Note

Remark on a Combinatorial Identity*

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In a recent paper [2], Gould has discussed the identity

\[ \sum_{k=0}^{\min(a,b)} \binom{x+y+k}{k} \binom{x+a-b}{a-k} \binom{y+b-a}{b-k} q^{(a-k)(b-k)} = \binom{x}{a} \binom{y}{b}, \]  

(1)

where

\[ \binom{x}{n} = \prod_{i=1}^{n} \frac{1 - q^{x-i+1}}{1 - q^i}. \]

It may be worthwhile remarking that (1) is implied by the q-analog of Saaschütz's formula [1, p. 68]:

\[ \sum_{k=0}^{n} \frac{(q^{-n})_k (x)_k (y)_k}{(q)_k (u)_k (v)_k} q^k = \frac{(u/x)_n (u/y)_n}{(u)_n (u/xy)_n}, \]

(2)

where

\[ (u)_k = (1 - u)(1 - qu) \cdots (1 - q^{k-1}u) \]

and

\[ uv = q^{1-n}xy. \]  

(3)

Indeed, it is easily verified that

\[ \binom{x+k}{k} = \frac{(q^{x+1})_k}{(q)_k}, \]

\[ \binom{x+a-b}{a-k} = (-1)^k q^{a k - (1/2) k (k-1)} \frac{(q^{a-b+1})_a}{(q)_a} \frac{(q^{-a})_k}{(q^{a-b+1})_k}. \]

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Thus (1) becomes, after simplification,
\[
\sum_{k=1}^{\min(a,b)} \frac{(q^{-a})_k (q^{-b})_k (q^{x+y+1})_k}{(q)_k (q^{x-b+1})_k (q^{y-a+1})_k} q^k = q^{-ab} \frac{(q^{x+1})_a (q^{y+1})_b}{(q^{x-b+1})_a (q^{y-a+1})_b}.
\]

(4)

It is clear that we may use (2) to evaluate the left-hand side of (4). Thus (4) reduces to
\[
\frac{(q^{-y-b})_a}{(q^{-y})_a} = q^{-ab} \frac{(q^{y+1})_b}{(q^{y-a+1})_b}.
\]

(5)

Since
\[
(q^{-y-b})_a = (-1)^a q^{-ay-ab+(1/2)a(a-1)}(q^{y+a+b+1})_a,
\]
\[
(q^{-y})_a = (-1)^a q^{-ay+(1/2)a(a-1)}(q^{y-a+1})_a,
\]
(5) becomes
\[
\frac{(q^{y-a+b+1})_a}{(q^{y-a+1})_a} = \frac{(q^{y+1})_b}{(q^{y-a+1})_b},
\]
that is,
\[
(q^{y-a+1})_b (q^{y-a+b+1})_a = (q^{y-a+1})_a (q^{y+1})_b,
\]
an obvious identity.

REFERENCES