42].[6]

The resulting values of the parameters are [2,5,6,42]:

$$\langle\alpha_s G^2\rangle = (6.8\pm 1.3)\times 10^{-2}\ \text{GeV}^4,$$
$$a_s\lambda^2 = -(6.5\pm 0.5)\times 10^{-2}\ \text{GeV}^2, \qquad (27)$$

where $a_s \equiv \alpha_s/\pi$ and where the value of the gluon condensate is about 2 times the original SVZ value as expected from Bell–Bertlmann analysis [43].[7]

One can also use the pseudoscalar LSR for extracting $a_s\lambda^2$ [4]. Studying the stability of $(m_u + m_d)$ with respect to the change of $\lambda^2$, one obtains, at the stability region in $\lambda$, a reduction of the value of light quark mass of about 5% and the corresponding $\lambda$-value:

$$a_s\lambda^2 = -(12\pm 6)\times 10^{-2}\ \text{GeV}^2, \qquad (28)$$

consistent with previous estimate from $e^+e^\to$ hadrons data though less accurate. Taking into account these uncertainties, we shall consider the conservative value:

$$a_s\lambda^2 = -(7\pm 3)\times 10^{-2}\ \text{GeV}^2. \qquad (29)$$

These results indicate that these power corrections are small though crucial for understanding the non-perturbative properties of QCD. One can also notice that the new quadratic correction can only slightly change the existing QSSR phenomenology because of its smallness.

### 7.3. QCD (spectral) sum rule scales

Remarkably enough, the simple model in Eq. (26) brings in a qualitative success, explaining various mass scales (details can be found in the original paper [4]) revealed by analysis of the sum rules [44]. To our knowledge, there is no alternative explanation of the numerical hierarchy of such different scales.

[6] Some estimates of $\langle\alpha_s G^2\rangle$ in the existing literature suffer from correlation with $\alpha_s$ and $m_Q$. We plan to reanalyze these sum rules.

[7] A detailed comparison with the SVZ result can be found in [2].

**Table 1**
QSSR predictions of the light quark masses in units of MeV to order $\alpha_s^3$ and including the $1/Q^2$ correction. For translating $(m_u + m_d)$ into $m_s$ (and vice versa), we have used the ChPT prediction for $m_s/(m_u + m_d)$.

| Channels | $(\bar{m}_u + \bar{m}_d)(2)$ | | $\bar{m}_s(2)$ | Ref. |
|---|---|---|---|---|
| LSR Pion | 8.6(2.1) | $\Longrightarrow$ | 107.4(22.0) | [2,4] |
| LSR Kaon | – | | 119.6(18.4) | [2] |
| $\tau$-decay | – | | 93(30) | [7] |
| $e^+e^-$ | – | | 104.3(15.4) | [8] |
| Average | 8.7(1.3) | $\Longleftarrow$ | 106.2(15.4) | |

### 7.4. Light quark masses

Effects of quadratic corrections in the determinations of the light quark masses have been studied in [2,4,8]. As anticipated previously, the absolute value of $|\lambda^2|$ tends to lower by 5–6% the values of $(m_u + m_d)$ and of $m_s$ running masses obtained from (pseudo)scalar sum rules, while it tends to increase the value of $m_s$ in $e^+e^-$ and in $\tau$-decay data. These different results and their first published average are summarized in Table 1.

These results are in better agreement with some recent lattice calculations based on non-perturbative normalization than the recent global average given in [2,8] which includes determinations without the $1/Q^2$-term.

### 7.5. $\alpha_s$ from $\tau$-decay

One of the sensitive places where the effect of the quadratic term can be important is the precise extraction of $\alpha_s$ from $\tau$-decays [24,25]. One of the authors has presented recently [6] analysis of the effect of the quadratic correction on the determination of $\alpha_s$. The result is:

$$\alpha_s(M_\tau) = 0.3249(29)_{\rm ex}(9)_{\rm st}(74)_{\rm nst}$$
$$\Longrightarrow \alpha_s(M_Z)\big|_\tau = 0.1192(4)_{\rm ex}(1)_{\rm st}(9)_{\rm nst}(2)_{\rm ev}, \qquad (30)$$

where the errors are due respectively to the data, to the standard and non-standard corrections and to the evolution from $M_\tau$ to $M_Z$. This value of $\alpha_s$ is in agreement with existing estimates [13,27,45–47] obtained using different appreciations of the non-perturbative contributions and of the large order perturbative series. This result agrees with the ones from the $Z$-width [27] and from a global fit of electroweak data at $\mathcal{O}(\alpha_s^4)$ [45]:

$$\alpha_s(M_Z)\big|_{N^3LO} = 0.1191(27)_{\rm exp}(1)_{\rm th}, \qquad (31)$$

and with the most recent world average [48]:

$$\alpha_s(M_Z)\big|_{\rm world} = 0.1189(10). \qquad (32)$$

One can notice that the $1/Q^2$ contribution tends to decrease the value of $\alpha_s$ obtained without this term and improves the agreement with the world average.

## 8. Conclusions

In conclusion, our main point here is that large-order perturbative and non-perturbative contributions mix up as a matter of principle. The duality between these corrections is expected theoretically.

The duality, however, was thought to be confined to the quadratic corrections. The most recent and intriguing development is that this perturbative–non-perturbative duality might extend to the quartic correction as well. Basing on the existence of the infrared renormalon in perturbation theory, one would not expect that the quartic correction is calculable via the long perturbative series. Therefore, it is a challenge to explain the numerical observations on the perturbative series [15–17].

The holographic approach [18] does suggest a mechanism for generating quartic corrections at short distances but much more is needed to be done to finally clarify the issue. In the holographic language the quadratic correction looks as a stringy correction.[8]

Taken at face values, these observations accumulate to a drastic change of expectations on behaviour of perturbative series at higher orders in pure gluonic sector. Instead of factorial divergences in the expansion coefficients and related power-like terms, there are emerging convergent and calculable series, or dual to the power-like terms. We scrutinized the newly emerging picture against the phenomenology and did not find any flaws.

Existence of the infrared renormalon has never been proven since it corresponds only to a subclass of all the graphs in a given order of the perturbation theory. However, within the field-theoretic formulation it is equally difficult to imagine that these graphs are cancelled. The dual, or stringy formulation provides an alternative view. Within these models it is the power corrections which are calculable most directly. They are coming from short distances. Therefore they should be calculable perturbatively within the field theory and one gets explanation why the infrared-renormalon graphs cancel.

Thus, small steps in the phenomenological analysis of the power corrections might accumulate to produce a new insight to the fundamental issues of QCD. It goes without saying that further checks of the novel picture are needed.

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