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Standard Young Tableaux of Height 4 and 5

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On donne des formules exactes pour le nombre de tableaux de Young standards ayant n cases et au plus k lignes pour $k = 4$ et $k = 5$. Comme corollaire de la preuve qui est bijective, on démontre que les séries énumératices correspondantes ne sont pas algébriques.

We give exact formulas for the number of standard Young tableaux having n cells and at most k rows in the cases $k = 4$ and $k = 5$. As a corollary to our bijective proof, we deduce that the corresponding generating functions are not algebraic.

1. INTRODUCTION

In 1961, Schensted [24] introduced a bijection (discovered earlier in a somewhat different form by Robinson [21]) between permutations of the symmetric group S_n and pairs of standard Young tableaux (see also [10], [14], [29]). The first simple expressions for the number of standard Young tableaux of a given shape were the Frobenius–Young formula [6, 30, 17] and the Frame–Robinson–Thrall hook formula [4]. Since 1954 many proofs of the hook formula have been given, using probabilistic (Greene, Nijenhuis and Wilf [11]) or completely combinatorial methods (Remmel [19], Remmel and Whitney [20], Gessel and Viennot [8], Zeilberger [32], Franzblau and Zeilberger [31]; see also [26], [7], [22]).

More recently, the number of standard Young tableaux has been studied according to the height of their shape. Regev [18] has given asymptotic values for these numbers and Stanley [27] has discussed the algebraic or differentiably finite nature of the corresponding generating functions.

The motivation for these works comes from many other areas besides combinatorics such as, for instance, the theory of symmetric functions, invariant theory, algebraic geometry, the theory of polynomial identities, the Procesi–Razmyslov theory of trace identities, and the theory of algorithms [15].

The purpose of the present paper is to give exact formulas for the number S_n^k of standard Young tableaux having n cells and at most k rows with $k = 4$ and $k = 5$. For $k = 2$ and $k = 3$, exact formulas are already known [18] since

$$S_{2n}^2 = \frac{(2n)!}{n!n!}, \qquad S_{2n+1}^2 = \frac{(2n+1)!}{n!(n+1)!} \qquad \text{and} \qquad S_n^3 = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} C_i = M_n$$

where C_n is the Catalan number $(2n)!/n!(n+1)!$ and M_n the Motzkin number. The corresponding generating functions are algebraic. We prove here the following results:

$$S_{2n-1}^4 = C_n C_n, \qquad S_{2n}^4 = C_n C_{n+1}$$

and

$$S_n^5 = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{3!n!(2i+2)!}{(n-2i)!i!(i+1)!(i+2)!(i+3)!}.$$

It is interesting to note that the numbers $C_n C_n$ and $C_n C_{n+1}$ appear in the enumeration of certain families of planar maps (Mullin [16], Tutte [28]), of alternating Baxter permutations (Dulucq [2], Cori, Dulucq and Viennot [1]) and of certain families of lattice paths (Gouyou-Beauchamps [9]).

The proofs of these results are purely combinatorial. In Section 3, we recall a bijection of Françon and Viennot [5] between involutions and labelled Motzkin words. In Section 4,

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we introduce the concept of stacks in an involution. We are thus able to give relations between the height of stacks and the length of decreasing sequences in an involution. In Section 5, we examine connections between stacks and involutions or, amounting to the same thing, between stacks and labelled Motzkin words. Therefore we can establish, in Section 6, a bijection between standard Young tableaux having at most 4 rows and pairs of non-crossing Dyck paths. As these combinatorial objects have already been enumerated in [9], we obtain the formulas for S_n^4 and S_n^5. To carry out this bijection we use ideas developed by Gessel and Viennot [8] for using lattice paths to interpret tableaux and certain determinants which count tableaux. Finally, in Section 7, we prove that the corresponding generating functions are not algebraic and so we give a partial answer to a question asked by Stanley [27].

2. Definitions

The set $\{1, 2, \ldots, n\}$ is denoted by $[n]$. T_n is the set of involutions of the symmetric group S_n on $[n]$.

We will say that a permutation σ of S_n has a decreasing sequence of length k if and only if there exist k integers $i_1, i_2, \ldots, i_k$ such that $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ and $\sigma(i_1) > \sigma(i_2) > \cdots > \sigma(i_k)$. T_n^k is the subset of T_n consisting of involutions without a decreasing sequence of length $k + 1$.

Let $\lambda = (\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_k)$ with $\lambda_1 \neq 0$ be a partition of the positive integer n. The Ferrers diagram F_λ of shape λ is the set of left-justified rows of cells with λ_i cells in the ith row (reading from the top to the bottom). A standard Young tableau (more simply, a Young tableau) of shape λ is a filling of the cells in F_λ with the numbers $1, 2, \ldots, n$ in such a way that the numbers increase in each row and column. We denote by Y_n^k the set of Young tableaux having n cells and at most k rows. Schensted [24] has proved that there exists a bijection between T_n^k and Y_n^k. We denote by S_n^k the number of elements of T_n^k and Y_n^k, and by $y_k(x)$ the power series $\Sigma_{n \geqslant 0} S_n^k x^n$.

We will use sets which are called alphabets, their elements being called letters. The main alphabets we use are $X = \{x, \bar{x}\}$ and $Y = \{y, x, \bar{x}\}$ and also the infinite alphabet $\bar{Y} = \{y, x, \bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots\}$. A word is a finite sequence of letters. The empty sequence (or the empty word) will be denoted Λ.

The set A^* of words on the alphabet A, or the free monoid generated by A, is equipped by the binary operation of concatenation of two words; thus a word can be considered as the concatenation of its letters. Of course, Λ is the neutral element for this operation.

The length of a word f, denoted by $|f|$, is the number of letters of f. For a letter x in the alphabet, $|f|_x$ denotes the number of letters of f that are equal to x. A word f' is a factor of a word f if there exist two words f_1 and f_2 such that $f = f_1 f' f_2$. If f_1 is the empty word, then f' is a left factor of f.

δ is the morphism of Y^* in $\mathbb{N}$ given by:

$$\delta(x) = 1, \qquad \delta(\bar{x}) = -1, \qquad \delta(y) = 0.$$

The Motzkin language M is the set of words of Y^* such that $\delta(f) = 0$ and for any left factor f' of f, $\delta(f') \geqslant 0$. It is well known that

$$|M \cap Y^n| = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2i)!\,i!\,(i+1)!} = M_n \qquad \text{(the Motzkin number)}.$$

The Dyck language D is $M \cap X^*$. It is also well known that

$$|D \cap X^{2n}| = \frac{(2n)!}{n!(n+1)!} = C_n \qquad \text{(the Catalan number)}.$$

$\bar{\delta}$ is the morphism of $\bar{Y}^*$ in $\mathbb{N}$ given by:

$$\bar{\delta}(x) = 1, \quad \bar{\delta}(y) = 0 \quad \text{and} \quad \bar{\delta}(\bar{x}_i) = -1 \qquad \text{for } i \geqslant 1.$$

The shuffle of two words f and g of a free monoid A^* is the subset denoted by $f \sqcup\!\sqcup\, g$ of all words h such that $h = f_1 g_1 f_2 g_2 \ldots f_n g_n, f_i, g_i \in A^*, f = f_1 f_2 \ldots f_n, g = g_1 g_2 \ldots g_n$.

3. Involutions and Labelled Motzkin Words

A labelled Motzkin word f is a word in $\bar{Y}^*$ which satisfies the three conditions:
(a) $\bar{\delta}(f) = 0$;
(b) for any left factor f' of f, $\bar{\delta}(f') \geqslant 0$;
(c) for any left factor f' of f such that $f = f'\bar{x}_i f''$, $\bar{\delta}(f') \geqslant i \geqslant 1$.

Françon and Viennot [5] have given a bijection between involutions on $[2n]$ and labelled Motzkin words of length $2n$. Flajolet [3] has widened this result to partitions (see also [23]).

Let us recall this bijection Ψ. σ is an involution of T_n; $\Psi(\sigma) = w$, where $w = w_1 w_2 \ldots w_n$ is defined, for $1 \leqslant i \leqslant n$, by:
(1) $w_i = y$ if $\sigma(i) = i$;
(2) $w_i = x$ if $\sigma(i) > i$;
(3) $w_i = x_k$ and $k = 1 + |\{j / j < \sigma(i) < i < \sigma(j)\}|$ if $\sigma(i) < i$.

Because of the definition of Ψ, it is easy to verify that w is a labelled Motzkin word. The reverse correspondence is given by the following algorithm:

Begin
  No letter of w is marked;
  For i from 1 *to n do*
    Begin
      If $w_i = \bar{x}_k$ *then*
        Begin
          Let j be the index of the kth not marked x in w;
          We say that this x is closed by $\bar{x}_k$;
          $\sigma(i) := j$; $\sigma(j) := i$;
          Mark w_j;
        End
      Else if $w_i = y$ *then* $\sigma(i) = i$;
    End
End

EXAMPLE. $\Psi(6\ 2\ 8\ 5\ 4\ 1\ 7\ 3\ 14\ 13\ 12\ 11\ 10\ 9) = xyxx\bar{x}_3\bar{x}_1 y\bar{x}_1 xxx\bar{x}_3\bar{x}_2\bar{x}_1$.

4. Stacks and Decreasing Sequences

A stack of height k in an involution σ is a set of k integers $i_1 < i_2 < \cdots < i_k$ such that $\sigma(i_1) > \sigma(i_2) > \cdots > \sigma(i_k) > i_k$.

Stacks of an involution can be graphically represented in a simple way by drawing a line between the points i and $\sigma(i)$ of the segment $[1, 2n]$ for each i such that $i < \sigma(i)$. Figure 1 illustrates stacks of the involution (6 2 8 5 4 1 7 3 14 13 12 11 10 9).

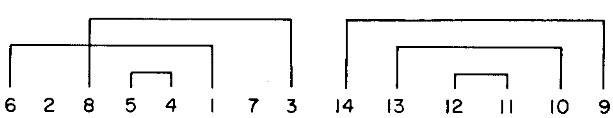

FIGURE 1.

We see one stack of height 3: 9, 10, 11 and five stacks of height 2: 1, 4; 3, 4; 9, 10; 10, 11 and 9, 11.

REMARK 1. A decreasing sequence contains at most one fixed point.

LEMMA 2. *An involution has a decreasing sequence of length* $2k$ *if and only if it has a stack of height* k.

PROOF. It is obvious that if an involution has a stack of height k, $i_1 < i_2 < \cdots < i_k$, then $\sigma(i_1) > \sigma(i_2) > \cdots > \sigma(i_k) > i_k > i_{k-1} > \cdots > i_1$ is a decreasing sequence of length $2k$.

Now suppose that $i_1 < i_2 < \cdots < i_{2k}$ is a decreasing sequence of length $2k$. So we have two cases:
(i) either $\sigma(i_k) > i_k$, in which case $\sigma(i_1) > \sigma(i_2) > \cdots > \sigma(i_k) > i_k$ and hence $i_1, i_2, \ldots, i_k$ is a stack of height k;
(ii) or $\sigma(i_k) \leqslant i_k$, in which case $\sigma(i_j) < i_j$ for $j > k$ and hence $\sigma(i_{2k}), \sigma(i_{2k-1}), \ldots, \sigma(i_{k+1})$ is a stack of height k.

So the lemma is proved. □

LEMMA 3. *If the generating function for fixed point free involutions having no decreasing sequence of length* $2k + 1$ *is* $\Sigma_{n\geqslant 0}\, a_{k,2n}\, x^{2n}$, *then the generating function for involutions having no decreasing sequence of length* $2k + 2$ *is*

$$\sum_{n\geqslant 0} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} a_{k,2i} x^n.$$

PROOF. It results from Remark 1 and Lemma 2 that any involution on $[n]$ having $n - 2i$ fixed points and no decreasing sequence of length $2k + 2$ can be seen as an element of the shuffle of $n - 2i$ fixed points and an involution without fixed point on $[2i]$ having no decreasing sequence of length $2k + 1$. Hence the relation between the two generating functions holds.

5. STACKS AND LABELLED MOTZKIN WORDS

t is the morphism of $\bar{Y}^*$ in $\mathbb{N}^*$ defined by:

$$t(x) = \Lambda, \quad t(y) = \Lambda, \quad t(\bar{x}_i) = i \qquad \text{for } i \geqslant 1.$$

For a labelled Motzkin word w, $t(w)$ will be called the track of w. Let σ be an involution on $[n]$. Suppose that $\Psi(\sigma) = w_1 w_2 \ldots w_n$ and that $t(\Psi(\sigma)) = t_1 t_2 \ldots t_m$ with $t_k \in \mathbb{N}$ and $t_k = t(w_{i_k})$ for $1 \leqslant k \leqslant m$. We denote τ_σ the mapping $k \to i_k$ from $[m]$ into $[n]$. So $\tau_\sigma(k)$ gives the rank in w of the kth letter of w belonging to $\{\bar{x}_1, \bar{x}_2, \ldots\}$.

EXAMPLE. $\sigma = (6\ 2\ 8\ 5\ 4\ 1\ 7\ 3\ 14\ 13\ 12\ 11\ 10\ 9)$;
$\Psi(\sigma) = xyxx\bar{x}_3\bar{x}_1 y\bar{x}_1 xxx\bar{x}_3\bar{x}_2\bar{x}_1$; $n = 14$;
$t(\Psi(\sigma)) = 3\ 1\ 1\ 3\ 2\ 1$; $m = 6$;
τ_σ: $1 \to 5, 2 \to 6, 3 \to 8, 4 \to 12, 5 \to 13, 6 \to 14$.

LEMMA 4. *Let* σ *be a fixed point free involution on* $[2n]$ *and let* w *be its image by* Ψ. *If* $t(w)$ *has* $k1^jl$ *as a factor with* $j \geqslant 0$ *and* $k - j > l > 1$ *then* σ *has a stack of height* 3.

PROOF. Let $t_1 t_2 \ldots t_n$ be the track of $w = w_1 w_2 \ldots w_{2n}$.

If $t(w)$ has a factor $k1^jl$ with $j \geqslant 0$ and $k - j > l > 1$, then there exists i such that $t_i = k, t_{i+j+1} = l$ and $t_{i+h} = 1$ for each h between 1 and j. As w is a labelled Motzkin word, w_{2n} is equal to $\bar{x}_1$, and t_n is equal to 1. So we can be sure of the existence of an integer u defined by $u = \text{Min}\{h/i + j + 1 < h \leqslant n \text{ and } t_h < l\}$. We denote by p, q and r the integers $\tau_\sigma(i)$, $\tau_\sigma(i + j + 1)$ and $\tau_\sigma(u)$. For all h such that $1 \leqslant h \leqslant j$, t_{i+h} is equal to 1. So the following equalities hold:

$$\sigma(\tau_\sigma(i + 1)) < \sigma(\tau_\sigma(i + 2)) < \cdots < \sigma(\tau_\sigma(i + j)) < \sigma(q) < \sigma(p) < p < \tau_\sigma(i + 1)$$
$$< \tau_\sigma(i + 2) < \cdots < \tau_\sigma(i + j) < q.$$

For all m such that $j + 1 < m < u$, t_m is not lower than l and hence $\sigma(q)$ is lower than $\sigma(\tau_\sigma(m))$. w_r being equal to $\bar{x}_p$ with $p < l$, $\sigma(r)$ is lower than $\sigma(q)$ and greater than $\sigma(\tau_\sigma(i + j))$. So we have exhibited six integers $\sigma(r) < \sigma(q) < \sigma(p) < p < q < r$ which form a stack of height 3. □

PROPOSITION 5. *Let σ be a fixed point free involution on $[2n]$ and let w be its image by Ψ. σ has no stack of height 3 if and only if $t(w)$ satisfies the two following conditions:*
(i) *if kl is a factor of $t(w)$ with $k > l$ then $l = 1$;*
(ii) *if $k1^{j+1}l$ is a factor of $t(w)$ with $l > 1$ then $k \leqslant j + l + 1$.*

PROOF. The only if part is given by Lemma 4.

For the if part, we suppose that σ is an involution having a stack of height 3. Hence there exist three integers $p < q < r$ such that $\sigma(r) < \sigma(q) < (p) < p$. By the definition of Ψ the following inequalities hold: $w_p > w_q > w_r \geqslant \bar{x}_1$, where we assume $\bar{x}_1 < \bar{x}_2 < \bar{x}_3 < \ldots$.

Let $t_1t_2 \ldots t_n$ be the track of w. We denote by i, j and k the three integers such that $\tau_\sigma(i) = p$, $\tau_\sigma(j) = q$ and $\tau_\sigma(k) = r$.

Let m be $|\{h/t_h = 1 \text{ and } i < h < j\}|$. By the definition of Ψ, m must be lower than $t_i - t_j$ because only $t_i - 1$ integers u are such that $\sigma(u) < \sigma(p) < p < u$, only $t_j - 1$ integers y are such that $\sigma(y) < \sigma(q) < \sigma(p) < p < q < y$ and we have at least m integers z such that $\sigma(z) < \sigma(q) < \sigma(p) < p < z < q$.

Suppose that the two conditions (i) and (ii) are satisfied between t_i and t_j. Then the factor $f = t_it_{i+1} \ldots t_j$ can be written $f = f_0g_1f_1g_2f_2 \ldots g_lf_l$, where:
(a) $l \geqslant 1$;
(b) f_α is an increasing sequence of integers greater than 1 for $0 \leqslant \alpha \leqslant l$ ($f_\alpha \neq \Lambda$);
(c) g_α is a sequence of λ_α letters 1 for $1 \leqslant \alpha \leqslant l$ ($\lambda_\alpha \geqslant 1$).

For $0 \leqslant \alpha \leqslant l$, we let a_α and b_α denote the first and last letters of f_α respectively. If $|f_\alpha| = 1$ then $a_\alpha = b_\alpha$. Note that $a_0 = t_i$ and $b_l = t_j$ and also that $b_\alpha \geqslant a_\alpha$.

As condition (ii) is satisfied in f, for $1 \leqslant \alpha \leqslant l$, we can write $b_{\alpha-1} - a_\alpha \leqslant \lambda_\alpha$. Thus $\Sigma_{\alpha=1}^{l} b_{\alpha-1} - a_\alpha \leqslant m$ or $b_0 - a_l + \Sigma_{\alpha=1}^{l-1} b_\alpha - a_\alpha \leqslant m$. We can minimize b_0 by $a_0 = t_i$, $-a_l$ by $-b_l = -t_j$ and $\Sigma_{\alpha=1}^{l-1} b_\alpha - a_\alpha$ by 0. We obtain the inequality $m \geqslant t_i - t_j$ and hence a contradiction. Therefore Proposition 5 is proved. □

COROLLARY 6. *An involution σ on $[n]$ has no decreasing sequence of length 5 if and only if:*
(a) *$t(\Psi(\sigma))$ satisfies conditions (i) and (ii) of Proposition 5;*
(b) *there are not five integers $1 \leqslant i < j < k < l < m \leqslant n$ such that $l = \sigma(j)$, $m = \sigma(i)$ and $k = \sigma(k)$.*

PROOF. This corollary results readily from Lemma 2 and Proposition 5. □

6. Involutions and Pairs of Non-crossing Dyck Left Factors

$D_{n,p}$ is the set of pairs (h, b) of Dyck left factors satisfying the three conditions ($n \geqslant p \geqslant 0$, n and p have the same parity):
(α) $|b| = |h| = n$;
(β) $\delta(b) = \delta(h) = p$;
(γ) for any left factors b' and h' of b and h such that $|b'| = |h'|$, $\delta(h') \geqslant \delta(b') \geqslant 0$.

An element of $D_{n,p}$ is called a pair of non-crossing Dyck left factors (PNDF for short). The number of elements of $D_{n,p}$ is denoted by $V_{n,p}$.

It is convenient to visualize words h and b as two random paths in a two-dimensional lattice. The letter x (resp. $\bar{x}$) corresponds to a north-east step (resp. south-east) from a point of positive integer coordinate (r, s) to a point $(r + 1, s + 1)$ (resp. $(r + 1, s - 1)$). These paths start from $(0, 0)$. Figure 2 shows the path corresponding to word $w = xxxxxx\bar{x}\bar{x}\bar{x}xx\bar{x}\bar{x}\bar{x}xx\bar{x}\bar{x}\bar{x}xxx\bar{x}\bar{x}\bar{x}$.

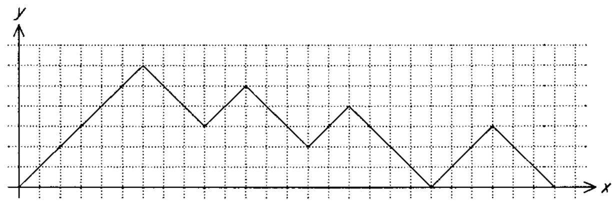

Figure 2.

$F_{n,p}$ (n and p have the same parity) is the set of involutions σ on $[n]$ such that:
(1) σ has p fixed points;
(2) σ has no decreasing sequence of length 5.

Proposition 7. *There exists a bijection between $F_{n,p}$ and $D_{n,p}$.*

Proof. By Corollary 6, we need only to establish a bijection φ between $D_{n,p}$ and the set of labelled Motzkin words w such that:
(a) $|w| = n$, $|w|_y = p$;
(b) $t(w)$ satisfies conditions (i) and (ii) of Proposition 5;
(c) if we denote $\Psi^{-1}(w)$ by σ, and if there exists r such $q = \sigma(r)$, $q < r$, $w_r = \bar{x}_i$, $i > 1$, then the word $w_q w_{q+1} \ldots w_r$ has no letter y (i.e. there is no fixed point under a stack of height 2).

φ is composed of two bijections, φ_1 and φ_2, such that $\varphi(w) = (\varphi_1(w), \varphi_2(w))$.

φ_1 is the morphism from $\bar{Y}^*$ in X^* defined by $\varphi_1(x) = \varphi_1(y) = x$ and $\varphi_1(\bar{x}_i) = \bar{x}$ for $i \geqslant 1$.

The bijection φ_2 is very easy to define for a word w which contains no letters y (i.e. fixed point free involutions). We give φ_2 in this subcase in order to enlighten the more difficult case of involutions with fixed points.

Let w be a labelled Motzkin word such that $|w| = 2n$, $|w|_y = 0$, and w satisfies condition b. Let $t = t_1 t_2 \ldots t_n$ be the track of w. We define recursively the sequence $f_0, f_1, \ldots, f_n$ of word on X, each f_i being a left factor of $\varphi_2(w)$:
(i) $f_0 = \Lambda$;
(ii) for $1 \leqslant i \leqslant n, f_i = f_{i-1}x^{t_i - \delta(f_{i-1})}\bar{x}$ if $\delta(f_{i-1}) < t_i$; $f_i = f_{i-1}\bar{x}$ if $t_i = 1$ and $\delta(f_{i-1}) > 0$.

Then $\varphi_2(w)$ will be f_n.

In other words, the path corresponding to $\varphi_2(w)$:
(i) climbs to the level t_i with a letter t_i and goes down to the level $t_i - 1$, if it is possible (i.e. if the path is on a level lower than t_i);
(ii) goes down one unit with letter $t_i = 1$ if the path is not on the zero level.

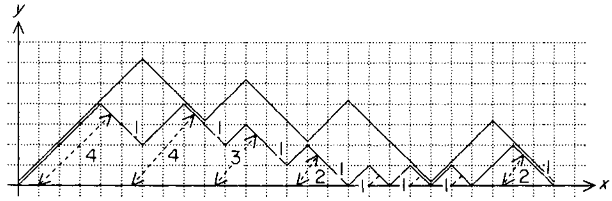

FIGURE 3.

Figure 3 shows $\varphi_1(w)$ and $\varphi_2(w)$ for the labelled Motzkin word

$$w = xxxxxxx\bar{x}_4\bar{x}_1\bar{x}_4xx\bar{x}_1\bar{x}_3\bar{x}_1xx\bar{x}_2\bar{x}_1\bar{x}_1\bar{x}_1xxx\bar{x}_1\bar{x}_2\bar{x}_1.$$

It is easy to verify that conditions (i) and (ii) of Proposition 5 imply that $t_i > \delta(f_{i-1})$ when $t_i > 1$ and that $(\varphi_1(w), \varphi_2(w))$ is a PNDF since w is a labelled Motzkin word.

Now we consider the general case.

The bijection φ is defined recursively according to n. There are the first values of φ (for $n = 0, 1, 2$ and 3):

$\varphi(\Lambda) = (\Lambda, \Lambda)$	$\varphi(x\bar{x}_1 y) = (x\bar{x}x, x\bar{x}x)$
$\varphi(y) = (x, x)$	$\varphi(xy\bar{x}_1) = (xx\bar{x}, x\bar{x}x)$
$\varphi(x\bar{x}_1) = (x\bar{x}, x\bar{x})$	$\varphi(yx\bar{x}_1) = (xx\bar{x}, xx\bar{x})$
$\varphi(yy) = (xx, xx)$	$\varphi(yyy) = (xxx, xxx)$

Assume that for any Motzkin labelled word w satisfying conditions (a), (b) and (c) such that $|w| < n$, $\varphi(x) = (h, b) = (\varphi_1(w), \varphi_2(w))$ satisfies the following conditions:

(1) (h, b) is a PNDF;

(2) if w begins with x^α and if the first letter $\bar{x}$ in w is $\bar{x}_r$, then b begins with x^λ, where $\lambda \geqslant \min(\alpha + 1, r)$;

(3) if $w = yw_1$ then $b = xb_1$, and for any left factor b' of b_1, $\delta(xb') \geqslant 1$.

Now let w be a Motzkin labelled word $= n$ and let (h, b) be its image of $b = \varphi_2(w)$; we have seven cases to consider. For each case, we remark on the way of builiding φ_2^{-1}.

First case. $w = yv$. Then $\varphi_2(w) = x\varphi_2(v)$. Remark: it will the only case where, for any non-empty left factor b' of $\varphi_2(w)$, $\delta(b') \geqslant 1$.

Second case. $w = xw_1\bar{x}_1w_2$, where $|w_1|_{\bar{x}} = 0$. The first letter $\bar{x}$ of w is $\bar{x}_1$. Then $\varphi_2(w) = x\bar{x}\varphi_2(w_1w_2)$. Remark: it will be the only case where $\varphi_2(w)$ begins with $x\bar{x}$. The length of the first left factor $x^i\bar{x}$ of $\varphi_1(w)$ is equal to $2 + |w_1|$.

Third case. $w = xw_1\bar{x}_jw_2\bar{x}_1w_3$ with $|w_1w_2|_{\bar{x}} = 0$. The first two letters $\bar{x}$ of w are $\bar{x}_j$ and $\bar{x}_1$. The first x is closed (see algorithm of Section 3) by the second $\bar{x}$. Then either $j = 2$ or $j > 2$.

Case 3.1: $j = 2$. The second x is closed by the first $\bar{x}$. The word w can be written $w = xy^\alpha xu_1\bar{x}_2w_2\bar{x}_1w_3$ (by condition (c), u_1 contains no y). Then $\varphi_2(w) = xx\bar{x}\bar{x}\varphi_2(y^\alpha u_1w_2w_3)$. Remark: the length of the first left factor $x^i\bar{x}$ of $\varphi_1(w)$ is equal to $3 + |y^\alpha u_1|$. The length of the first left factor $x^i\bar{x}x^k\bar{x}$ of $\varphi_1(w)$ is equal to $4 + |y^\alpha u_1w_2|$.

Case 3.2: $j > 2$ and w begins with xy. We exhibit the letter x that closes $\bar{x}_j$ by splitting w in $w = xy^pxu_1xu_2\bar{x}_jw_2\bar{x}_1w_3$ with $|xy^pxu_1x|_x = j$. Then $\varphi_2(w) = xx\bar{x}\bar{x}\varphi_2(u_1xy^pu_2w_2w_3)$. Remark: by condition (c), u_2 contains no y. The length of the first left factor $x^i\bar{x}$ of $\varphi_1(w)$ is equal to $3 + |u_1xy^pu_2|$. The length of the first left factor $x^i\bar{x}x^k\bar{x}$ is equal to $4 + |u_1xy^pu_2w_2|$.

Case 3.3: $j > 2$ and w begins with xx. We have $w = xw_1\bar{x}_jw_2\bar{x}_1w_3$. According to conditions (a), (b) and (c), let b' be $\varphi_2(w_1\bar{x}_{j-1}w_2w_3)$. Suppose that $b' = x^r\bar{x}b''$. Then we have $r \geqslant 2$ according to inductive condition (2) because $\min(2, j-1) \geqslant 2$. Then $\varphi_2(w) = x^{r+1}\bar{x}\bar{x}b''$. Remark: the length of the first left factor $x^i\bar{x}x^k\bar{x}$ of $\varphi_1(w)$ is equal to $2 + |w_1\bar{x}_{j-1}w_2|$. We have also $|w_2|_{\bar{x}} = 0$.

Fourth case. $w = xw_1\bar{x}_j w_2 \bar{x}_k w_3$ with $k \geqslant j \geqslant 2$ and $|w_1 w_2|_{\bar{x}} = 0$. The first two $\bar{x}$ of w are $\bar{x}_j$ and $\bar{x}_k$. We exhibit the x that closes $\bar{x}_j$ by splitting w in $w = xu_1 xu_2 \bar{x}_j w_2 \bar{x}_k w_3$ with $|xu_1 x|_x = j$. According to inductive condition (c), $u_2 = x^\alpha$ $(\alpha \geqslant 0)$. We have two cases to consider according to whether the x that closes $\bar{x}_k$ belongs to u_2 or w_2.

Case 4.1: $w = xu_1 xx^\alpha \bar{x}_j w_2 \bar{x}_k w_3$ with $\alpha > k - j$. The x that closes $\bar{x}_k$ belongs to u_2. According to condition (c), w_2 contains no y and so $w_2 = x^\lambda$ $(\lambda \geqslant 0)$. Assume $k - j = i$ (so $i \geqslant 0$ and $\alpha \geqslant i + 1$). Let w' be $xx^i u_1 x^{\alpha - i} x^\lambda \bar{x}_k w_3$. By inductive hypothesis, we have $\varphi_2(w') = x^r \bar{x} b_1$ with $r \geqslant \min(i + 2, k)$. Since $k - j = i$ and $j \geqslant 2$, inequality $r \geqslant i + 2$ holds. Then $\varphi_2(w) = x^{r-i} \bar{x} x^{i+1} \bar{x} b_1$. Remark: the length of the first left factor $x^q \bar{x}$ of $\varphi_1(w)$ is equal to $2 + |x^{i+1} u_1 x^{\alpha - i}|$.

Case 4.2: $w = xu_1 xx^\alpha \bar{x}_j w_2 \bar{x}_k w_3$ with $\alpha \leqslant k - j$. The x that closes $\bar{x}_k$ belongs to w_2. Let w' be $xx^\alpha u_1 w_2 \bar{x}_k w_3$. By inductive hypothesis, we have $\varphi_2(w) = x^r \bar{x} b_1$ with $r \geqslant \min(\alpha + 2, k)$. Since $\alpha \leqslant k - j$ and $2 \leqslant j$, inequality $r \geqslant \alpha + 2$ holds. Then $\varphi_2(w) = x^{r-\alpha} \bar{x} x^{\alpha+1} \bar{x} b_1$. Remark: the length of the first left factor $x^q \bar{x}$ of $\varphi_1(w)$ is equal to $2 + |x^{\alpha+1} u_1|$.

For each case it is very easy to check that (h, b) satisfies conditions (1), (2) and (3) of the inductive hypothesis. So the definition of φ is consistent.

Now we define the inverse correspondence to prove that φ is a bijection. We again use an induction on the length of h and b.

Assume that for any PNDF (h, b) such that $|h| < n$, the word $w = \varphi^{-1}(h, b)$ satisfies the following conditions:

(1) w satisfies conditions (a), (b) and (c);

(2) $h = \varphi_1(w)$;

(3) if $b = x^k \bar{x} b_1$ with $k \geqslant 2$ then $w = x^{k-1} w_1 \bar{x}_j w_2$ with $2 \leqslant k \leqslant j$ and $|w_1|_{\bar{x}} = 0$.

Let (h, b) be a PNDF with $|h| = n$ and let w be its image by φ^{-1}. We again meet seven cases to consider. Each case of the φ^{-1} definition corresponds to any case of the φ definition that is numbered in the same way.

First case. $(h, b) = (xh_1, xb_1)$ and, for any left factor b' of b_1, $\delta(xb') \geqslant 1$. Then $w = y\varphi^{-1}(h_1, b_1)$. Remark: for the following cases h and b begin with x and there exists a non-empty left factor b' of b such that $\delta(b') = 0$.

Second case. $(h, b) = (x^{m+1} \bar{x} h_1, x\bar{x} b_1)$ with $m \geqslant 0$. b begins with $x\bar{x}$. Recursively let $w_1 w_2$ be $\varphi^{-1}(x^m h_1, b_1)$ with $|w_1| = m$ and hence, according to condition (2), $|w_1|_{\bar{x}} = 0$. Then $w = xw_1 \bar{x}_1 w_2$.

Third case. $(h, b) = (x^{m+1} \bar{x} x^q \bar{x} h_1, x^{k+1} \bar{x} \bar{x} b_1)$ with $m \geqslant k \geqslant 1$ and $q \geqslant 0$. The first two $\bar{x}$ of b are consecutives. We have two cases: $k = 1$ or $k > 1$. If $k = 1$, recursively let $xw_1 w_2 \bar{x}_1 w_3$ be $\varphi^{-1}(x^{m+q} \bar{x} h_1, x\bar{x} b_1)$ with $|w_1| = m - 1$ and $|w_2| = q$. Condition 2 implies that $|w_1 w_2|_{\bar{x}} = 0$.

Case 3.1: $k = 1$ and $w_1 = y^p u_1$ with $|u_1|_y = 0$ and $p \geqslant 0$. All the letters y in w_1 are at the beginning. Then $w = xy^p xu_1 \bar{x}_2 w_2 \bar{x}_1 w_3$. Notice that inductive hypothesis (3) holds: $|xy^p xu_1|_{\bar{x}} = 0$.

Case 3.2: $k = 1$ and there exist $j > 2$ and $p > 0$ such that $w_1 = u_1 xy^p u_2$ with $|u_2|_y = 0$ and $|u_1|_x = j - 3$. There exists at least one y in w_1 that follows at least one x. Then $w = xy^p u_1 xu_2 \bar{x}_j w_2 \bar{x}_1 w_3$. Notice that inductive hypothesis (3) holds: $|xy^p u_1 xu_2|_{\bar{x}} = 0$.

Case 3.3: $k > 1$. Recursively let $w_1 \bar{x}_{j-1} w_2 w_3$ be $\varphi^{-1}(x^m \bar{x} x^q h_1, x^k \bar{x} b_1)$ with $|w_1| = m$, $|w_2| = q$ and by induction $w_1 = x^{k-1} u_1$ $(m \geqslant j - 1 \geqslant k \geqslant 2$ and $|u_1|_{\bar{x}} = 0)$. Then $w = x^k u_1 \bar{x}_j w_2 \bar{x}_1 w_3$. Notice that $j > 2$ and that inductive hypothesis (3) holds: $m + 1 \geqslant j \geqslant k + 1 \geqslant 3$ and $|x^k u_1|_{\bar{x}} = 0$.

Fourth case. $(h, b) = (x^{m+1} \bar{x} x^q \bar{x} h_1, x^{r-p} xx^{p+1} \bar{x} b_1)$ with $m + q \geqslant r$, $m + 1 \geqslant r - p \geqslant 2$ and $p \geqslant 0$. The first two $\bar{x}$ of b are separated by at least one x. Recursively let $xw_1 w_2 \bar{x}_k w_3$ be $\varphi^{-1}(x^{m+q} \bar{x} h_1, x^r \bar{x} b_1)$ with $|w_2| = q$, $|w_1| = m - 1$ and $|w_1 w_2|_{\bar{x}} = 0$. By induction, we have $w_1 = x^{r-2} v_1$ with $m + q \geqslant k \geqslant r \geqslant 2$, and since $r - 2 \geqslant p$, we can write w_1 as $w_1 = x^{r-2-p} x^p v_1$.

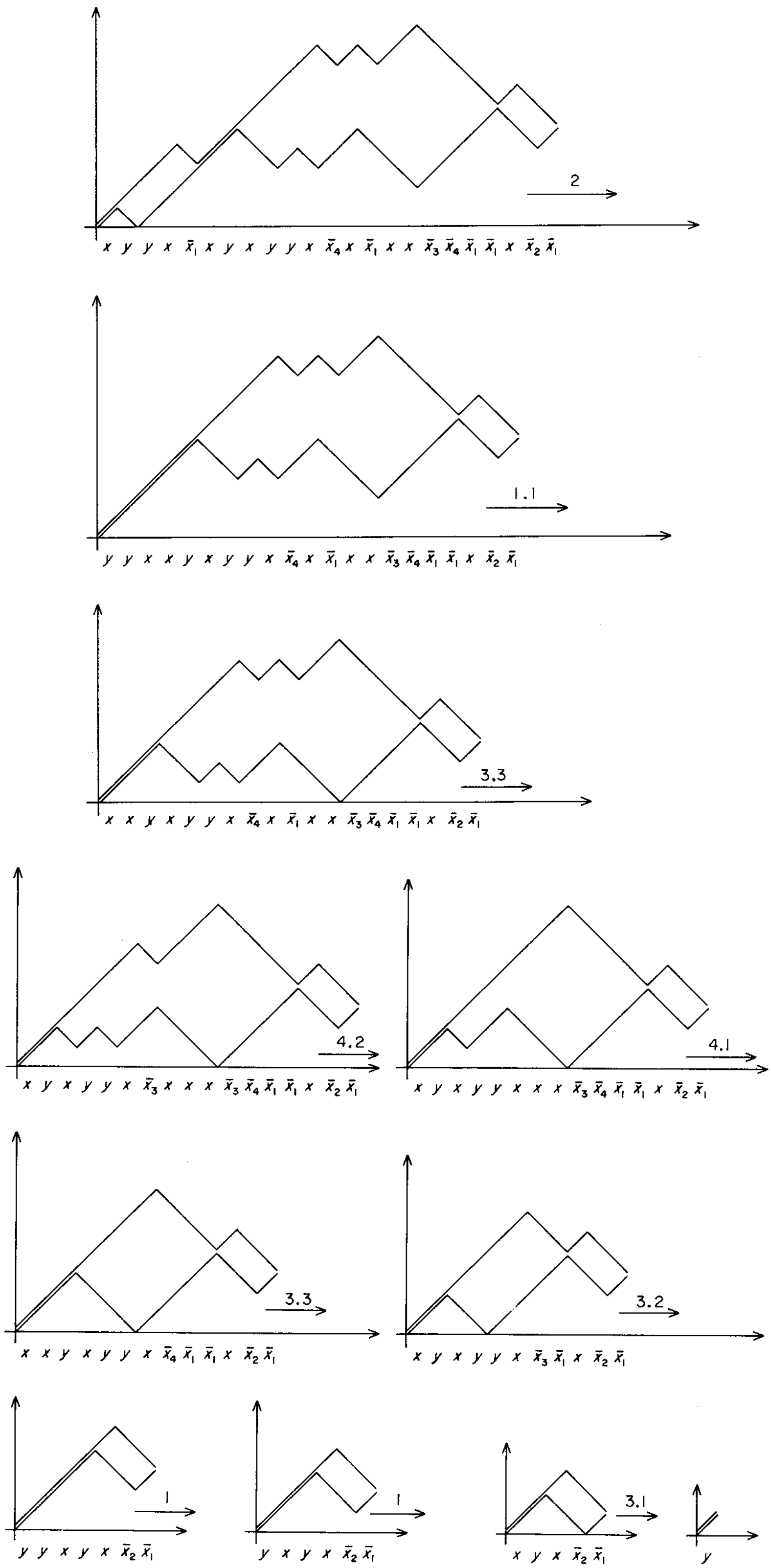
2
x y y x x̄₁ x y x y y x x̄₄ x x̄₁ x x x̄₃ x̄₄ x̄₁ x̄₁ x x̄₂ x̄₁
1.1
y y x x y x y y x x̄₄ x x̄₁ x x x̄₃ x̄₄ x̄₁ x̄₁ x x̄₂ x̄₁
3.3
x x y x y y x x̄₄ x x̄₁ x x x̄₃ x̄₄ x̄₁ x̄₁ x x̄₂ x̄₁
4.2
x y x y y x x̄₃ x x x x̄₃ x̄₄ x̄₁ x̄₁ x x̄₂ x̄₁
4.1
x y x y y x x x x̄₃ x̄₄ x̄₁ x̄₁ x x̄₂ x̄₁
3.3
x x y x y y x x̄₄ x̄₁ x̄₁ x x̄₂ x̄₁
3.2
x y x y y x x̄₃ x̄₁ x x̄₂ x̄₁
1
y y x y x x̄₂ x̄₁
1
y x y x x̄₂ x̄₁
3.1
x y x x̄₂ x̄₁
y

FIGURE 4.

Case 4.1: $|w_1|_x \geqslant k - 1$. The x that is closed by $\bar{x}_k$ belongs to w_1. According to condition (c), $w_2 = x^q$.

We notice that $k \leqslant 1 + |w_1|_x$ implies that $k \leqslant m$. In w_1, we exhibit the letter x that closes $\bar{x}_k$ by splitting w_1 in $w_1 = x^{r-2-p}x^p u_1 x u_2$ with $|x^{r-2-p}x^p u_1|_x = k - 2$. According to condition (c), $u_2 = x^{\lambda - p}$ with $\lambda \geqslant p \geqslant 0$. Let j be $k - p$ ($j \geqslant 2$ because $k \geqslant r \geqslant p + 2$). Hence $|x^{r-2-p}u_1|_x = k - p - 2 = j - 2$. Then $w = x^{r-p-1}u_1 x^{\lambda+1}\bar{x}_j x^q \bar{x}_k w_3$. We observe that inductive hypothesis (3) holds: w begins with $r - p - 1$ letters x with $2 \leqslant r - p \leqslant j$ and $|u_1 x^{\lambda+1}|_{\bar{x}} = 0$.

Case 4.2: $|w_1|_x < k - 1$. The x that is closed by $\bar{x}_k$ belongs to w_2. Let j be $|w_1|_x + 2 - p$. Since $k - 2 \geqslant |w_1|_x \geqslant r - 2$, we have $k \geqslant k - p \geqslant j \geqslant r - p \geqslant 2$ and hence $k \geqslant j \geqslant 2$. Then $w = x^{r-p-1}v_1 x^{p+1}\bar{x}_j w_2 \bar{x}_k w_3$. We observe that inductive hypothesis 3 holds: w begins with $r - p - 1$ letters x with $2 \leqslant r - p \leqslant j$ and $|v_1 x^{p+1}|_{\bar{x}} = 0$.

For each case it is very easy to check that w satisfies conditions (1), (2) and (3) of the inductive hypothesis and that $w = \varphi^{-1}(\varphi(w))$.

So Proposition 7 is proved. □

Figure 4 gives an example of $\varphi(w)$ for:

$$w = xyyx\bar{x}_1 xyxyyx\bar{x}_4 x\bar{x}_1 xx\bar{x}_3\bar{x}_4\bar{x}_1\bar{x}_1 x\bar{x}_2\bar{x}_1$$

This Motzkin labelled word corresponds to the involution:

(5, 2, 3, 14, 1, 19, 7, 20, 9, 10, 12, 11, 17, 4, 23, 18, 13, 16, 6, 8, 22, 21, 15)

(see Fig. 5) and to the standard Young tableau of Fig. 6.

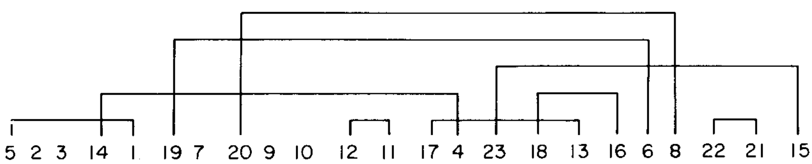

FIGURE 5.

14	19	20	23					
5	12	17	18	22				
2	7	9	10	16				
1	3	4	6	8	11	13	15	21

FIGURE 6.

Figure 7 gives the first values of $F_{n,p}$.

COROLLARY 8. *The number of Young tableaux having n cells, at most* 4 *rows and p columns of odd height is*

$$\frac{n!(n+2)!(p+3)!}{\left(\frac{n-p}{2}\right)!\left(\frac{n-p}{2}+1\right)!p!\left(\frac{n+p}{2}+2\right)!\left(\frac{n+p}{2}+3\right)!}$$

(*n and p have the same parity*).

$n\backslash p$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	1	0	1								
3	0	3	0	1							
4	3	0	6	0	1						
5	0	14	0	10	0	1					
6	14	0	40	0	15	0	1				
7	0	84	0	90	0	21	0	1			
8	84	0	300	0	175	0	28	0	1		
9	0	594	0	825	0	308	0	36	0	1	
10	594	0	2475	0	1925	0	504	0	45	0	1

FIGURE 7.

PROOF. It is well known that the number of fixed points in an involution is equal to the number of columns of odd height in the corresponding Young tableau (Schützenberger [25]). On the other hand, a bijective proof of the equality

$$V_{n,p} = \frac{n!(n+2)!(p+3)!}{\left(\frac{n-p}{2}\right)!\left(\frac{n-p}{2}+1\right)!p!\left(\frac{n+p}{2}+2\right)!\left(\frac{n+p}{2}+3\right)!}$$

is given in [9].

COROLLARY 9. $S^4_{2n-1} = C_n C_n$ *and* $S^4_{2n} = C_n C_{n+1}$.

PROOF. According to [9], we can say that $S^4_{2n-1} = \Sigma^n_{p=1} V_{2n-1,2p-1} = C_n C_n$ and $S^4_{2n} = \Sigma^n_{p=0} V_{2n,2p} = C_n C_{n+1}$. The first values of S^4_n are $(0 \leqslant n \leqslant 18)$: 1, 1, 2, 4, 10, 25, 70, 196, 588, 1764, 5544, 17424, 56628, 184041, 613470, 2044900, 6952660, 23639044 and 81662162. □

COROLLARY 10. *The number* $G_{n,n-2k}$ *of Young tableaux having n cells, at most 5 rows and* $n-2k$ *columns of odd height is*

$$\frac{n!3!(2k+2)!}{(n-2k)!k!(k+1)!(k+2)!(k+3)!}$$

PROOF. From Lemma 3 and Corollary 8, it is easy to prove that this number is

$$\binom{n}{2k} V_{2k,0}.$$

Figure 8 gives the first values of $G_{n,p}$. □

$n\backslash p$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	1	0	1								
3	0	3	0	1							
4	3	0	6	0	1						
5	0	15	0	10	0	1					
6	14	0	45	0	15	0	1				
7	0	98	0	105	0	21	0	1			
8	84	0	382	0	210	0	28	0	1		
9	0	756	0	1176	0	378	0	36	0	1	
10	594	0	3780	0	2940	0	630	0	45	0	1

FIGURE 8.

From Lemma 3 and Corollary 10 we also obtain:

COROLLARY 11.

$$S_n^5 = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{3!n!(2k+2)!}{(n-2k)!k!(k+1)!(k+2)!(k+3)!}.$$

The first values of S_n^5 are ($0 \leqslant n \leqslant 18$): 1, 1, 2, 4, 10, 26, 75, 225, 715, 2347, 7990, 27 908, 99 991, 365 587, 1 362 310, 5 159 208, 19 831 101, 77 233 517 and 304 423 574.

7. THE POWER SERIES $y_4(x)$ AND $y_5(x)$

PROPOSITION 12. *The power series $y_4(x)$ and $y_5(x)$ are not algebraic.*

PROOF. Suppose that $y_4(x)$ is algebraic. Then the Hadamard product $g(x)$ of $y_4(x)$ with the rational power series $x/(1 - x^2)$ is algebraic (cf. [13]). Now

$$g(x) = \sum_{n \geqslant 0} C_n C_n x^{2n} = \sum_{n \geqslant 0} \frac{(2n)!(2n)!}{n!(n+1)!n!(n+1)!} x^{2n}$$

and the power series:

$$f(x) = (x(x^2 g(x))')' = 4x \sum_{n \geqslant 0} \frac{(2n)!(2n)!}{n!n!n!n!} = 4^{2n+1} x \sum_{n \geqslant 0} \left(\frac{(2n-1)(2n-3)\ldots 3\,1}{(2n)(2n-2)\ldots 4\,2} \right)^2 x^{2n}$$

is algebraic. But the power series

$$J(x) = \sum_{n \geqslant 0} \left(\frac{(2n-1)(2n-3)\ldots 3\,1}{(2n)(2n-2)\ldots 4\,2} \right)^2 x^{2n}$$

is well known to be not algebraic. So we have a contradiction. Hence $y_4(x)$ is not algebraic (see [13]).

Suppose that $y_5(x)$ is algebraic. Then the power series:

$$h(x) = \frac{1}{1+x} y_5\left(\frac{1}{1+x}\right) = \sum_{n\geqslant 0} \frac{3!(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!} x^{2n}$$

is also algebraic and so the power series:

$$l(x) = (x^3(x(x^4(xh(x))')')')' = 24x^3 \sum_{n\geqslant 0} \left(\frac{(2n+2)!}{(n+1)!(n+1)!}\right)^2 x^{2n+2}$$

$$= 24x^3 \left(-1 + \sum_{n\geqslant 0} \left(\frac{(2n-1)(2n-3)\ldots 3\,1}{(2n)(2n-2)\ldots 4\,2}\right)^2 (4x)^{2n}\right).$$

But as the power series $J(x)$ is not algebraic, we have again a contradiction. Hence $y_5(x)$ is not algebraic. □

REMARK. $y_4(x)$ and $y_5(x)$ are differentiably finite as has been shown by Stanley and Zeilberger (cf. [27]) in the general case, since coefficients of $g(x)$ satisfy the relation $(n+2)^2 C_{n+1} C_{n+1} - 4(2n+1)^2 C_n C_n = 0$ and coefficients h_n of $h(x) = \Sigma_{n\geqslant 0} h_n x^n$ satisfy the relation $(n+4)(n+3)h_{n+1} - 4(2n+3)(2n+1)h_n = 0$.

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