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POLYNOMIAL FUNCTORS AND WREATH PRODUCTS

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Introduction

The theory of "invariant matrices" was the subject of Schur's dissertation in 1901 [8a] and his first major mathematical achievement. The problem which Schur posed and solved completely was to find all functions F from $m\times m$ matrices to $r\times r$ matrices such that

(i) the r^2 entries of $F(A)$ are polynomial functions of the m^2 entries of A – i.e., F is a polynomial mapping from the space of $m\times m$ matrices to the space of $r\times r$ matrices;

(ii) $F(AB)=F(A)F(B)$ for all $m\times m$ matrices A, B. Schur called $F(A)$ an "invariant form" or "invariant matrix" of A; he showed that each such F could be split up into homogeneous components, and that the homogeneous F of degree n are in a natural one-one correspondence with the representations of the symmetric group S_n, provided that $n\leqslant m$.

In 1927 [8b] Schur returned to this question, and showed that all the irreducible invariant matrices of degree n could be obtained by decomposing the nth tensor (or Kronecker) power $T^n(A)=A\otimes\cdots\otimes A$. To state this result in modern language, let V be a finite-dimensional vector space over a field k of characteristic 0, and let $T^n(V)=V\otimes\cdots\otimes V$ be its nth tensor power over k. The symmetric group S_n acts on $T^n(V)$ by permuting the factors, so that we have a finite-dimensional representation of S_n, which we decompose into its isotypic components:

$$T^n(V)\cong\bigoplus_\pi \mathrm{Hom}_{kS_n}(E_\pi, T^n(V))\otimes_k E_\pi=\bigoplus_\pi F_\pi(V)\otimes_k E_\pi,$$

say, functorially in V: here the E_π are the distinct irreducible kS_n-modules. If now $A: V\to V$ is a linear transformation, then $F_\pi(A): F_\pi(V)\to F_\pi(V)$ is an irreducible invariant matrix of A, homogeneous of degree n, and all the irreducible invariant matrices of degree n, up to equivalence, arise in this way. Thus an invariant matrix in Schur's sense defines a *functor* F on the category $\mathbf{V}_k$ of finite-dimensional k-vector spaces, which is *polynomial* in the sense that for each pair of vector spaces U, V, the mapping $F:\mathrm{Hom}(U, V)\to\mathrm{Hom}(F(U), F(V))$ is a polynomial mapping between these vector spaces.

We shall take this observation as our starting point. The purpose of this paper is to investigate polynomial functors between more general categories than the category $\mathbf{V}_k$.

For the adjective "polynomial" to make sense, we need from the beginning an underlying field of scalars. If k is any field, an additive category $\mathbf{A}$ in Grothendieck's sense [5] (i.e., admitting finite direct sums) will be called *k-linear* if each object of $\mathbf{A}$ admits scalar multiplication by the elements of k, satisfying the obvious conditions (see §1 for the precise definition), which imply that $\mathrm{Hom}_{\mathbf{A}}(X, Y)$ is a k-vector space for any two objects X, Y in $\mathbf{A}$.

Now let $\mathbf{A}, \mathbf{B}$ be k-linear categories, where k is an infinite field. A (covariant) functor $F: \mathbf{A} \to \mathbf{B}$ will be said to be *polynomial* if for all X, Y in $\mathbf{A}$ the mapping $F: \mathrm{Hom}_{\mathbf{A}}(X, Y) \to \mathrm{Hom}_{\mathbf{B}}(F(X), F(Y))$ is a polynomial mapping. We shall also need to assume that the category $\mathbf{B}$ is pseudo-abelian (or Karoubian) [3]: this means that idempotent morphisms have images and hence determine direct sum decompositions – briefly, "idempotents split".

The basic idea (following Schur) is that polynomial functors $F: \mathbf{A} \to \mathbf{B}$ can be treated in the same way as polynomials. First we show that F is the direct sum of its homogeneous parts (Section 2), and hence we may assume from now on that F is homogeneous of degree $n > 0$ (the case $n = 0$ being trivial). Next, just as a homogeneous polynomial can be linearized, so can our functor F: the *linearization* L_F of F (see Section 3 for definition) is a multilinear functor of n variables, i.e. a functor from $\mathbf{A}^n = \mathbf{A} \times \cdots \times \mathbf{A}$ to $\mathbf{B}$ which is k-linear in each variable.

In Section 4 we define an action of the symmetric group S_n on $L_F^{(n)}(X) = L_F(X, \ldots, X)$, where X is any object of $\mathbf{A}$, and (because we shall want to divide by $n!$) we assume from now on that the field k has characteristic 0. Then the subobject $(L_F^{(n)}(X))^{S_n}$ of S_n-invariants is defined, and our first main result (4.10) is that the functor F can be reconstructed from its linearization L_F:

Theorem 1. *F is isomorphic to the functor $X \mapsto (L_F^{(n)}(X))^{S_n}$.*

This result reduces the study of homogeneous polynomial functors to that of multilinear functors, and the next step would therefore be to classify the latter. We do not attempt this task in any great generality; from now on we take $\mathbf{A}$ to be the category $\mathbf{V}_A$ of finitely-generated projective left A-modules, where A is a k-algebra.

At this stage tensor products enter naturally. Suppose $L: \mathbf{V}_A^n \to \mathbf{B}$ is k-linear in each variable, and let $T^n(A) = A \otimes \cdots \otimes A$ be the nth tensor power of A over k. Then $T^n(A)$ acts on $L^{(n)}(A) = L(A, \ldots, A)$ on the right, and thus $L^{(n)}(A)$ is a right $T^n(A)$-module object in the category $\mathbf{B}$. With an appropriate definition of tensor product in this context, we show (5.6) that

Theorem 2. *L is isomorphic to the functor*

$$(P_1, \ldots, P_n) \mapsto L^{(n)}(A) \otimes_{T^n(A)} (P_1 \otimes_k \cdots \otimes_k P_n).$$

From Theorems 1 and 2 we have immediately

Theorem 3. *Let $F: \mathbf{V}_A \to \mathbf{B}$ be a homogeneous polynomial functor of degree $n > 0$. Then F is isomorphic to the functor*

$$P \to (L_F^{(n)}(A) \otimes_{T^n(A)} T^n(P))^{S_n}.$$

In this situation the symmetric group S_n acts on all three ingredients: the k-algebra $T^n(A)$, the right $T^n(A)$-module $L_F^{(n)}(A)$, and the left $T^n(A)$-module $T^n(P)$, and in the two last cases the action of S_n is compatible with the module structure. This leads us in Section 6 to the second main theme of this paper, namely wreath products. The vector space $T^n(A) \otimes_k kS_n$ is a k-algebra for the multiplication $(a \otimes s)(b \otimes t) = a \cdot s(b) \otimes st$, where $a, b \in T^n(A)$ and $s, t \in S_n$; this k-algebra we call the *wreath product* of A with S_n, and denote it by $A \sim S_n$. (If $A = k\Gamma$ is the algebra of a group Γ, then $A \sim S_n$ is the group algebra of the wreath product $\Gamma \sim S_n$.) In this way we may regard the nth tensor power (over k) as a functor from $\mathbf{V}_A$ to $\mathbf{V}_{A \sim S_n}$, and Theorem 3 then takes the form ((6.4), (6.5)):

Theorem 4. *Let $F: \mathbf{V}_A \to \mathbf{B}$ be a homogeneous polynomial functor of degree $n > 0$, and let $M = L_F^{(n)}(A)$. Then $F \cong U_M \circ T^n$, where $T^n: \mathbf{V}_A \to \mathbf{V}_{A \sim S_n}$ is the nth tensor power, and $U_M: \mathbf{V}_{A \sim S_n} \to \mathbf{B}$ is the k-linear functor $Q \to M \otimes_{A \sim S_n} Q$.*

Moreover the functors $\alpha: F \mapsto L_F^{(n)}(A)$, $\beta: M \mapsto U_M \circ T^n$ constitute an equivalence of the category of homogeneous polynomial functors $F: \mathbf{V}_A \to \mathbf{B}$ of degree n with the category of right $A \sim S_n$-module objects in $\mathbf{B}$.

Suppose now that $\mathbf{B}$ is the category $\mathbf{V}_k$ of finite-dimensional k-vector spaces. Then it follows from Theorem 4 that the classification of irreducible polynomial functors $F: \mathbf{V}_A \to \mathbf{V}_k$ of degree n is tantamount to the classification of simple $A \sim S_n$-modules, finite-dimensional over k; and this is a particular case of the classification of simple modules over twisted group rings, which we discuss in the Appendix. In this way we obtain (7.2), on the assumption that k is algebraically closed (and of characteristic 0):

Theorem 5. *Every irreducible polynomial functor $F: \mathbf{V}_A \to \mathbf{V}_k$ is isomorphic to a tensor product of functors of the form*

$$P \mapsto V \otimes_{kS_m} T^m(E \otimes_A P)$$

where E (resp. *V*) *is a finite-dimensional simple right A-module* (resp. *kS_m-module*), *and no two of the E's are isomorphic. Moreover, this factorization of F as a tensor product is unique* (*up to the order of the factors*).

Next, in Section 8, we turn our attention to the Grothendieck group $K(\mathbf{P}_A)$ of the category $\mathbf{P}_A$ of polynomial functors $F: \mathbf{V}_A \to \mathbf{V}_k$ of bounded degree. $K(\mathbf{P}_A)$ is a commutative graded ring, the multiplication being tensor product of functors;

moreover, it is a λ-ring, $\lambda^i(F)$ being the composition of F with the ith exterior power $\bigwedge^i: \mathbf{V}_k \to \mathbf{V}_k$. The structure of $K(\mathbf{P}_A)$ is given (8.1) by

Theorem 6. *$K(\mathbf{P}_A)$ is the free λ-ring generated by the classes of the functors $P \mapsto E \otimes_A P$, where E runs through a complete set of non-isomorphic finite-dimensional simple right A-modules.*

Finally, in Section 9 we specialize to the case where $A = kG$ is the group algebra of a finite group G, and k is the field of complex numbers. Then the ring $K(\mathbf{P}_A)$ may be canonically identified with the direct sum $R(G) = \bigoplus_{n \geq 0} R(G_n)$, where $G_n = G \sim S_n$ is the wreath product of G with S_n, and $R(G_n)$ is the Grothendieck group of kG_n-modules. By Theorem 6, $R(G)$ is the free λ-ring generated by G^*, the set of isomorphism classes of simple kG-modules.

We next introduce $C(G) = \bigoplus_{n \geq 0} C(G_n)$, where $C(G_n)$ is the space of k-valued class functions on G_n, and we proceed to make $C(G)$ into a λ-ring. The multiplication is defined by means of induction of class functions, and the λ-structure by means of the Adams operations ψ^n: the details are in Section 9. It then appears that $C(G)$ is the free k-λ-algebra generated by G_*, the set of conjugacy classes of G.

Now let $\chi: R(G) \to C(G)$ be the linear mapping which takes each representation to its character. The basic fact (9.8) is

Theorem 7. *$\chi: R(G) \to C(G)$ is a homomorphism of λ-rings.*

It is this fact which underlies the computation of the character tables of the wreath products $G_n = G \sim S_n$ (Specht [9]). We shall not attempt to reproduce the details in this introduction; the reader will find them at the end of Section 9.

The paper ends with an appendix on twisted group rings, which include the wreath products $A \sim S_n$ as particular cases. Here we have restricted ourselves to the results we need in the body of the paper, and have not attempted a complete account.

Finally, it should perhaps be said that the methods of this paper are elementary throughout, and demand from the reader no more than a general familiarity with functorial linear algebra, as expounded (for example) in Bourbaki's *Algèbre*.

1. Polynomial functors

Let $\mathbf{A}$ be an additive category in the sense of Grothendieck [5]. A *field of scalars* for $\mathbf{A}$ is any subfield of the ring of endomorphisms of the identity functor $1_{\mathbf{A}}$ of $\mathbf{A}$, or more precisely is an embedding of a field k in this ring. Thus each element $\lambda \in k$ determines a morphism $1_{\mathbf{A}} \to 1_{\mathbf{A}}$; that is to say, for each object X in $\mathbf{A}$ we have a morphism $\lambda_X: X \to X$ such that 1_X is the identity morphism, $\lambda_X + \mu_X = (\lambda + \mu)_X$ and $\lambda_X \mu_X = (\lambda\mu)_X$ for all $\lambda, \mu \in k$, and $f\lambda_X = \lambda_Y f$ for all morphisms $f: X \to Y$ in $\mathbf{A}$. We shall often write λ in place of λ_X, whenever the context permits.

Since composition of morphisms in an additive category is bilinear, we have $(\lambda+\mu)f=\lambda f+\mu f$ and $\lambda(f+g)=\lambda f+\lambda g$ for all $\lambda, \mu \in k$ and $f, g: X \to Y$ in $\mathbf{A}$. Hence $\mathrm{Hom}_{\mathbf{A}}(X, Y)$ is a k-vector space.

A category $\mathbf{A}$ as above will be called a *k-linear* category.

Let $\mathbf{A}, \mathbf{B}$ be k-linear categories. A (covariant) functor $F: \mathbf{A} \to \mathbf{B}$ is said to be *polynomial* if, for each pair of objects X, Y in $\mathbf{A}$, the mapping $F: \mathrm{Hom}_{\mathbf{A}}(X, Y) \to \mathrm{Hom}_{\mathbf{B}}(FX, FY)$ is a polynomial mapping between these k-vector spaces. We may express this condition as follows:

(1.1) *Given any finite sequence of morphisms $f_i: X \to Y$ in $\mathbf{A}$ and elements $\lambda_i \in k$ $(1 \leq i \leq r)$, the morphism $F(\lambda_1 f_1 + \cdots + \lambda_r f_r)$ is a polynomial in $\lambda_1, \ldots, \lambda_r$ with coefficients in* $\mathrm{Hom}_{\mathbf{B}}(FX, FY)$ *(depending on $f_1, \ldots, f_r$ and F).*

We shall assume throughout that the field k is infinite (and from Section 4 onwards that it has characteristic 0). The coefficient morphisms in (1.1) are then uniquely determined. We shall also assume that the category $\mathbf{B}$ is pseudo-abelian (see the introduction).

2. Homogeneity

Let $F: \mathbf{A} \to \mathbf{B}$ be a polynomial functor. For each object X in $\mathbf{A}$, the morphism $F(\lambda_X)$ will be a polynomial in λ with coefficients in $\mathrm{End}_{\mathbf{B}}(FX)$ independent of λ, say

$$F(\lambda_X) = \sum_{n \geq 0} u_n(X)\lambda^n \tag{2.1}$$

where $u_n(X) \in \mathrm{End}_{\mathbf{B}}(FX)$. Since $F((\lambda\mu)_X) = F(\lambda_X \mu_X) = F(\lambda_X)F(\mu_X)$, we have

$$\sum_{n \geq 0} u_n(X)(\lambda\mu)^n = \Big(\sum_{n \geq 0} u_n(X)\lambda^n\Big)\Big(\sum_{n \geq 0} u_n(X)\mu^n\Big)$$

for all $\lambda, \mu \in k$. Because k is infinite, it follows that $u_n(X)^2 = u_n(X)$ for all $n \geq 0$, and that $u_m(X)u_n(X) = 0$ for $m \neq n$. Also, by taking $\lambda = 1$ in (2.1), we have $\sum u_n(X) = F(1_X) = 1_{FX}$. Since the category $\mathbf{B}$ is pseudo-abelian, it follows that the morphisms $u_n(X)$ determine a finite direct sum decomposition of $F(X)$, say

$$F(X) = \bigoplus_{n \geq 0} F_n(X)$$

where $F_n(X)$ is the image of $u_n(X)$.

Moreover, if $f: X \to Y$ is any morphism in $\mathbf{A}$, we have $F(f)F(\lambda_X) = F(\lambda_Y)F(f)$ for all $\lambda \in k$, from which and (2.1) it follows (again because k is infinite) that $F(f)u_n(X) = u_n(Y)F(f)$ for each $n \geq 0$, in other words that each u_n is an endomorphism of the functor F. Hence $F(f)$ induces by restriction morphisms $F_f(f): F_n(X) \to F_n(Y)$, and each F_n is a *functor*, which is clearly polynomial. Consequently we have a direct decomposition of the functor F:

(2.2) $$F = \bigoplus_{n \geq 0} F_n$$

in which each F_n is a *homogeneous* polynomial functor of degree n. For $F_n(\lambda) = \lambda^n$ and hence, in the situation of (1.1),

$$F_n(\lambda\lambda_1 f_1 + \cdots + \lambda\lambda_r f_r) = F_n(\lambda)F_n(\lambda_1 f_1 + \cdots + \lambda_r f_r)$$
$$= \lambda^n F_n(\lambda_1 f_1 + \cdots + \lambda_r f_r)$$

which shows that $F_n(\lambda_1 f_1 + \cdots + \lambda_r f_r)$ is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_r$.

The F_n are called the *homogeneous parts* of F.

(2.3) **Remarks.** (1) When $n = 0$, we have $F_0(\lambda) = 1$ for all $\lambda \in k$, so that in particular $F_0(0) = F_0(1)$. It follows that, for all morphisms $f: X \to Y$, $F_0(f) = F_0(0_{XY})$ and hence is independent of f, and is an isomorphism of $F_0(X)$ onto $F_0(Y)$. Hence all the objects $F_0(X)$ are canonically isomorphic.

(2) Next, when $n = 1$, we have $F_1(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 g_1 + \lambda_2 g_2$ say; taking (λ_1, λ_2) to be $(1, 0)$, $(0, 1)$ successively, we see that $g_i = F_1(f_i)$ and hence that $F_1(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 F_1(f_1) + \lambda_2 F_2(f_2)$: in other words, F_1 is *k-linear* and in particular *additive*.

(3) The direct sum (2.2) may well have infinitely many non-zero components, although for any given object X we shall have $F_n(X) = 0$ for all sufficiently large n, because the sum (2.1) is finite. An example is the exterior algebra functor on the category of finite-dimensional vector spaces over k.

If $F_n = 0$ for all sufficiently large n, we shall say that F has *bounded degree*.

More generally, let $\mathbf{A}_1, \ldots, \mathbf{A}_r$ be k-linear categories, and let $\mathbf{P} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_r$ be the product category, whose objects are all sequences $X = (X_1, \ldots, X_r)$, where X_i is an object of $\mathbf{A}_i$ for $1 \leq i \leq r$, and $\mathrm{Hom}_{\mathbf{P}}(X, Y) = \bigoplus_{i=1}^r \mathrm{Hom}_{\mathbf{A}_i}(X_i, Y_i)$. The category $\mathbf{P}$ is k-linear. Now let F be a polynomial functor from $\mathbf{P}$ to a pseudo-abelian category $\mathbf{B}$. If $\lambda_1, \ldots, \lambda_r \in k$, then $F((\lambda_1)_{X_1}, \ldots, (\lambda_r)_{X_r})$ will be a polynomial in $\lambda_1, \ldots, \lambda_r$ with coefficients in $\mathrm{End}_{\mathbf{B}}(FX)$, say

(2.4) $$F(\lambda_1, \ldots, \lambda_r) = \sum_{m_1, \ldots, m_r} u_{m_1, \ldots, m_r}(X_1, \ldots, X_r)\lambda_1^{m_1} \cdots \lambda_r^{m_r}.$$

Exactly as before, we see that the $u_{m_1, \ldots, m_r}$ are endomorphisms of the functor F, and that if we denote the image of $u_{m_1, \ldots, m_r}(X_1, \ldots, X_r)$ by $F_{m_1, \ldots, m_r}(X_1, \ldots, X_r)$, then the $F_{m_1, \ldots, m_r}$ are subfunctors of F giving a direct decomposition

(2.5) $$F = \bigoplus_{m_1, \ldots, m_r} F_{m_1, \ldots, m_r},$$

summed over all $(m_1, \ldots, m_r) \in \mathbf{N}^r$. Each $F_{m_1, \ldots, m_r}$ is homogeneous of multidegree $(m_1, \ldots, m_r)$, i.e.

$$F_{m_1, \ldots, m_r}(\lambda_1, \ldots, \lambda_r) = \lambda_1^{m_1} \cdots \lambda_r^{m_r}.$$

3. Linearization

Again let $F: \mathbf{A} \to \mathbf{B}$ be a polynomial functor. In view of the decomposition (2.2), we shall assume from now on that F is *homogeneous* of degree $n > 0$. The considerations at the end of Section 2 apply to the functor $F': \mathbf{A}^n \to \mathbf{B}$ defined by $F'(X_1, \ldots, X_n) = F(X_1 \oplus \cdots \oplus X_n)$, and show that there exists a direct decomposition, functorial in each variable,

$$F(X_1 \oplus \cdots \oplus X_n) = \bigoplus F'_{m_1,\ldots,m_n}(X_1, \ldots, X_n)$$

summed over all $(m_1, \ldots, m_n) \in \mathbf{N}^n$ such that $m_1 + \cdots + m_n = n$.

Our main interest will be in the functor $F'_{1,\ldots,1}$, the image of the functorial morphism $u_{1,\ldots,1}$ (see (2.4)). For brevity, we shall write L_F in place of $F_{1,\ldots,1}$, and v in place of $u_{1,\ldots,1}$. We call L_F the *linearization* of F. (Another name for it is the *nth deviation* of F, see (3.2) below.) The functor $L_F: \mathbf{A}^n \to \mathbf{B}$ is homogeneous and k-linear in each variable (and therefore additive in each variable (2.3)).

To recapitulate the definitions of L_F and v, let $Y = X_1 \oplus \cdots \oplus X_n$. Then there exist monomorphisms $i_\alpha: X_\alpha \to Y$ and epimorphisms $p_\alpha: Y \to X_\alpha$ for $1 \leqslant \alpha \leqslant n$, such that

$$(3.1) \qquad p_\alpha i_\alpha = 1_{X_\alpha}, \qquad p_\alpha i_\beta = 0 \quad \text{if } \alpha \neq \beta, \qquad \sum_{\alpha=1}^{n} i_\alpha p_\alpha = 1_Y.$$

For each $\lambda = (\lambda_1, \ldots, \lambda_n) \in k^n$, let (λ) denote the morphism $\sum \lambda_\alpha i_\alpha p_\alpha : Y \to Y$, so that (λ) acts as scalar multiplication by λ_α on the component X_α. Then $v(X_1, \ldots, X_n)$ is the coefficient of $\lambda_1 \ldots \lambda_n$ in $F((\lambda))$, and $L_F(X_1, \ldots, X_n)$ is the image of $v(X_1, \ldots, X_n)$, and is a direct summand of $F(X_1 \oplus \cdots \oplus X_n)$.

(3.2) **Remark.** For each subset E of $\{1, 2, \ldots, n\}$, let

$$\psi_E = \sum_{\alpha \in E} i_\alpha p_\alpha$$

so that, in the notation introduced above, $\psi_E = (\mu)$ where $\mu_\alpha = 1$ or 0 according as α does or does not belong to E. As in (2.4), let $u_{m_1,\ldots,m_n}$ be the coefficient of $\lambda_1^{m_1} \cdots \lambda_n^{m_n}$ in $F((\lambda))$, and let $\phi_E = \sum u_{m_1,\ldots,m_n}$ summed over those $(m_1, \ldots, m_n) \in \mathbf{N}^n$ with support equal to E (i.e. such that $m_\alpha > 0$ if and only if $\alpha \in E$). Then it is clear that

$$F(\psi_E) = \sum_{D \subset E} \phi_D$$

for each subset E of $\{1, 2, \ldots, n\}$. Solving these equations for the ϕ's, we obtain

$$\phi_D = \sum_{E \subset D} (-1)^{|D-E|} F(\psi_E).$$

In particular, when $D = \{1, 2, \ldots, n\}$, we have $\phi_D = u_{1,\ldots,1} = v$, and therefore

$$v = \sum_E (-1)^{n-|E|} F(\psi_E),$$

summed over all subsets E of $\{1, 2, \ldots, n\}$. This formula shows that L_F is the 'nth deviation' of F (see Epstein [4]).

(3.3) **Example.** Suppose that $\mathbf{A} = \mathbf{B} = \mathbf{V}_k$, the category of finite-dimensional vector spaces over k, and that F is the nth exterior power $\bigwedge^n$, which is a homogeneous polynomial functor of degree n. In this case we have

$$F(X_1 \oplus \cdots \oplus X_n) = \bigoplus_{m_1, \ldots, m_n} \bigwedge^{m_1}(X_1) \otimes \cdots \otimes \bigwedge^{m_n}(X_n)$$

summed over all $(m_1, \ldots, m_n) \in \mathbf{N}^n$ such that $m_1 + \cdots + m_n = n$, and hence

$$L_F(X_1, \ldots, X_n) = \bigwedge^1(X_1) \otimes \cdots \otimes \bigwedge^1(X_n) = X_1 \otimes \cdots \otimes X_n.$$

4. The action of the symmetric group

Assume from now on that k has characteristic 0.

With $F: \mathbf{A} \to \mathbf{B}$ as before, consider in particular

$$L_F^{(n)}(X) = L_F(X, \ldots, X)$$

for any object X in $\mathbf{A}$. For each permutation s in the symmetric group S_n there exists a morphism $s = s_X : X^n \to X^n$ (where $X^n = X \oplus \cdots \oplus X$) which permutes the summands according to s, namely

$$s_X = \sum_{\alpha=1}^{n} i_{s(\alpha)} P_\alpha \tag{4.1}$$

in the notation of (3.1). If as before we write $\lambda = (\lambda_1, \ldots, \lambda_n) \in k^n$ and $(\lambda) = \sum \lambda_\alpha i_\alpha p_\alpha$, then a simple calculation shows that $s(\lambda) = (s\lambda) \cdot s$, where $s\lambda = (\lambda_{s^{-1}(1)}, \cdots, \lambda_{s^{-1}(n)})$, and hence that $F(s)F((\lambda)) = F((s\lambda))F(s)$. By picking out the coefficient of $\lambda_1 \cdots \lambda_n$ on either side, we see that

$$F(s)v = vF(s) \tag{4.2}$$

from which it follows that $F(s)$ induces by restriction an endomorphism $\tilde{F}(s)$ of $L_F^{(n)}$. Explicitly, if

$$j = j_X : L_F^{(n)}(X) \to F(X^n), \qquad q = q_X : F(X^n) \to L_F^{(n)}(X) \tag{4.3}$$

are the injection and projection associated with the direct summand $L_F^{(n)}(X)$ of $F(X^n)$, so that $qj = 1$ and $jq = v$, then

$$\tilde{F}(s) = qF(s)j. \tag{4.4}$$

It follows that $\tilde{F}(st) = \tilde{F}(s)\tilde{F}(t)$ for $s, t \in S_n$, so that we have a representation of S_n on $L_F^{(n)}(X)$, functorial in X.

Our aim now will be to show that this representation of S_n determines the functor F up to isomorphism; more precisely, that there exists a functorial isomorphism of $F(X)$ onto the subobject of S_n-invariants of $L_F^{(n)}(X)$.

(4.5) **Example.** In the Example (3.3) we have $L_F^{(n)}(X) = T^n(X)$, the nth tensor power of X over k, and the action of S_n on $L_F^{(n)}(X)$ in this case is given by

$$\tilde{F}(s)(x_1 \otimes \cdots \otimes x_n) = \varepsilon(s) x_{s^{-1}(1)} \otimes \cdots \otimes x_{s^{-1}(n)},$$

where $\varepsilon(s)$ is the sign (± 1) of the permutation s, and $x_1, \ldots, x_n$ are elements of X. Hence $L_F^{(n)}(X)^{S_n}$ is the space of skew-symmetric tensors in $T^n(X)$, which is isomorphic to $\bigwedge^n(X)$ since k has characteristic 0.

We consider morphisms $f: X^n \to X^n$ of the form $f = \sum_{\alpha,\beta} \xi_{\alpha\beta} i_\alpha p_\beta$, where the $\xi_{\alpha\beta}$ are elements of k. Since F is homogeneous of degree n, $F(f)$ will be a homogeneous polynomial of degree n in the n^2 variables $\xi_{\alpha\beta}$, with coefficients in End $F(X^n)$. For each $s \in S_n$, let w_s denote the coefficient of $\xi_{s(1)1} \cdots \xi_{s(n)n}$ in $F(f)$, and let

(4.6) $\quad i = \sum i_\alpha : X \to X^n, \qquad p = \sum p_\alpha : X^n \to X.$

Then we have

(4.7) $\quad vF(ip)v = \sum_{s \in S_n} F(s)v.$

Proof. By (4.2), v commutes with $F(s)$; also $v^2 = v$, so that $F(s)v = vF(s)v$, which by definition is the coefficient of $\lambda_1 \cdots \lambda_n \mu_1 \cdots \mu_n$ in

$$F((\lambda))F(s)F((\mu)) = F((\lambda)s(\mu)) = F\Big(\sum_\alpha \lambda_{s(\alpha)} \mu_\alpha i_{s(\alpha)} p_\alpha\Big).$$

This coefficient is clearly w_s, so that we have $w_s = F(s)v$.

Again, $vF(ip)v$ is the coefficient of $\lambda_1 \cdots \lambda_n \mu_1 \cdots \mu_n$ in

$$F((\lambda))F(ip)F((\mu)) = F((\lambda)ip(\mu)) = F\Big(\sum_{\alpha,\beta} \lambda_\alpha \mu_\beta i_\alpha p_\beta\Big)$$

and this coefficient is clearly $\sum_{s \in S_n} w_s = \sum_s F(s)v$. □

We now define two morphisms of functors:

$$\xi = qF(i) : F \to L_F^{(n)}, \qquad \eta = F(p)j : L_F^{(n)} \to F$$

(j and q were defined in (4.3)). Then

(4.8) $\quad \eta\xi = n!$ (i.e., scalar multiplication by $n!$)

$$\xi\eta = \sum_{s \in S_n} \tilde{F}(s).$$

Proof. We have $\eta\xi = F(p)jqF(i) = F(p)vF(i)$, which is the coefficient of $\lambda_1 \cdots \lambda_n$ in $F(p)F((\lambda))F(i) = F(p(\lambda)i)$. Now $p(\lambda)i : X \to X$ is scalar multiplication by $\lambda_1 + \cdots + \lambda_n$, so that $F(p(\lambda)i)$ is scalar multiplication by $(\lambda_1 + \cdots + \lambda_n)^n$, and the coefficient of $\lambda_1 \cdots \lambda_n$ is therefore $n!$, as asserted.

Next, $\xi\eta = qF(i)F(p)j$, so that by (4.3) and (4.7)

$$j\xi\eta q = vF(ip)v = \sum_s F(s)v$$

and hence by (4.4)

$$\xi\eta = \sum_s qF(s)j = \sum_s \tilde{F}(s). \quad \square$$

If a finite group G acts on an object Y in an *abelian* category, and if scalar multiplication by the order $|G|$ of G is an automorphism of Y, then the subobject

$$Y^G = \bigcap_{g\in G} \operatorname{Ker}(1-g)$$

is a direct summand of Y and is equal to the image of the projection

$$\sigma = \frac{1}{|G|} \sum_{g\in G} g.$$

For since $(1-g)\sigma = 0$ for each $g \in G$, we have $\operatorname{Im}(\sigma) \subset \operatorname{Ker}(1-g)$ and hence $\operatorname{Im}(\sigma) \subset Y^G$; and on the other hand

$$Y^G = \bigcap_{g\in G} \operatorname{Ker}(1-g) \subset \operatorname{Ker} \sum_g (1-g)$$

which is equal to $\operatorname{Ker}(1-\sigma)$ because $|G|$ is invertible in Y, and hence is equal to $\operatorname{Im}(\sigma)$ because $\sigma^2 = \sigma$; consequently $Y^G \subset \operatorname{Im}(\sigma)$.

If the category is only pseudo-abelian, we *define* Y^G to be the image of the projection σ. In our situation, G is the symmetric group, acting on $L_F^{(n)}(X)$ via $\tilde{F}$; since the characteristic of k is zero (this is the first point at which we have made use of this assumption) scalar multiplication by $n!$ is an automorphism, and therefore $L_F^{(n)}(X)^{S_n}$ is defined and is a direct summand of $L_F^{(n)}(X)$. Let

$$\varepsilon : L_F^{(n)}(X)^{S_n} \to L_F^{(n)}(X); \qquad \pi : L_F^{(n)}(X) \to L_F^{(n)}(X)^{S_n}$$

be the associated injection and projection, so that $\pi\varepsilon = 1$ and

$$\text{(4.9)} \qquad \varepsilon\pi = \sigma = \frac{1}{n!} \sum_{s\in S_n} \tilde{F}(s) = \frac{1}{n!}\xi\eta$$

by (4.8).

(4.10) **Theorem.** *$F(X)$ is functorially isomorphic to $L_F^{(n)}(X)^{S_n}$. More precisely, the morphisms*

$$\xi' = \pi\xi : F(X) \to L_F^{(n)}(X)^{S_n}, \qquad \eta' = \eta\varepsilon : L_F^{(n)}(X)^{S_n} \to F(X)$$

are functorial isomorphisms such that $\xi'\eta' = n!$ and $\eta'\xi' = n!$.

Proof. This is a direct consequence of (4.8) and (4.9), since we have

$$\xi'\eta' = \pi\xi\eta\varepsilon = n!\,\pi\varepsilon\pi\varepsilon = n!$$

and

$$\eta'\xi' = \eta\varepsilon\pi\xi = \frac{1}{n!}\eta\xi\eta\xi = n!. \quad \square$$

It follows from (4.10) that every homogeneous polynomial functor of degree $n > 0$ from $\mathbf{A}$ to $\mathbf{B}$ is of the form

$$X \to L(X, \ldots, X)^{S_n},$$

where $L: \mathbf{A}^n \to \mathbf{B}$ is k-linear in each variable, and is acted on by S_n. The next step, therefore, would be to classify such functors. We shall not attempt to do this in any great generality; from now onwards we shall take $\mathbf{A}$ to be the category $\mathbf{V}_A$ of finitely generated projective left A-modules, where A is a k-algebra.

5. Tensor products and linear functors

Let A be a ring, $\mathbf{B}$ an additive category. A *left A-module* object in $\mathbf{B}$ is an object M of $\mathbf{B}$ together with a ring homomorphism of A into $\mathrm{End}_{\mathbf{B}}(M)$. Equivalently, $N \to \mathrm{Hom}_{\mathbf{B}}(M, N)$ is a functor from $\mathbf{B}$ to the category of right A-modules and homomorphisms. Likewise we define a *right A-module* object in $\mathbf{B}$. If $\mathbf{B}$ is a category of left modules over a ring B, a right A-module object in $\mathbf{B}$ is the same thing as a (B, A)-bimodule.

Let M be a right A-module object in $\mathbf{B}$. Then for each object N in $\mathbf{B}$ the abelian group $\mathrm{Hom}_{\mathbf{B}}(M, N)$ is a left A-module, and therefore if P is any left A-module the abelian group $\mathrm{Hom}_A(P, \mathrm{Hom}_{\mathbf{B}}(M, N))$ is defined. If there exists an object E in $\mathbf{B}$ and for each N an isomorphism

$$\mathrm{Hom}_{\mathbf{B}}(E, N) \tilde{\to} \mathrm{Hom}_A(P, \mathrm{Hom}_{\mathbf{B}}(M, N))$$

functorial in N, we shall say that E is a *tensor product* of M and P over A, and write $E = M \otimes_A P$. The tensor product, when it exists, is therefore defined by

(5.1) $$\mathrm{Hom}_{\mathbf{B}}(M \otimes_A P, N) \cong \mathrm{Hom}_A(P, \mathrm{Hom}_{\mathbf{B}}(M, N))$$

and (since it represents a functor) is unique up to isomorphism. Note also that M and P lie in general in different categories.

Now let A, B be k-algebras, let $C = A \otimes_k B$ and let M be a right C-module object in the category $\mathbf{B}$. The canonical homomorphisms $a \to a \otimes 1$ and $b \to 1 \otimes b$ of A and B into C determine by restriction of scalars A-module and B-module structures on M. Let P be a left A-module and Q a left B-module, so that $P \otimes_k Q$ is a left C-module. The isomorphism (5.1) shows that the abelian group $\mathrm{Hom}_{\mathbf{B}}(M \otimes_A P, N)$ has a left B-module structure for each object N in $\mathbf{B}$, so that $M \otimes_A P$, when it exists, is a right B-module object in $\mathbf{B}$. In this situation we have

(5.2) $$M \otimes_{A \otimes_k B} (P \otimes_k Q) \cong (M \otimes_A P) \otimes_B Q$$

in the sense that if one side exists, so does the other and they are isomorphic. For by (5.1)

$$\mathrm{Hom}_{\mathbf{B}}(M \otimes_C (P \otimes_k Q), N) \cong \mathrm{Hom}_C(P \otimes_k Q, \mathrm{Hom}_{\mathbf{B}}(M, N))$$

and

$$\mathrm{Hom}_{\mathbf{B}}((M \otimes_A P) \otimes_B Q, N) \cong \mathrm{Hom}_B(Q, \mathrm{Hom}_A(P, \mathrm{Hom}_{\mathbf{B}}(M, N)))$$

and each of the $\mathbf{Z}$-modules on the right is isomorphic to the $\mathbf{Z}$-module of functions from $P \times Q$ to $\mathrm{Hom}_{\mathbf{B}}(M, N)$ which are A-linear in the first factor and B-linear in the second.

We shall now take up the problem raised at the end of Section 4, namely that of finding all functors $L: \mathbf{V}_A^n \to \mathbf{B}$ which are k-linear in each variable, where A is any k-algebra and $\mathbf{B}$ is as before a pseudo-abelian k-linear category.

We begin with the case $n = 1$. Let then $L: \mathbf{V}_A \to \mathbf{B}$ be a k-linear functor (hence additive). For each left A-module P let $e = e_P : P \to \mathrm{Hom}_A(A, P)$ be the canonical isomorphism, so that $e(p)a = ap$ for $a \in A$ and $p \in P$. In particular, taking $P = A$, $e(a)$ is right multiplication by a; applying the functor L, we obtain $L(e(a)) : L(A) \to L(A)$ for each $a \in A$, and hence $L(A)$ is a right A-module object in $\mathbf{B}$.

Now let N be any object in $\mathbf{B}$, and let

(5.3) $$\psi = \psi(P) : \mathrm{Hom}_{\mathbf{B}}(L(P), N) \to \mathrm{Hom}_A(P, \mathrm{Hom}_{\mathbf{B}}(L(A), N))$$

be the homomorphism of $\mathbf{Z}$-modules defined by $\psi(f)(p) = f \circ L(e(p))$.

(5.4) *For each $P \in \mathbf{V}_A$, $\psi(P)$ is an isomorphism.*

Proof. It is easily checked that $\psi(A)$ is the isomorphism e. Since ψ is a morphism between additive functors, it follows that $\psi(A^n) = \psi(A)^n$ is an isomorphism. If now $P \in \mathbf{V}_A$, there exists $Q \in \mathbf{V}_A$ such that $P \oplus Q \cong A^n$ for some $n \geq 0$; hence $\psi(P \oplus Q)$ is an isomorphism. But $\psi(P \oplus Q) = \psi(P) \oplus \psi(Q)$, hence $\psi(P)$ is an isomorphism. $\square$

From (5.1) and (5.4) it follows that $L(P)$ is a tensor product of $L(A)$ and P over A:

$$L(P) = L(A) \otimes_A P$$

functorially in P. In other words,

(5.5) $$L = U_{L(A)}$$

where $U_{L(A)}$ is the functor $P \mapsto L(A) \otimes_A P$.

We consider now the general case. Let $L: \mathbf{V}_A^n \to \mathbf{B}$ be k-linear in each variable, and let $T^n(A)$ be the nth tensor power of A over k. Then $L^{(n)}(A) = L(A, \ldots, A)$ is a right $T^n(A)$-module object in $\mathbf{B}$, the action of $a_1 \otimes \cdots \otimes a_n \in T^n(A)$ being the morphism $L(e(a_1), \ldots, e(a_n))$.

(5.6) *There exists an isomorphism of functors*

$$L(P_1, \ldots, P_n) \cong L^{(n)}A \otimes_{T^n(A)} (P_1 \otimes_k \cdots \otimes_k P_n).$$

Proof. We shall write out the proof for $n = 2$. Regarding P_1 as fixed and P_2 as variable, we have from (5.5)

$$\begin{aligned} L(P_1, P_2) &\cong L(P_1, A) \otimes_A P_2 \\ &\cong (L(A, A) \otimes_A P_1) \otimes_A P_2 \quad \text{by (5.5) again} \\ &\cong L^{(2)}(A) \otimes_{T^2(A)} (P_1 \otimes_k P_2) \quad \text{by (5.2).} \quad \square \end{aligned}$$

From (4.10) and (5.6) we have immediately

(5.7) **Theorem.** *Let $F: \mathbf{V}_A \to \mathbf{B}$ be a polynomial functor, homogeneous of degree $n > 0$. Then there exists an isomorphism of functors*

$$F(P) \cong (L_F^{(n)}(A) \otimes_{T^n(A)} T^n(P))^{S_n},$$

where $L_F^{(n)}(A) = L_F(A, \ldots, A)$, and L_F is the linearization (Section 3) of F, and $T^n(P)$ (resp. $T^n(A)$) is the nth tensor power of P (resp. A) over k.

Remarks. (1) We have assumed throughout, for convenience of exposition, that the functor F is covariant. There is no difficulty in dealing with contravariant functors: if $F: \mathbf{V}_A \to \mathbf{B}$ is a contravariant polynomial functor, then $F^*: P \mapsto F(P^*)$ is covariant, where $P^* = \mathrm{Hom}_A(P, A)$ is the dual of P, and is a finitely generated projective *right* A-module; $L_F^{(n)}(A)$ is now a *left* $T^n(A)$-module object in $\mathbf{B}$, and in place of (5.7) we have

$$F(P) \cong (T^n(P^*) \otimes_{T^n(A)} L_F^{(n)}(A))^{S_n}.$$

(2) In (5.7) the symmetric group S_n acts on the k-algebra $T^n(A)$ and the $T^n(A)$-module $T^n(P)$, by permuting the factors in these tensor products; also S_n acts on the right $T^n(A)$-module object $L_F^{(n)}(A)$; and in each case the action of S_n is compatible with the module structure, so that for example $s(ap) = s(a)s(p)$ for $a \in T^n(A)$, $p \in T^n(P)$ and $s \in S_n$. In the next section we shall consider this situation more generally.

6. Wreath products

Let G be a subgroup of S_n. The group G acts on $T^n(A)$ by permuting the factors:

$$s(a_1 \otimes \cdots \otimes a_n) = a_{s^{-1}(1)} \otimes \cdots \otimes a_{s^{-1}(n)}.$$

Let kG be the group algebra of G over k, and define a multiplication in the vector space $T^n(A)\otimes_k kG$ by the rule

$$(a\otimes s)(b\otimes t)=a\cdot s(b)\otimes st$$

for $a, b\in T^n(A)$ and $s, t\in G$. The resulting k-algebra we shall call the *wreath product of* A *with* G, and denote by $A\sim G$. It is associative and has an identity element $1\otimes\cdots\otimes 1\otimes e$, where 1 (resp. e) is the identity element of A (resp. G).

In particular, if A is the group algebra $k\Gamma$ of a group Γ, then $T^n(A)$ may be identified with the group algebra of $\Gamma^n=\Gamma\times\cdots\times\Gamma$, and $A\sim G$ with the group algebra of the wreath product $\Gamma\sim G$ of Γ with G:

$$(6.1)\qquad (k\Gamma)\sim G\cong k(\Gamma\sim G).$$

(To be quite explicit, we define $\Gamma\sim G$ to be the group whose elements are pairs (γ, s), where $\gamma=(\gamma_1,\ldots,\gamma_n)\in\Gamma^n$ and $s\in G$ with multiplication defined by $(\gamma, s)(\delta, t)=(\gamma\cdot s\delta, st)$, where $s\delta=(\delta_{s^{-1}(1)},\ldots,\delta_{s^{-1}(n)})$. The isomorphism (6.1) then identifies (γ, s) with $\gamma_1\otimes\cdots\otimes\gamma_n\otimes s$.)

In view of (6.1), we may leave out the brackets and write $k\Gamma\sim G$ unambiguously.

If also H is a subgroup of a symmetric group S_p, then $G\sim H$ is a subgroup of $S_n\times S_p$, hence (up to conjugacy) a subgroup of S_{n+p}, we have

$$(6.2)\qquad (A\sim G)\otimes_k(A\sim H)\cong A\sim(G\times H)$$

by identifying $(a\otimes s)\otimes(b\otimes t)$ with $(a\otimes b)\otimes(s,t)$ where $a\in T^n(A)$, $b\in T^p(A)$, $s\in G$ and $t\in H$.

Again, the wreath product $G\sim H$ is a subgroup of $S_n\sim S_p$, which in turn may be identified (up to conjugacy) with a subgroup of S_{np} ([6, p. 31]), and then we have

$$(6.3)\qquad (A\sim G)\sim H\cong A\sim(G\sim H).$$

For

$$\begin{aligned}(A\sim G)\sim H&=T^p(T^n(A)\otimes_k kG)\otimes_k kH\\ &\cong T^{np}(A)\otimes_k T^p(kG)\otimes_k kH\\ &\cong T^{np}(A)\otimes_k(kG\sim H)=A\sim(G\sim H).\end{aligned}$$

We may therefore write $A\sim G\sim H$ unambiguously.

Now let M be a left A-module. Then $T^n(M)$ is a left $T^n(A)$-module on which S_n acts by permuting the factors, so that we have $s(am)=s(a)s(m)$ for $s\in S_n$, $a\in T^n(A)$ and $m\in T^n(M)$. We may therefore regard $T^n(M)$ as a left $A\sim S_n$-module by defining $(a\otimes s)m=a\cdot s(m)$. If M is finitely-generated and projective, it is a direct summand of say A^r, and hence $T^n(M)$ is a direct summand of $T^n(A^r)$, which is a free $T^n(A)$-module: hence $T^n(M)$ is finitely-generated and projective as a $T^n(A)$-module, and therefore by (A.3) also as an $A\sim S_n$-module. In other words, T^n is a functor from $\mathbf{V}_A$ to $\mathbf{V}_{A\sim S_n}$.

If now V is a left kS_n-module, then $A \sim S_n$ acts on $T^n(M) \otimes_k V$ by the rule $(a \otimes s)(m \otimes v) = a \cdot s(m) \otimes s(v)$, where $a \in T^n(A)$, $m \in T^n(M)$, $s \in S_n$ and $v \in V$. The resulting $A \sim S_n$-module we call the *wreath project of M with V*, and denote by $M \sim V$.

Let $F: \mathbf{V}_A \to \mathbf{B}$ be a homogeneous polynomial functor of degree n; then $L_F^{(n)}(A)$ is a right $T^n(A)$-module object in $\mathbf{B}$, on which S_n acts via $\tilde{F}$ (Section 4), hence is a right $A \sim S_n$-module object in $\mathbf{B}$, if we define the right action of $s \in S_n$ to be $\tilde{F}(s^{-1})$. (Explicitly, if $\mathbf{B}$ is a category of modules, the right action of $A \sim S_n$ on $L_F^{(n)}(A)$ is given by $x(a \otimes s) = s^{-1}(xa) = s^{-1}(x)s^{-1}(a)$ for $a \in T^n(A)$, $s \in S_n$ and $x \in L_F^{(n)}(A)$.) From (A.2) we have

$$(L_F^{(n)}(A) \otimes_{T^n(A)} T^n(P))^{S_n} \cong L_F^{(n)}(A) \otimes_{A \sim S_n} T^n(P),$$

and therefore we can restate (5.7) as follows:

(6.4) **Theorem.** *Let $F: \mathbf{V}_A \to \mathbf{B}$ be a homogeneous polynomial functor of degree $n > 0$, and let $M = L_F^{(n)}(A)$. Then*

$$F \cong U_M \circ T^n$$

where $T^n: \mathbf{V}_A \to \mathbf{V}_{A \sim S_n}$ is the nth tensor power functor, and $U_M: \mathbf{V}_{A \sim S_n} \to \mathbf{B}$ is the k-linear functor (Section 5) defined by

$$U_M(Q) = M \otimes_{A \sim S_n} Q.$$

In other words, every homogeneous polynomial functor of degree n is obtained by composing T^n with a linear functor.

Now let $\mathbf{P}_n$ denote the category of homogeneous polynomial functors $F: \mathbf{V}_A \to \mathbf{B}$ of degree n, and let $\mathbf{M}_n$ denote the category of right $A \sim S_n$-module objects M in the category $\mathbf{B}$ for which the tensor product $M \otimes_{A \sim S_n} T^n(P)$ exists for all P in $\mathbf{V}_A$ (this condition will be automatically satisfied if $\mathbf{B}$ is a category of modules), the morphisms in $\mathbf{M}_n$ being the morphisms in $\mathbf{B}$ which commute with the action of $A \sim S_n$.

(6.5) **Theorem.** *The functors $\alpha: \mathbf{P}_n \to \mathbf{M}_n$ and $\beta: \mathbf{M}_n \to \mathbf{P}_n$ defined by*

$$\alpha(F) = L_F^{(n)}(A), \qquad \beta(M) = U_M \circ T^n$$

constitute an equivalence of categories.

Proof. We have $\beta \circ \alpha \cong 1_{\mathbf{P}_n}$ by (6.4), and we have to verify that $\alpha \circ \beta \cong 1_{\mathbf{M}_n}$. If $\beta(M) = F$, we have

$$F(P_1 \oplus \cdots \oplus P_n) = M \otimes_{A \sim S_n} T^n(P_1 \oplus \cdots \oplus P_n)$$

and therefore, from the definition of L_F in Section 3,

$$L_F(P_1, \ldots, P_n) = M \otimes_{A \sim S_n} \left(\bigoplus_{s \in S_n} P_{s(1)} \otimes \cdots \otimes P_{s(n)} \right)$$

so that

$$L_F^{(n)}(A) = L_F(A, \ldots, A) \cong M \otimes_{A \sim S_n} (A \sim S_n) \cong M. \quad \square$$

It follows from (6.5) that the functors α and β establish a one-one correspondence between the isomorphism classes of homogeneous polynomial functors of degree n from $\mathbf{V}_A$ to $\mathbf{B}$, and the isomorphism classes of right $A \sim S_n$-module objects in the category $\mathbf{B}$. We shall now examine some of the properties of this correspondence, taking the category $\mathbf{B}$ now to be $\mathbf{V}_B$, where B is a k-algebra.

(6.6) **Tensor product.** Let F, $G: \mathbf{V}_A \to \mathbf{V}_B$ be homogeneous polynomial functors, of degrees m and n respectively. Let $F \otimes G$ be the functor defined by

$$(F \otimes G)(P) = F(P) \otimes_k G(P);$$

it is homogeneous polynomial of degree $m + n$, with values in the category $\mathbf{V}_{B \otimes_k B}$. If $L_F^{(m)}(A) = M$ and $L_G^{(n)}(A) = N$, we have

$$\begin{aligned}(F \otimes G)(P) &\cong (M \otimes_{A \sim S_m} T^m(P)) \otimes_k (N \otimes_{A \sim S_n} T^n(P)) \\ &\cong (M \otimes_k N) \otimes_{A \sim (S_m \times S_n)} T^{m+n}(P)\end{aligned}$$

by use of the isomorphism (6.2). Hence $F \otimes G$ corresponds to the $(B \otimes_k B, A \sim S_{m+n})$-bimodule

$$M \cdot N = (M \otimes_k N) \otimes_{A \sim (S_m \times S_n)} (A \sim S_{m+n})$$

which we call the *induction product* of M and N. Since tensor products are commutative and associative (up to isomorphism), so also are induction products.

(6.7) **Composition.** Let C be another k-algebra and let $F: \mathbf{V}_A \to \mathbf{V}_B$, $G: \mathbf{V}_B \to \mathbf{V}_C$ be homogeneous polynomial functors of degrees m, n respectively. Then $G \circ F: \mathbf{V}_A \to \mathbf{V}_C$ is a homogeneous polynomial functor of degree mn. If $L_F^{(m)}(A) = M$ and $L_G^{(n)}(B) = N$, so that M is a $(B, A \sim S_m)$-bimodule and N is a $(C, B \sim S_n)$-bimodule, then we have

$$\begin{aligned}(G \circ F)(P) &\cong N \otimes_{B \sim S_n} T^n(M \otimes_{A \sim S_m} T^m(P)) \\ &\cong (N \otimes_{B \sim S_n} T^n(M)) \otimes_{A \sim S_m \sim S_n} T^{mn}(P)\end{aligned}$$

so that $G \circ F$ corresponds to the $(C, A \sim S_{mn})$-bimodule

$$N \circ M = (N \otimes_{B \sim S_n} T^n(M)) \otimes_{A \sim S_m \sim S_n} (A \sim S_{mn})$$

which we call the *composition product* (or *plethysm*) of M with N. Since composition of functors is associative, so is this composition product (up to isomorphism).

The two products just defined satisfy the "distributive law"

(6.8) $\quad (N_1 \circ M) \cdot (N_2 \circ M) \cong (N_1 \cdot N_2) \circ M.$

For the corresponding relation for the functors is

(6.8') $\quad (G_1 \circ F) \otimes (G_2 \circ F) = (G_1 \otimes G_2) \circ F,$

which is obvious from the definitions.

(6.9) **Remark.** In the situation of (6.7), if F is linear (so that $m = 1$ and $F = U_M$ in the notation of (5.5)), and if $B = C = k$, then M is a right A-module, finite-dimensional over k; N is a finite-dimensional kS_n-module; and we have

$$\begin{aligned}(G \circ F)(P) &\cong N \otimes_{kS_n} T^n(M \otimes_A P)\\ &\cong (N \otimes_k (T^n(M) \otimes_{T^n(A)} T^n(P)))^{S_n}\\ &\cong (N \otimes_k T^n(M)) \otimes_{A \sim S_n} T^n(P),\end{aligned}$$

the last two isomorphisms by virtue of (A.2). Hence in this case $G \circ F$ corresponds to the right $A \sim S_n$-module $T^n(M) \otimes_k N = M \sim N$, i.e. to the *wreath product* of M by N.

7. Irreducible polynomial functors

A functor $F: \mathbf{V}_A \to \mathbf{V}_B$ is said to be *irreducible* if $F \neq 0$ and F has no subfunctors other than 0 and F. If F is an irreducible polynomial functor, the decomposition of Section 2 shows that F must be homogeneous, of degree say n; discarding the uninteresting case $n = 0$, it then follows that F corresponds, in the correspondence established in Section 6, to a $(B, A \sim S_n)$-bimodule which is finitely-generated and projective as a left B-module and is *simple* as a right $A \sim S_n$-module. In particular, when $A = B = k$, the irreducible polynomial functors $\mathbf{V}_k \to \mathbf{V}_k$ of degree n correspond to the simple kS_n-modules, i.e. to the irreducible k-representations of S_n.

We shall assume from now on that $B = k$ and that k is algebraically closed (and of course of characteristic 0). The irreducible linear functors from $\mathbf{V}_A$ to $\mathbf{V}_k$ correspond, up to isomorphism, to the simple right A-modules which are finite-dimensional over k. Let A^* denote the set of isomorphism classes of finite-dimensional simple right A-modules, and for each $\alpha \in A^*$ choose a representative E_α of the class α. Let $L_\alpha: \mathbf{V}_A \to \mathbf{V}_k$ denote the corresponding linear functor, so that

(7.1) $\quad L_\alpha(P) = E_\alpha \otimes_A P.$

In particular, we shall use this notation when $A = kS_m$: to each class $\pi \in S_m^*$ $(= (kS_m)^*)$ there corresponds a linear functor $L_\pi: \mathbf{V}_{kS_m} \to \mathbf{V}_k$, defined by $L_\pi(P) = E_\pi \otimes_{kS_m} P$.

It follows from (A.5) that the finite-dimensional simple right $A \sim S_n$-modules are induction products of wreath products of the form

$$(E_{\alpha_1} \sim E_{\rho_1}) \cdot \cdots \cdot (E_{\alpha_r} \sim E_{\rho_r})$$

where $\alpha_1, \ldots, \alpha_r \in A^*$ are distinct classes, and $\rho_i \in S^*_{m_i}$ $(1 \leq i \leq r)$ with $m_1 + \cdots + m_r = n$. Since the induction product is commutative, the order of the factors is immaterial. From (6.9), the wreath product $E_\alpha \sim E_\rho$ $(\alpha \in A^*, \rho \in S^*_m)$ corresponds to the functor

$$P \to E_\rho \otimes_{kS_m} T^m(E_\alpha \otimes_A P)$$

i.e. to the functor $L_\rho \circ T^m \circ L_\alpha$. Hence:

(7.2) **Theorem.** *Every irreducible polynomial functor $F: \mathbf{V}_A \to \mathbf{V}_k$ is isomorphic to a tensor product*

$$\bigotimes_{i=1}^{r} (L_{\rho_i} \circ T^{m_i} \circ L_{\alpha_i})$$

where the $\alpha_i \in A^*$ are all distinct, the m_i are positive integers and $\rho_i \in S^*_{m_i}$. Moreover, this factorization of F is unique (up to the order of the factors).

(7.3) **Remark.** The irreducible representations of S_n are customarily indexed by the partitions of n, and it is convenient to identify $\coprod_{n \geq 0} S^*_n$ with the set of all partitions (including the empty partition). It then follows from (7.2) that the irreducible polynomial functors $F: \mathbf{V}_A \to \mathbf{V}_k$ are indexed by the partition-valued functions p on A^* such that

$$\|p\| = \sum_{\alpha \in A^*} |p(\alpha)| < \infty.$$

In particular, when $A = k$, the irreducible polynomial functors $\mathbf{V}_k \to \mathbf{V}_k$ are indexed by partitions π. We shall denote by F_π the functor corresponding to π, so that

(7.4) $$F_\pi(V) = E_\pi \otimes_{kS_n} T^n(V)$$

where $n = |\pi|$ and E_π is the simple kS_n-module indexed by π.

8. The Grothendieck ring

Let $\mathbf{P}_A$ (resp. $\mathbf{P}_A^{(n)}$) denote the category of polynomial functors $F: \mathbf{V}_A \to \mathbf{V}_k$ of bounded degree (2.3) (resp. homogeneous of degree n). Since the category $\mathbf{V}_k$ is abelian, so are the categories $\mathbf{P}_A$ and $\mathbf{P}_A^{(n)}$. Let $K(\mathbf{P}_A)$, $K(\mathbf{P}_A^{(n)})$ denote their respective Grothendieck groups. We shall denote by $[F]$ the class of F in $K(\mathbf{P}_A)$.

$K(\mathbf{P}_A)$ has plenty of structure. First of all, it is graded: indeed, it is clear from Section 2 that

$$K(\mathbf{P}_A) = \bigoplus_{n \geq 0} K(\mathbf{P}_A^{(n)}).$$

Next, it is a commutative ring, the ring structure being defined by the tensor product

(6.6), namely

$$[F]\cdot[G]=[F\otimes G]$$

and the ring structure respects the grading. Moreover, $K(\mathbf{P}_A)$ is a λ-ring, the λ-operations being defined by

$$\lambda^i[F]=[\bigwedge^i\circ F],$$

where $\bigwedge^i:\mathbf{V}_k\to\mathbf{V}_k$ is the ith exterior power functor.

$K(\mathbf{P}_A)$ also carries a scalar product. If F, G are polynomial functors, it is clear that the set $\mathrm{Hom}(F,G)$ of functorial morphisms from F to G has the structure of a k-vector space. If F and G are homogeneous of degrees m and n respectively, and $\phi\in\mathrm{Hom}(F,G)$, then we have $\lambda^m\cdot\phi=\phi\cdot\lambda^n$ for all $\lambda\in k$, so that $\phi=0$ unless $m=n$; and if $m=n$, then $\mathrm{Hom}(F,G)$ is finite-dimensional over k, because it is isomorphic to $\mathrm{Hom}_{A\sim S_n}(M,N)$, where M and N are the $A\sim S_n$-modules which correspond to F and G respectively. So we may define a $\mathbf{Z}$-valued scalar product in $K(\mathbf{P}_A)$ by

$$\langle F,G\rangle=\dim_k\mathrm{Hom}(F,G)=\sum_{n\geqslant 0}\dim_k\mathrm{Hom}(F_n,G_n)$$

where F_n, G_n are the homogeneous parts (Section 2) of F, G respectively.

(8.1) **Theorem.** *$K(\mathbf{P}_A)$ is freely generated as a λ-ring* (over $\mathbf{Z}$) *by the classes $[L_\alpha]$, $\alpha\in A^*$, in the notation of Section 7.*

Proof. For any k-algebra B, let $R(B)$ denote the Grothendieck group of the category of finite-dimensional right B-modules. It follows from the Jordan–Hölder theorem that $R(B)$ may be canonically identified with the free $\mathbf{Z}$-module generated by B^*.

Now from (6.5) it follows that $K(\mathbf{P}_A^{(n)})\cong R(A\sim S_n)$, the irreducible functors corresponding to the simple modules. Hence $K(\mathbf{P}_A^{(n)})$ is freely generated as $\mathbf{Z}$-module by the classes of the irreducible functors, and the same is therefore true of $K(\mathbf{P}_A)$.

For each $\alpha\in A^*$, let K_α be the $\mathbf{Z}$-submodule of $K(\mathbf{P}_A)$ generated by the classes $[L_\rho\circ T^m\circ L_\alpha]$, where m is any integer $\geqslant 0$ and $\rho\in S_m^*$. From (7.2) it follows that

$$(8.2)\qquad K(\mathbf{P}_A)\cong\bigotimes_{\alpha\in A^*}K_\alpha.$$

Moreover, each K_α is a subring of $K(\mathbf{P}_A)$, by virtue of the distributive law (6.8′), which also shows that the linear mapping from $K(\mathbf{P}_k)\to K_\alpha$ defined by $[F]\to[F\circ L_\alpha]$ is a ring homomorphism; it is in fact an isomorphism, because K_α (resp. $K(\mathbf{P}_k)$) is freely generated as a $\mathbf{Z}$-module by the classes $[L_\rho\circ T^m\circ L_\alpha]$ (resp. $[L_\rho\circ T^m]$). Hence we have

$$(8.3)\qquad K(\mathbf{P}_k)\cong K_\alpha\quad\text{for each }\alpha\in A^*.$$

From (8.2) and (8.3), it follows that the proof of (8.1) reduces to the special case $A=k$: that is to say, we are reduced to showing that $K(\mathbf{P}_k)$ is the free λ-ring generated by the class $[1_{\mathbf{V}_k}]$ of the identity functor. This is a known result ([1, 7]), but for completeness we include the proof.

Let $t_1, t_2, \ldots$ be an infinite sequence of independent variables and for each $r \geq 1$ let

$$\Lambda_r = \mathbf{Z}[t_1, \ldots, t_r]^{S_r}$$

be the $\mathbf{Z}$-algebra of symmetric polynomials in $t_1, \ldots, t_r$. This is a graded ring, say

$$\Lambda_r = \bigoplus_{n \geq 0} \Lambda_r^{(n)}$$

where $\Lambda_r^{(n)}$ consists of the homogeneous symmetric polynomials (including 0) of degree n. The homomorphism $\mathbf{Z}[t_1, \ldots, t_{r+1}] \to \mathbf{Z}[t_1, \ldots, t_r]$ which sends t_{r+1} to 0 and the other t_i to themselves defines on restriction to Λ_{r+1} a surjective homomorphism $q_r : \Lambda_{r+1} \to \Lambda_r$ of graded rings. We set

$$\Lambda = \varprojlim \Lambda_r,$$

the inverse limit being taken in the category of *graded* rings, so that $\Lambda = \bigoplus_{n \geq 0} \Lambda^{(n)}$, where $\Lambda^{(n)} = \varprojlim \Lambda_r^{(n)}$. Then by the fundamental theorem on symmetric functions we have $\Lambda = \mathbf{Z}[e_1, e_2, \ldots]$, where the e's are the elementary symmetric functions in the variables t_i ([7, Chapter I]). We make Λ into a λ-ring by defining $\lambda^n(e_1) = e_n$ for all $n \geq 1$, and then Λ is the free λ-ring over $\mathbf{Z}$ generated by e_1.

Now let $F : \mathbf{V}_k \to \mathbf{V}_k$ be a polynomial functor. For each $\lambda = (\lambda_1, \ldots, \lambda_r) \in k^r$ let (λ) denote the diagonal endomorphism of k^r with eigenvalues $\lambda_1, \ldots, \lambda_r$. Then trace $F((\lambda))$ will be a polynomial in $\lambda_1, \ldots, \lambda_r$, which is symmetric because trace $F(s(\lambda)s^{-1}) = \text{trace } F((\lambda))$ for all permutations $s \in S_r$. Since the trace is an additive function, it determines a mapping

$$\mathrm{ch}_r : K(\mathbf{P}_k) \to \Lambda_r,$$

namely $\mathrm{ch}_r[F]\,(\lambda_1, \ldots, \lambda_r) = \text{trace } F((\lambda))$. Since the trace of a tensor product is the product of the traces, ch_r is a homomorphism of graded rings. Moreover, it is clear from our definitions that $\mathrm{ch}_r = q_r \circ \mathrm{ch}_{r+1}$; hence, letting $r \to \infty$, we obtain a homomorphism of graded rings

$$\mathrm{ch} : K(\mathbf{P}_k) \to \Lambda$$

called the *characteristic homomorphism*. In particular, $\mathrm{ch}[1_{\mathbf{V}_k}] = e_1$, and more generally $\mathrm{ch}[\bigwedge^n] = e_n$. Likewise, the images under ch of the symmetric powers S^n are the complete symmetric functions h_n.

To complete the proof of (8.1) it is enough to show that

(8.4) $\mathrm{ch} : K(\mathbf{P}_k) \to \Lambda$ *is an isomorphism of* λ*-rings*

Proof. [1] We have already remarked that $\mathrm{ch}[\bigwedge^n] = e_n$, and that $\Lambda = \mathbf{Z}[e_1, e_2, \ldots]$, from which it follows that ch is surjective. Now the rank of $\Lambda^{(n)}$ as $\mathbf{Z}$-module is equal to the number of monomials in the e_i of total degree n, hence is the number $p(n)$ of partitions of n. On the other hand, the rank of $K(\mathbf{P}_k^{(n)})$, the component of degree n in $K(\mathbf{P}_k)$, is by (6.5) equal to the rank of $R(S_n)$, i.e. to the number of irreducible representations of S_n, hence is also equal to $p(n)$. Consequently $\mathrm{ch}: K(\mathbf{P}_k^{(n)}) \to \Lambda^{(n)}$ is a surjective homomorphism between free $\mathbf{Z}$-modules of the same rank, hence is an isomorphism. Finally, the fact that $[\bigwedge^n]$ corresponds to $e_n = \lambda^n(e_1)$ shows that ch is an isomorphism of λ-rings. $\square$

There is another description [1] of the characteristic homomorphism which we shall now explain. Each element s of the symmetric group S_n acts on $T^n(k^r)$ by permuting the factors in this tensor product, hence $s \cdot T^n((\lambda))$ is a linear transformation of $T^n(k^r)$, where as before (λ) denotes the diagonal endomorphism of k^r with eigenvalues $\lambda_1, \ldots, \lambda_r$. Define $\Delta_{n,r}(s) \in \Lambda_r^{(n)}$ by

$$\Delta_{n,r}(s)(\lambda) = \mathrm{trace}(sT^n((\lambda))).$$

Since s commutes with $T^n((\lambda))$ it follows that $\Delta_{n,r}$ is a class function on S_n with values in $\Lambda_r^{(n)}$. Also it is easily seen that $\Delta_{n,r}(s) = q_r \circ \Delta_{n,r+1}(s)$, so that the $\Delta_{n,r}(s)$ for fixed n and s and varying r define an element $\Delta_n(s) \in \Lambda^{(n)}$, and that Δ_n is a class function on S_n with values in $\Lambda^{(n)}$.

Now let $F: \mathbf{V}_k \to \mathbf{V}_k$ be a homogeneous polynomial functor of degree n, let $E = L_F^{(n)}(k)$ be the corresponding kS_n-module, and let χ_E be the character of E. Then we have

(8.5) $\qquad \mathrm{ch}[F] = \langle \chi_E, \Delta_n \rangle$

where $\langle\, ,\rangle$ is the usual scalar product of class functions on S_n:

$$\langle u, v\rangle = \frac{1}{n!} \sum_{s \in S_n} u(s)v(s^{-1}) = \frac{1}{n!} \sum_{s \in S_n} u(s)v(s).$$

To prove (8.5), let $M = E \otimes_k T^n(k^r)$, so that $M^{S_n} = F(k^r)$, and let σ, ϕ be the endomorphisms of the vector space M defined by

$$\sigma = \frac{1}{n!} \sum_{s \in S_n} s \otimes s, \qquad \phi = 1_E \otimes T^n((\lambda)).$$

Then σ is a projection of M onto M^{S_n}; also σ commutes with ϕ because, as we have already remarked, the action of $s \in S_n$ on $T^n(k^r)$ commutes with $T^n((\lambda))$. Hence

$$\mathrm{trace}(\sigma\phi) = \mathrm{trace}(\phi \,|\, M^{S_n}).$$

But $\mathrm{trace}(\phi \,|\, M^{S_n}) = \mathrm{trace}\, F((\lambda)) = \mathrm{ch}_r[F](\lambda_1, \ldots, \lambda_r)$, and on the other hand

$$\begin{aligned} \mathrm{trace}(\sigma\phi) &= \frac{1}{n!} \sum_s \mathrm{trace}(s \otimes sT^n((\lambda))) \\ &= \langle \chi_E, \Delta_{n,r}(\lambda_1, \ldots, \lambda_r)\rangle. \end{aligned}$$

Letting $r \to \infty$, we obtain (8.5). $\square$

As in (7.4) let E_π (π a partition of n) be the simple kS_n-module indexed by π, and let $F_\pi: \mathbf{V}_k \to V_k$ be the corresponding functor. Let χ^π be the character of E_π. Since the χ^π form an orthonormal basis for the class functions on S_n, it follows from (8.5) that

(8.6) $$\Delta_n = \sum_\pi \mathrm{ch}[F_\pi] \cdot \chi^\pi.$$

Now $\mathrm{ch}[F_\pi]$ is an isobaric polynomial in the elementary symmetric functions e_n, say

$$\mathrm{ch}[F_\pi] = f_\pi(e_1, e_2, \ldots).$$

As a symmetric function in the variables t_i, this is the *S-function* [7] corresponding to the partition π. If now R is any λ-ring and x is any element of R, we define the *Schur operation* s_π on R by

$$s_\pi(x) = f_\pi(x, \lambda^2(x), \lambda^3(x), \ldots).$$

It remains for us to compute the symmetric functions $\Delta_n(s)$ for each permutation $s \in S_n$. For this purpose we shall use the following lemma, whose proof we leave to the reader:

(8.7) *Let $f_1, \ldots, f_n: V \to V$ be linear transformations of a finite-dimensional vector space V, and let $\phi: T^n(V) \to T^n(V)$ be the linear transformation defined by*

$$\phi(v_1 \otimes \cdots \otimes v_n) = f_1(v_n) \otimes f_2(v_1) \otimes \cdots \otimes f_n(v_{n-1}).$$

Then $\mathrm{trace}(\phi) = \mathrm{trace}(f_n f_{n-1} \cdots f_1)$. □

Suppose first that s is the n-cycle $(1\ 2 \cdots n)$, and apply (8.7) with $f_1 = \cdots = f_n = (\lambda)$ and $V = k^r$. We obtain

$$\mathrm{trace}(sT^n((\lambda)) = \mathrm{trace}((\lambda)^n) = \sum_i \lambda_i^n$$

so that

(8.8) $$\Delta_n(s) = \sum t_i^n = p_n, \quad \text{say},$$

when s is an n-cycle.

If now s is a product of disjoint cycles s_i of orders $n_1, n_2, \ldots$, where $\nu = (n_1, n_2, \ldots)$ is a partition of n, then $sT^n((\lambda))$ is the tensor product of the $s_i T^{n_i}((\lambda))$, and therefore its trace is the product of the traces of the factors: consequently

(8.9) *If $s \in S_n$ has cycle-type $\nu = (n_1, n_2, \ldots)$, then*

$$\Delta_n(s) = p_{n_1} p_{n_2} \cdots = p_\nu \quad \textit{say},$$

a product of power sums. □

Now in the λ-ring Λ, p_n is $\psi^n(e_1)$, where the ψ's are the Adams operations. Hence it follows from (8.6) and (8.8) that the ψ^n are expressed in terms of the Schur operations s_π by

$$\psi^n = \sum_\pi \chi^\pi_{(n)} s_\pi \tag{8.10}$$

where the sum is over the partitions π of n, and $\chi^\pi_{(n)}$ is the value of the character χ^π at an n-cycle. More generally, if we define operations ψ^ν for each partition $\nu = (n_1, n_2, \dots)$ by $\psi^\nu(x) = \psi^{n_1}(x)\psi^{n_2}(x)\dots$, then from (8.6) and (8.9) we have

$$\psi^\nu = \sum_\pi \chi^\pi_\nu s_\pi \tag{8.11}$$

where χ^π_ν is the value of χ^π at elements of cycle-type ν. Of course (8.10) and (8.11) are not new: they are originally due to Frobenius.

Remark. The graded ring $K(\mathbf{P}_A)$ also carries the structure of an (associative, commutative) graded Hopf algebra. The comultiplication may be defined as follows: if $F: \mathbf{V}_A \to \mathbf{V}_k$ is a polynomial functor, define ΔF by $\Delta F(P_1, P_2) = F(P_1 \oplus P_2)$; then ΔF is a polynomial functor on the category $\mathbf{V}_{A\times A} = \mathbf{V}_A \times \mathbf{V}_A$, hence $[F] \to [\Delta F]$ is a mapping

$$K(\mathbf{P}_A) \to K(\mathbf{P}_A \times \mathbf{P}_A) = K(\mathbf{P}_A) \otimes K(\mathbf{P}_A),$$

which is easily verified to have the required properties.

9. Characters of wreath products

In this final section we shall apply the results of Section 8 to the situation in which k is the field of complex numbers and A is the group algebra kG of a finite group G. (In fact, everything will work provided that k is a splitting field for G contained in $\mathbf{C}$.) The category $\mathbf{V}_A$ is then the category of all finite-dimensional kG-modules. We denote by G^* the set of isomorphism classes of simple kG-modules, and for each $\gamma \in G^*$ we choose a representative E_γ of the class γ.

Let G_n denote the wreath product $G \sim S_n$. (In particular, G_0 is the group with one element, and $G_1 = G$.) Let $R(G_n)$ denote the Grothendieck group of the category $\mathbf{V}_{kG_n}$; we shall identify $R(G_n)$ with the free $\mathbf{Z}$-module generated by G_n^*. The direct sum

$$R(G) = \bigoplus_{n\geq 0} R(G_n)$$

is a commutative graded ring with respect to the induction product defined in Section 6: if $\alpha \in G_m^*$, $\beta \in G_n^*$, then $\alpha\beta$ is the class of the kG_{m+n}-module obtained by inducing $E_\alpha \otimes_k E_\beta$ from $G_m \times G_n$ to G_{m+n}. The ring $R(G)$ carries a $\mathbf{Z}$-valued scalar product – the intertwining number – relative to which the union of the G_n^* forms an orthonormal basis of $R(G)$.

As we have seen in Section 8, $R(G)$ may also be regarded as the Grothendieck group of the category of polynomial functors of bounded degree from $\mathbf{V}_{kG}$ to $\mathbf{V}_k$. As such it has the structure of a λ-ring and by (8.1) we may canonically identify $R(G)$ with the free λ-ring generated by G^*. In other words, $R(G)$ is freely generated as $\mathbf{Z}$-algebra by the elements $\lambda^n(\gamma)$ $(n \geqslant 1,\ \gamma \in G^*)$, and hence is freely generated as $\mathbf{Z}$-module by the products

(9.1) $$s_\alpha = \prod_{\gamma \in G^*} s_{\alpha(\gamma)}(\gamma)$$

where α runs through all partition-valued functions on G^*, and the $s_{\alpha(\gamma)}$ are the Schur operations described in Section 8. The generators (9.1) correspond to the irreducible polynomial functors $F_\alpha : \mathbf{V}_{kG} \to \mathbf{V}_k$, or equivalently to the simple kG_n-modules E_α, where $n = \sum_\gamma |\alpha(\gamma)|$: namely $s_\alpha = [E_\alpha]$.

Next we consider class-functions on the groups G_n. Let $C(G_n)$ denote the k-vector space of functions $f : G_n \to k$ which are constant on each conjugacy class, and let

$$C(G) = \bigoplus_{n \geqslant 0} C(G_n).$$

We define a product in $C(G)$ as follows. If $f \in C(G_m)$ and $g \in C(G_n)$, then $f \otimes g : (x, y) \mapsto f(x)g(y)$ is a class function on $G_m \times G_n$, and we define fg to be the class function on G_{m+n} obtained by inducing $f \otimes g$ from $G_m \times G_n$ to G_{m+n}. In this way $C(G)$ acquires the structure of a commutative, associative, graded k-algebra, whose identity element 1 is the characteristic function of G_0. The algebra $C(G)$ also carries a hermitian scalar product $\langle f, g\rangle$: if f, g are homogeneous, $\langle f, g\rangle$ is defined to be zero unless their degrees are equal, and if $f, g \in G_n$ we define

$$\langle f, g\rangle = \langle f, g\rangle_{G_n} = \frac{1}{|G_n|} \sum_{x \in G_n} f(x)\overline{g(x)}.$$

We recall the classification ([6, 9]) of the conjugacy classes of the wreath product $G_n = G \sim S_n$. An element of G_n is of the form (x, s) where $x = (x_1, \ldots, x_n) \in G^n$ and $s \in S_n$. Express s as a product of disjoint cycles: if $z = (i_1 \cdots i_r)$ is one of these cycles, the element $x_{i_r} \cdots x_{i_2} x_{i_1}$ of G is determined up to conjugacy in G by x and z, and we denote its conjugacy class in G by $c(x, z)$. Now let G_* denote the set of conjugacy classes in G. The element $(x, s) \in G_n$ determines a partition-valued function μ on G_* by the following rule: for each $c \in G_*$, the parts of the partition $\mu(c)$ are the lengths of the cycles z in s such that $c(x, z) = c$. Clearly $\|\mu\| = \sum_c |\mu(c)| = n$. Call μ the *type* of (x, s) in G_n. It is well-known, and not hard to verify, that two elements of G_n have the same type if and only if they are conjugate in G_n, and that all partition-valued functions μ on G_* such that $\|\mu\| = n$ occur as types.

For each such μ, let $\psi^\mu \in C(G_n)$ be the unique function such that

(9.2) *For each $f \in C(G_n)$, $\langle f, \psi^\mu\rangle$ is the value of f at elements of type μ.*

In other words, $\psi^\mu(x, s)$ is equal to the order of the centralizer of (x, s) in G_n, if (x, s) has type μ; and is zero otherwise.

If μ, ν are partition-valued functions on G_*, define $\mu \cup \nu$ by $(\mu \cup \nu)(c) = \mu(c) \cup \nu(c)$, the *union* of the partitions $\mu(c), \nu(c)$, for each $c \in G_*$. With this definition we have

(9.3) $\quad \psi^\mu \psi^\nu = \psi^{\mu \cup \nu}$.

Proof. Let $\|\mu\| = m$, $\|\nu\| = n$, so that $\|\mu \cup \nu\| = m + n$. By Frobenius reciprocity,

$$\langle f, \psi^\mu \psi^\nu \rangle_{G_{m+n}} = \langle f \,|\, G_m \times G_n, \psi^\mu \otimes \psi^\nu \rangle_{G_m \times G_n}$$

which by (9.2) is equal to the value of f at an element $(x, s) \times (y, t)$ of $G_m \times G_n$, where (x, s) has type μ and (y, t) type ν. The embedding of $G_m \times G_n$ in G_{m+n} replaces $(x, s) \times (y, t)$ by $(x \times y, s \times t)$ (up to conjugacy in G_{m+n}), and it is clear from the description of conjugacy types that the type of this element is $\mu \cup \nu$. Hence $\langle f, \psi^\mu \psi^\nu \rangle$ is the value of f at elements of type $\mu \cup \nu$ in G_{m+n}, and so (9.3) follows from (9.2) $\square$

We shall now define a λ-ring structure on the k-algebra $C(G)$. For this purpose it is enough to define the Adams operations ψ^n. For each class $c \in G_*$ and each integer $n \geq 1$, let c_n be the class of elements $(x, s) \in G_n$ such that s is an n-cycle and $c(x, s) = c$; this class is described by the partition-valued function $\nu = \nu_{n,c}$ on G_* such that $\nu(c) = n$ and $\nu(c') = 0$ for $c' \neq c$. Define

$$\psi^n(c) = \psi^{\nu_{n,c}}$$

so that by (9.2) we have

(9.4) $\langle f, \psi^n(c) \rangle$ *is the value of* $f \in C(G_n)$ *at the class* c_n.

Now let μ be any partition-valued function on G_*. By expressing each partition $\mu(c) = (\mu_1(c), \mu_2(c), \ldots)$ as the union of the one-part partitions $(\mu_i(c))$, it follows from (9.3) that

(9.5) $$\psi^\mu = \prod_{c,i} \psi^{\mu_i(c)}(c) = \prod_c \psi^{\mu(c)}(c)$$

in the notation of (8.11), the product being over all $c \in G_*$. Since the ψ^μ form a k-basis of $C(G)$, it is clear that

(9.6) $C(G)$ *is the free* λ*-ring over* k *generated by* G_*.

(Of course we define $\psi^n(x) = x$ for all $x \in k$ and all $n \geq 1$.)

(9.7) *Let* $f \in C(G_m)$, $g \in C(G_n)$, *where* $m > 0$ *and* $n > 0$. *Then*

$$\langle fg, \psi^{m+n}(c) \rangle = 0 \quad \textit{for all } c \in G_*.$$

Proof. By Frobenius reciprocity this scalar product is equal to

$$\langle f \otimes g, \psi^{m+n}(c) | G_m \times G_n \rangle_{G_m \times G_n}$$

which is zero because the class c_{m+n} in G_{m+n} does not meet $G_m \times G_n$. $\square$

Now let

$$\chi : R(G) \to C(G)$$

be the $\mathbf{Z}$-linear mapping which assigns to each representation its character. From the definitions it is clear that χ is an isometry and an injective homomorphism of graded rings, and that it induces an isometric isomorphism of graded k-algebras: $k \otimes R(G) \xrightarrow{\sim} C(G)$. The basic fact, which will allow us to compute the irreducible characters of the wreath products G_n, is that χ also respects the λ-structures:

(9.8) **Theorem.** *χ is a homomorphism of λ-rings.*

Proof. Since the functions $\chi(\gamma)$ for $\gamma \in G^*$ form an orthonormal basis of $C(G)$, we have

$$\psi^1(c) = \sum_{\gamma \in G^*} \langle \psi^1(c), \chi(\gamma) \rangle \chi(\gamma)$$

and therefore, since the ψ^n are additive,

$$\psi^n(c) = \sum_\gamma \langle \psi^1(c), \chi(\gamma) \rangle \psi^n(\chi(\gamma)).$$

To prove the theorem it is enough to show that $\psi^n(\chi(\gamma)) = \chi(\psi^n(\gamma))$ for all $\gamma \in G^*$ and $n \geq 1$, or equivalently that

$$\psi^n(c) = \sum_\gamma \langle \psi^1(c), \chi(\gamma) \rangle \chi(\psi^n(\gamma)). \tag{9.9}$$

Now the functions $\chi(s_\alpha)$ such that $\|\alpha\| = n$ (9.1) form an orthonormal basis of $C(G_n)$, and therefore

$$\psi^n(c) = \sum_{\|\alpha\|=n} \langle \psi^n(c), \chi(s_\alpha) \rangle \chi(s_\alpha).$$

But $\chi(s_\alpha) = \prod_\gamma (s_{\alpha(\gamma)}(\gamma))$, and by (9.7) the scalar product $\langle \psi^n(c), \chi(s_\alpha) \rangle$ therefore vanishes unless s_α is of the form $s_\pi(\gamma)$ for some $\gamma \in G^*$ and some partition π of n. Hence

$$\psi^n(c) = \sum_{\gamma, \pi} \langle \psi^n(c), \chi(s_\pi(\gamma)) \rangle \chi(s_\pi(\gamma)). \tag{1}$$

Now $\langle \chi(s_\pi(\gamma)), \psi^n(c) \rangle$ is by our definitions the value of the character of the G_n-module $E_\pi \otimes_k T^n(E_\gamma)$ at an element $(x, s) \in G_n$, where $s = (1\ 2 \cdots n)$ and $x = (x_1, \ldots, x_n) \in G^n$ is such that $x_n x_{n-1} \cdots x_1 \in c$. The action of (x, s) on this module is given by

$$(x, s) \cdot (v \otimes u_1 \otimes \cdots \otimes u_n) = sv \otimes x_1 u_n \otimes x_2 u_1 \otimes \cdots \otimes x_n u_{n-1}$$

and therefore by (8.7) we have

$$\begin{aligned}\langle \psi(s_\pi(\gamma)), \psi^n(c)\rangle &= \operatorname{trace}((x, s), E_\pi \otimes_k T^n(E_\gamma)) \\ &= \operatorname{trace}(s, E_\pi) \cdot \operatorname{trace}(x_n x_{n-1} \cdots x_1, E_\gamma) \\ &= \chi^\pi_{(n)} \langle \chi(\gamma), \psi^1(c)\rangle.\end{aligned}$$

Hence from (1) we have

$$\psi^n(c) = \sum_\gamma \langle \psi^1(c), \chi(\gamma)\rangle \sum_\pi \chi^\pi_{(n)} \chi(s_\pi(\gamma))$$

in which the second sum is

$$\chi\Big(\sum_\pi \chi^\pi_{(n)} s_\pi(\gamma)\Big) = \chi(\psi^n(\gamma))$$

by (8.10). Hence finally we obtain

$$\psi^n(c) = \sum_\gamma \langle \psi^1(c), \chi(\gamma)\rangle \chi(\psi^n(\gamma))$$

which proves (9.9) and hence also (9.8). □

The computation of the irreducible characters of G_n is an immediate consequence of (9.8), or rather of the equivalent statement (9.9). Let (χ^γ_c) be the character table of G, so that

$$\chi^\gamma_c = \langle \chi(\gamma), \psi^1(c)\rangle = \overline{\langle \psi^1(c), \chi(\gamma)\rangle}$$

is the value of the character $\chi(\gamma)$ at elements of the class $c \in G_*$. For each $c \in G_*$ and $n \geq 1$, define

$$\phi^n(c) = \sum_{\gamma \in G^*} \chi^\gamma_c \psi^n(\gamma) \in k \otimes R(G_n)$$

so that $\chi(\phi^n(c)) = \overline{\psi^n(c)}$. Then define, for each partition $\nu = (n_1, n_2, \ldots)$

$$\phi^\nu(c) = \phi^{n_1}(c)\phi^{n_2}(c) \cdots$$

and for each partition-valued function μ on G_* define

$$\phi^\mu = \prod_{c \in G_*} \phi^{\mu(c)}(c)$$

so that we have $\chi(\phi^\mu) = \overline{\psi^\mu} \in C(G_n)$ (where $n = \|\mu\|$).

As before, for each partition-valued function α on G^* such that $\|\alpha\| = n$, let $s_\alpha \in G^*_n$ be the irreducible representation (9.1) parametrized by α, and let (X^α_μ) be the character-table of G_n, so that

$$X^{\alpha}_{\mu} = \langle \chi(s_\alpha), \psi^{\mu} \rangle$$

is the value of $\chi(s_\alpha)$ at elements of type μ.

(9.10) **Theorem.** *The irreducible characters of the wreath product $G_n = G \sim S_n$ are determined by the set of equations*

$$\phi^{\mu} = \sum_{\alpha} X^{\alpha}_{\mu} s_\alpha :$$

in other words, the character table (X^{α}_{μ}) of G_n is the transition matrix between the two k-bases (ϕ^{μ}) and (s_α) of $k \otimes R(G_n)$.

Proof. We have

$$\begin{aligned}\langle \phi^{\mu}, s_\alpha \rangle &= \langle s_\alpha, \overline{\phi^{\mu}} \rangle = \langle \chi(s_\alpha), \chi(\overline{\phi^{\mu}}) \rangle \\ &= \langle \chi(s_\alpha), \psi^{\mu} \rangle = X^{\alpha}_{\mu};\end{aligned}$$

since the s_α form an orthonormal basis of $R(G_n)$, (9.10) follows directly. □

Remarks. (1) The characters of wreath products $G \sim S_n$ were first worked out by Specht [9]. Curiously, (9.10), which is a direct generalization of Frobenius's formula (8.11) for the characters of S_n, does not occur in Specht's paper, although the reverse set of equations does, namely (in our notation)

$$s_\alpha = \sum_{\mu} z_{\mu}^{-1} X^{\alpha}_{\mu} \phi^{\mu}$$

where z_μ is the order of the centralizer of an element of type μ in G_n. In the particular case where G has order 2 (so that G_n is the hyperoctahedral group), (9.10) occurs in A. Young [10].

(2) The degree of the representation s_α of G_n is equal to

$$n! \prod_{\gamma \in G^*} d_{\gamma}^{|\alpha(\gamma)|} / h_{\alpha(\gamma)}$$

where d_γ is the degree of γ, and $h_{\alpha(\gamma)}$ is the product of the hook-lengths of the partition $\alpha(\gamma)$ [7].

Appendix: Twisted group rings

(1) Let R be a ring and let G be a finite group which acts on R as a group of automorphisms. Consider the free left R-module R^G on G as basis, the elements of which are formal sums $\sum_{g \in G} a_g \cdot g$ with coefficients $a_g \in R$. Define a multiplication in

R^G by the rule

$$\Big(\sum_g a_g \cdot g\Big)\Big(\sum_h b_h \cdot h\Big) = \sum_{g,h} a_g \cdot g(b_h) \cdot gh.$$

In this way R^G becomes an associative ring with identity element $1 \cdot e$, where 1 and e are the identity elements of R and G respectively. We denote this ring by $R \rtimes G$ and call it the *twisted group ring* of G over R. The mapping $a \to a \cdot e$ embeds R as a subring of $R \rtimes G$.

Examples. (1) Let N be a group on which G acts as a group of automorphisms, and let $N \rtimes G$ denote the semidirect product of N with G. If k is any commutative ring, the group algebra of $N \rtimes G$ over k is $(kN) \rtimes G$.

(2) Let A be a k-algebra, $R = T^n(A)$ the nth tensor power of A over k. The symmetric group S_n acts on R by permuting the factors, and $T^n(A) \rtimes S_n$ is what we have called the *wreath product* $A \sim S_n$ (Section 6).

Let M be a left R-module in which G acts in such a way that $g(x+y) = g(x) + g(y)$, $g(ax) = g(a)g(x)$ for $a \in R$, $g \in G$ and $x, y \in M$. Then M becomes a left $R \rtimes G$-module if we define $(a \cdot g)x = a \cdot g(x)$. Conversely, any left $R \rtimes G$-module may be regarded as a left R-module (by restriction of scalars) on which G acts as above, by defining $g(x)$ to be $(1 \cdot g)x$. Likewise for right modules. In particular, R itself is a left and right $R \rtimes G$-module.

Let M, N be left $R \rtimes G$-modules. Then G acts on $\mathrm{Hom}_R(M, N)$ in the usual way: $(g \cdot \phi)(x) = g(\phi(g^{-1}(x)))$ for $g \in G$, $\phi : M \to N$ and $x \in M$, and it is immediately verified that

(A.1) $\quad (\mathrm{Hom}_R(M, N))^G = \mathrm{Hom}_{R \rtimes G}(M, N).$

(2) Suppose in this section that the order $|G|$ of G is a unit in R. Let $\mathbf{B}$ be a pseudo-abelian category, let M be a right $R \rtimes G$-module object in $\mathbf{B}$, and let N be a left $R \rtimes G$-module. Then, with tensor products defined as in Section 5, we have

(A.2) $\quad (M \otimes_R N)^G \cong M \otimes_{R \rtimes G} N$

in the sense that if one side exists, so does the other and they are canonically isomorphic.

Proof. Let Y be an object of $\mathbf{B}$ on which G acts, and suppose that scalar multiplication by $|G|$ is an automorphism of Y. Then (Section 4) Y^G is defined to be the image of the idempotent morphism

$$\sigma = |G|^{-1} \sum_{g \in G} g.$$

Now let X be any object of $\mathbf{B}$. Then (since Hom is additive) $\mathrm{Hom}_{\mathbf{B}}(Y^G, X)$ is the image of σ acting on $\mathrm{Hom}_{\mathbf{B}}(Y, X)$, and therefore

$$\mathrm{Hom}_{\mathbf{B}}(Y^G, X) = (\mathrm{Hom}_{\mathbf{B}}(Y, X))^G.$$

Taking $Y = M \otimes_R N$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{B}}((M \otimes_R N)^G, X) &= (\mathrm{Hom}_{\mathbf{B}}(M \otimes_R N), X))^G \\ &= \mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbf{B}}(M, X)))^G \\ &= \mathrm{Hom}_{R \rtimes G}(N, \mathrm{Hom}_{\mathbf{B}}(M, X)) \\ &= \mathrm{Hom}_{\mathbf{B}}(M \otimes_{R \rtimes G} N, X)\end{aligned}$$

which proves (A.2). □

(A.3) *Let P be a left $R \rtimes G$-module which is projective as a left R-module. Then P is projective as a left $R \rtimes G$-module.*

Proof. By choosing a set of generators of P we obtain an exact sequence

$$(*) \qquad 0 \to Q \xrightarrow{\alpha} F \xrightarrow{\beta} P \to 0$$

of $R \rtimes G$-modules and homomorphisms, where F is free. Since P is R-projective, there exists an R-homomorphism $\gamma: P \to F$ such that $\beta\gamma = 1_P$. Let

$$\gamma^* = |G|^{-1} \sum_{g \in G} g\gamma g^{-1},$$

then γ^* is an $R \rtimes G$-homomorphism and $\beta\gamma^* = 1_P$. Hence the exact sequence $(*)$ splits over $R \rtimes G$, and therefore P is $R \rtimes G$-projective. □

(3) Let E be an R-module. Since G acts on R, we can twist the action of R on E by an element $g \in G$. To be precise, let ${}^gE$ denote the R-module whose underlying additive group is E, but with scalar multiplication defined by $(a, v) \mapsto g^{-1}(a)v$ for $a \in R$ and $v \in E$. The submodules of ${}^gE$ are the same as the submodules of E, so that ${}^gE$ is simple if E is simple. In this way G acts on the set R^* of isomorphism classes of simple R-modules. The set of $g \in G$ such that ${}^gE$ is R-isomorphic to E is a subgroup H of G, called the *inertia group* of E.

Suppose now that the ring R is an algebra over an algebraically closed field k, and that G fixes each element of k. All modules will be assumed to be left modules, finite-dimensional over k. Our aim here is to describe (up to isomorphism) all simple $R \rtimes G$-modules.

Let M be a simple $R \rtimes G$-module. Consider M as an R-module by restriction of scalars, and let E be a simple R-submodule of M. For each $g \in G$, the subspace gE is a simple R-module of M, isomorphic to ${}^gE$, and $\sum_{g \in G} gE$ is a nonzero $R \rtimes G$-submodule of M, hence is the whole of M. It follows that M is R-semisimple and that every simple R-submodule of M is isomorphic to ${}^gE$ for some $g \in G$.

Let H be as above the inertia group of E, and put $N = \sum_{h \in H} hE$, which is an $R \rtimes H$-module. If $g_1, \ldots, g_r$ are left coset representatives of H in G, then the g_iN are the isotypic components of M as an R-module, and we have

$$M = \bigoplus_{i=1}^{r} g_iN \cong (R \rtimes G) \otimes_{R \rtimes H} N$$

which we write as

(1) $\qquad M \cong \mathrm{ind}_H^G(N).$

The $R \rtimes H$-module N is a sum of R-submodules isomorphic to E, hence from [2, p. 15, Théorème 1] and the fact that $\mathrm{End}_R(E) = k$ by Schur's lemma (since k is algebraically closed) we have

(2) $\qquad N \cong V \otimes_k E$

where

$$V = \mathrm{Hom}_R(E, N) = \mathrm{Hom}_R(E, M)$$

is a kH-module, with H acting via its action on N. From (1) and (2), it follows that

$$M \cong \mathrm{ind}_H^G(V \otimes_k E).$$

Moreover, the kH-module V must be *simple*, for if V' is a kH-submodule of V then $M' = \mathrm{ind}_H^G(V' \otimes_k E)$ will be an $R \rtimes G$-submodule of M. Hence

(A.4) *Every simple $R \rtimes G$-submodule M is (up to isomorphism) of the form $\mathrm{ind}_H^G(V \otimes_k E)$, where E is a simple R-module, H is the inertia group of E, and V is a simple kH-module.*

We remarked above that every simple R-submodule of M is isomorphic to ${}^gE$ for some $g \in G$. Had we started off with $E' = {}^gE$ in place of E, we should have obtained an isomorphism

$$M \cong \mathrm{ind}_{H'}^G(V' \otimes_k E')$$

where $H' = gHg^{-1}$ and $V' = \mathrm{Hom}_R(E', gN)$ is the simple kH'-module obtained from V by twisting the action of H by g.

Conversely, every $R \rtimes G$-module M as described in (A.4) is simple. For if M' is an $R \rtimes G$-submodule of M, then M' is R-semisimple, hence is the direct sum of its isotypic components $M' \cap g_iN = g_iN'$, where $N' = M' \cap N$, and $M' = V' \otimes_k E$ where $V' = \mathrm{Hom}_R(E, N')$ is a kH-submodule of V. Since V is simple, we have $V' = 0$ or V and correspondingly $M' = 0$ or M.

Finally, suppose that $M_i = \mathrm{ind}_{H_i}^G(V_i \otimes_k E_i)$ $(i = 1, 2)$ are isomorphic simple $R \rtimes G$-modules. Then E_1, being a simple R-submodule of M_1, must be isomorphic to ${}^gE_2$ for some $g \in G$. Replacing E_2 by ${}^gE_2$, we may assume that $E_1 = E_2 = E$ say, and therefore $H_1 = H_2 = H$ say. Since $V_i = \mathrm{Hom}_R(E, M_i)$, it follows that V_1 and V_2 are isomorphic kH-modules.

To summarize:

(A.5) **Theorem.** *Choose respresentatives E_α of the orbits of G in R^*, and let H_α be the inertia group of E_α. Let $V_{\lambda,\alpha}$ run through a complete set of simple kH_α-modules. Then the $R \times G$-modules*

$$\mathrm{ind}_{H_\alpha}^{G}(V_{\lambda,\alpha} \otimes_k E_\alpha) \quad (\alpha \in R^*/G, \lambda \in H_\alpha^*)$$

form a complete set of nonisomorphic simple $R \rtimes G$-modules.

The particular case of this result which is used in Section 7 is that in which $R = T^n(A)$ and $G = S_n$ (Example 2). Since k is algebraically closed, we have $R^* = (A^*)^n$ ([2, p. 94]), the Cartesian product of n copies of A^*, on which S_n acts by permuting the factors. For each $\alpha \in (A^*)^n$ the inertia group H_α is (up to conjugacy in S_n) of the form $S_\nu = S_{n_1} \times \cdots \times S_{n_r}$, where $n_1 + \cdots + n_r = n$, and hence by (A.5) the simple $A \sim S_n$-modules are of the form

$$\mathrm{ind}_{S_\nu}^{S_n}(V \otimes_k T^{n_1}(E_{\alpha_1}) \otimes \cdots \otimes R^{n_r}(E_{\alpha_r})) \tag{3}$$

for all choices of *distinct* $\alpha_1, \ldots, \alpha_r \in A^*$, integers $n_1, \ldots, n_r$ such that $\sum n_i = n$, and simple kS_ν-modules V. Each such V is a tensor product $V_1 \otimes_k \cdots \otimes_k V_r$ where each V_i is a simple kS_{n_i}-module, and therefore the module (3) can be written as

$$(A \sim S_n) \otimes_{A \sim S_\nu} \left(\bigotimes_{i=1}^{r} (V_i \otimes T^{n_i}(E_{\alpha_i})) \right)$$

which, in the terminology introduced in Section 6, is an induction product of wreath products

$$(E_{\alpha_1} \sim V_1) \cdot \cdots \cdot (E_{\alpha_r} \sim V_r).$$

References

[1] M.F. Atiyah, Power operations in K-theory, Quarterly J. of Math., Oxford (2) 17 (1966) 165–193.
[2] N. Bourbaki, Algèbre, Chapter 8: Modules et anneaux semi-simples (Hermann, Paris, 1958).
[3] M. Demazure, Séminaire Bourbaki 365, Springer Lecture Notes in Mathematics, Vol. 180 (Springer-Verlag, Berlin, 1971) 19–38.
[4] D.B.A. Epstein, Group representations and functors, Amer. J. Math. 91 (1969) 395–414.
[5] A. Grothendieck, Sur quelques points d'algèbre homologique, Tohoku Math. J. (2) 9 (1957) 119–221.
[6] A. Kerber, Representations of Permutation Groups I, Springer Lecture Notes in Mathematics, Vol. 240 (Springer-Verlag, Berlin, 1971).
[7] I.G. Macdonald, Symmetric functions and Hall polynomials, to appear.
[8a] I. Schur, Über eine Klasse von Matrizen, die sich einer gegebenen Matrix zuordnen lassen, Ges. Abhandlungen Vol. 1, 1–70.
[8b] I. Schur, Über die rationalen Darstellungen der allgemeinen linearen Gruppe, Ges. Abhandlungen, Vol. 3, 68–85.
[9] W. Specht, Eine Verallgemeinerung der symmetrischen Gruppe, Schriften Math. Seminar Berlin 1 (1932) 1–32.
[10] A. Young, On quantitative substitutional analysis (5th paper), Proc. London Math. Soc. 31 (1930) 273–288.