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HOW TO HANDLE DEPENDENCE WITH THE ANALYTIC HIERARCHY PROCESS

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Abstract—Hierarchic and network systems are discussed as basic frameworks of unstructured problems modeled by the Analytic Hierarchy Process. A hierarchy represents a linear chain of interactions, whereas a network allows for feedback in the form of cycles and loops. A theory is provided for the priorities of a network system of which those of a hierarchy are shown to be a special case. Practical applications are illustrated.

1. HIERARCHIES AND NETWORKS

A hierarchy is a simple structure used to represent the simplest type of functional (contextual or semantic) dependence of one level or component of a system on another in a sequential manner. It is also a convenient way to decompose a complex problem in search of cause–effect explanations in steps which form a linear chain. One result of this approach is to assume the functional independence of an upper part, component or cluster from its lower parts. This often does not imply its structural independence from the lower parts which involves information on the number of elements, their measurements etc. But there is a more general way to structure a problem involving functional dependence. It allows for feedback between components. It is a network system of which a hierarchy is a special case. In both hierarchies and networks the elements component may also be dependent on each other [1]. Figure 1 below shows two diagrams which depict the structural difference between the two frameworks. In this figure, a loop means that there is inner dependence of elements within a component.

A nonlinear network can be used to identify relationships among components using one's own thoughts, relatively free of rules. It is especially suited for modeling dependence relations. Such a network approach makes it possible to represent and analyze interactions and also to synthesize their mutual effects by a single logical procedure.

For emphasis we note again that in the nonlinear network diagram or system with feedback below, there are two kinds of dependence: that between components, but in a way which allows for feedback circuits; and the other, the interdependence within a component combined with feedback between components. We have called these respectively outer and inner dependence.

If the criteria cannot be compared with respect to an overall objective because of lack of information, they can be compared in terms of the alternatives. The systems approach can then be used to replace the hierarchic approach.

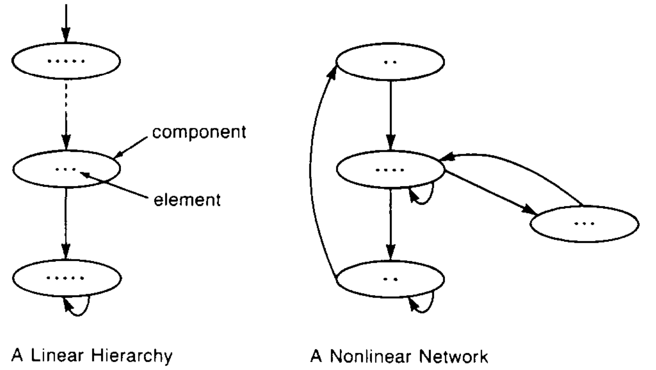

Fig. 1. A linear hierarchy and a nonlinear network. A → B means that A dominates B or that B depends on A.

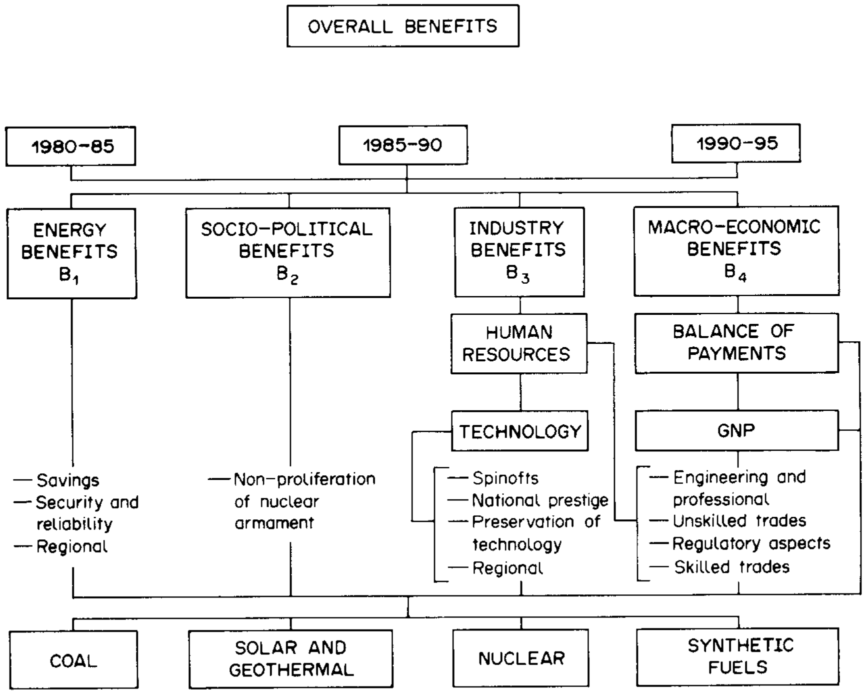

Fig. 2A. Benefits hierarchy for power generation alternatives.

2. THE SUPERMATRIX

Briefly the system prioritization approach begins with what is known as a supermatrix of blocks of interaction among components. Each column of a block is the eigenvector of priorities of the impact of a component on an element in the system. These eigenvectors are obtained from individual matrices of paired comparisons: One set for comparing criteria in terms of alternatives by answering the question "Given the alternative, how much more important is one criterion than another for that alternative?"; the other set for comparing alternatives in terms of criteria by answering the question "Given the criterion, how much more important is one alternative than another for that criterion?"

There are problems in which the future must be factored into the decisions taken in the present. In that case one lays out different time horizons and different criteria (or scenarios) likely to prevail during one or the other of these time periods. They are essentially a discrete characterization of the situation. In a situation like this, one needs to set priorities on the criteria for each time period. One also needs to set priorities for the time periods for each criterion by entering a judgment as to the time during which the criterion is most likely to prevail. The resulting priority vectors are then entered as the columns of a supermatrix representing the interactions of the two levels of the hierarchy. It has the form

$$A = \begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}.$$

Without details we present Figs 2A and 2B to show the benefits and costs hierarchies of a power generation problem in which there is dependence between the time period level and the major criteria level. The columns of the submatrix A_{21} correspond to the priority vectors of the criteria in terms of each time period arranged in the proper order and the columns of A_{12} correspond to the priority vectors of the time periods in terms of the criteria; the matrix A is column stochastic. In this manner components which depend on one another have impacts which appear in two blocks [2]. The overall, or limiting, priorities are obtained from the following supermatrix of

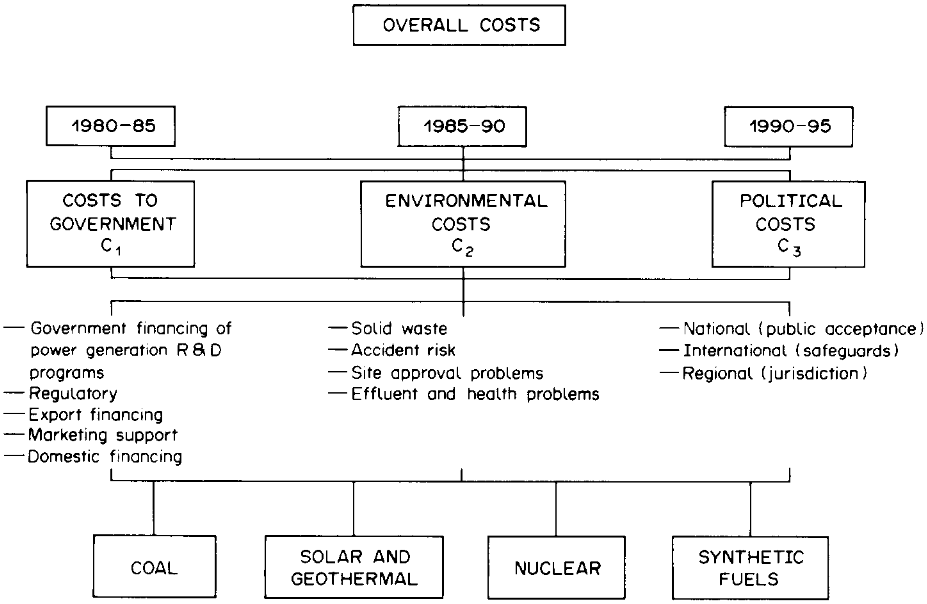

Fig. 2B. Costs hierarchy for power generation alternatives.

interactions. If we denote the four criteria by C_1, C_2, C_3 and C_4 and the alternatives by A, B and C we have the following stochastic supermatrix:

$$W = \begin{array}{c} C_1 \\ C_2 \\ C_3 \\ C_4 \\ A \\ B \\ C \end{array} \overset{\begin{array}{ccccccc} C_1 & C_2 & C_3 & C_4 & A & B & C \end{array}}{\begin{bmatrix} 0 & 0 & 0 & 0 & 0.6279 & 0.6279 & 0.6279 \\ 0 & 0 & 0 & 0 & 0.0942 & 0.0942 & 0.0942 \\ 0 & 0 & 0 & 0 & 0.2060 & 0.2060 & 0.2060 \\ 0 & 0 & 0 & 0 & 0.0719 & 0.0719 & 0.0719 \\ 0.250 & 0.500 & 0.556 & 0.545 & 0 & 0 & 0 \\ 0.333 & 0.333 & 0.286 & 0.273 & 0 & 0 & 0 \\ 0.417 & 0.167 & 0.158 & 0.182 & 0 & 0 & 0 \end{bmatrix}}.$$

As we shall see later this is an irreducible imprimitive matrix. The limiting priorities of the criteria and of the alternatives are obtained by solving the eigenvalue problem $W\mathbf{w} = \mathbf{w}$ and normalizing the first four components of $\mathbf{w}$ for the priorities of the criteria and the last three for the priorities of the alternatives. This gives, respectively, (0.6279, 0.0942, 0.2060, 0.0719) and (0.3578, 0.3190, 0.3232). In this case the same results could have been obtained, respectively, as the first four nonzero elements from any of the first four columns of

$$\lim_{k \to \infty} (W^2)^k$$

and the last three nonzero elements from its last three columns.

The above results were used to carry the analysis through to the lowest level of both hierarchies over different time horizons, resulting in Table 1. This table gives a summary of overall benefits and costs of the alternatives, their ratios and marginal ratios for decision purposes. Here the nuclear alternative is favored.

An interesting application of the feedback concept has been used in the analysis of terrorism where a hierarchy was used whose bottom level of alternatives is linked to its top level of criteria giving rise to a cyclic hierarchy known as a holarchy. The supermatrix application was essential in deriving the priorities for the courses of action to be followed [3].

Table 1

	Benefits	Cost	Total benefits/cost	Marginal $\frac{B_i - B_j}{C_i - C_j}$
Coal	0.137	0.152	0.901	0.901
Solar and geothermal	0.169	0.216	0.782	—
Nuclear	0.288	0.210	1.371	2.603
Syn-fuels	0.406	0.432	0.940	1.097

3. SYSTEMS WITH FEEDBACK

Let us consider the general situation of a system in which components affect each other [4]. It is thus desirable to obtain priorities for the impacts of the elements on all other elements in the system. To do this for each criterion we must perform pairwise comparisons for each component with respect to the elements in each other component with which it interacts. The resulting eigenvector of each matrix is entered as part of a column of a supermatrix along whose side and at its top all the elements are listed to obtain a measure of their interaction. Thus all eigenvectors representing the impact of the elements of a component on one element of another component are arranged in a single column next to their corresponding elements. In this manner we fill out the entire matrix. Next, all eigenvectors corresponding to a pair of interacting components are weighted by the priorities of the component which creates the impact. These priorities are obtained from a separate pairwise comparison matrix of the components. The judgments of that matrix are entered by answering the question: "Given a component of the system which other component affects it most with respect to the given criterion and how strongly?", i.e. which is the most important component affecting it? The resulting weighted supermatrix has each column adding to unity. One must make sure that the sum is precisely equal to unity.

We are interested in two types of priorities. Those that give the influence or impact of one element on any other element in the system are known as the impact priorities. We are also interested in the absolute priority of any element regardless of which elements it influences. Generally we seek limiting values of the two kinds of priorities. Calculation of these priorities shows where existing trends might lead if there is no change in preferences which affects the priorities. By experimenting with the process of modifying priorities and noting their limiting trends, we may be able to steer a system towards a more desired outcome.

Now for the formal definitions. The discussion below parallels the theory of Markov chains, as given by Gantmacher [5], adapted for our purpose. If w_{ij} is the impact priority of the ith element on the jth element in the system then

$$w_{ij}^{(1)} = w_{ij},$$

$$w_{ij}^{(2)} = \sum_m w_{im} w_{mj},$$

$$w_{ij}^{(h+k)} = \sum_m w_{im} w_{mj}^{(k)}$$

and

$$w_{ij}^{(h+k)} = \sum_m w_{im}^{(h)} w_{mj}^{(k)}.$$

The sum of the impact priorities along all possible paths from a given element gives the priorities of an element. This amounts to raising the matrix W to powers. (The last expression is equivalent to $W^{h+k} = W^h W^k$.)

Given that the initial priority of the ith element is $w_i^{(0)}$, we have the following absolute priority of the jth element in paths of length $k \neq 0$:

$$w_j^{(k)} = \sum_i w_i^{(0)} w_{ij}^{(k)}.$$

The problem is to find the limiting impact priority (LIP) matrix W^∞ and the limiting absolute priority (LAP) vector $\mathbf{w}^\infty$ as $k \to \infty$. (For a priority system we may also be interested in determining priorities for finite values of k. This does not present problems of existence, as does the limiting case.) Of particular interest is to determine when the LAP priority is independent of the initial priorities $w_i^{(0)}$. Such independence is called the ergodicity of the system.

The following is a classification of elements useful in characterizing a system. The reader may wish to go on to the actual discussion of existence and construction of LIP and LAP solutions. The element j can be reached from the element i if for some integer $k \geqslant 1$, $w_{ij}^{(k)} > 0$, where $W^k = (w_{ij}^{(k)})$. Here W^k gives the *k-reach* of each element. A subset of elements C of a system is *closed* (opposite definition to that for Markov chains [6]) if $w_{ij}^{(k)} = 0$, whenever $i \in C$ and $j \notin C$. It follows that no element can be reached from any element not in C. The subset C is minimal if it contains no proper closed subset of elements. A set of elements which forms a minimal closed subset corresponds to what is known as an irreducible matrix, the system or subsystem itself is called irreducible. A system is called decomposable if it has two or more closed sets.

If we *initially* start with the jth element for some fixed j and denote its first impact on itself in a path of length $k \geqslant 1$ by $f_j^{(k)}$, we have

$$f_j^{(1)} = w_{jj}^{(1)}, \qquad f_j^{(2)} = w_{jj}^{(2)} - f_j^{(1)} w_{jj}^{(1)} \ldots f_j^{(k)} = w_{jj}^{(k)} - f_j^{(1)} w_{jj}^{(k-1)} + \cdots - f_j^{(k-1)} w_{jj}^{(1)}$$

and

$$f_j = \sum_{k=1}^{\infty} f_j^{(k)}$$

gives the cumulative impact of j on itself. The mean impact (of j on itself) is given by $u_j = \sum_{k=0}^{\infty} k f_j^{(k)}$.

According to priority influence we have the following (the new terms introduced below are essential, as we are not dealing with time transitions):

(1) If $f_j = 1$, j is called an *enduring* (recurrent) element. Thus an element is enduring if the sum of its impact priorities on itself in a single step (by a loop) in two steps (through a cycle involving one other element), in three steps involving two other elements etc. is equal to unity.

(2) If $f_j > 1$, j is called transitory (transient). An element j that is either enduring or transitory is called cyclic (periodic) with cyclicity c if u has values $c, 2c, 3c, \ldots$ where c is the greatest integer *greater than* unity with this property ($w_{ij}^{(k)} = 0$, where k is not divisible by c). An enduring element j for which u_j is an infinite is called *fading* (null). An enduring element j that is neither cyclic nor fading (i.e. $u_j > \infty$) is called *sustaining* (ergodic).

For either a transitory or a fading element j, $w_{ij}^{(k)} \to 0$ for every i. If one element in an irreducible subsystem is cyclic with cyclicity c, all the elements in that subsystem are cyclic with cyclicity c. It is known that if j is a sustaining element, then as $k \to \infty$, $w_{jj}^{(k)} \to 1/u_j$; j is a fading element if this number of elements of an irreducible subsystem are all transitory or all enduring and the system itself is called transitory or enduring, respectively.

Remark

The following expression always exists whether a system is irreducible or not (in the former case its values are known and are as indicated):

$$\lim_{m\to\infty}\sum_{k=0}^{m-1} w_{ij}^{(k)} = \begin{cases} 0 & \text{if } i \text{ and } j \text{ are transitory} \\ 1/u & \text{if } i \text{ and } j \text{ are enduring.} \end{cases}$$

All finite systems of elements must have at least one sustaining element which generates a closed irreducible subset of elements. Since the enduring elements of a finite system are all sustaining the block (or component) thus generated is called sustaining.

If j is cyclic with cyclicity $c > 1$, then $w_{jj}^{(k)} = 0$ if k is not a multiple of c and $w_{jj}^{(m)} \to c/u$ as $m \to \infty$; $k = mc$, m positive and c the largest integer for which $k = mc$ holds.

We stated earlier that reducibility and primitivity play an important role in proving the existence of LIP and LAP. We now give a few basic facts relating these concepts which will be useful in the ensuing discussion.

A nonnegative irreducible matrix is primitive if it has a unique principal eigenvalue. If the matrix has another eigenvalue with the same modulus as the principal eigenvalue, it is called imprimitive.

If the principal eigenvalue has multiplicity greater than unity (equal to unity), but there are no other eigenvalues of the same modulus as the principal eigenvalue, then the matrix is called proper (regular).

A primitive matrix is always regular and hence proper but not conversely, e.g. the identity matrix which has unity as an eigenvalue of multiplicity equal to the order of the matrix. A matrix is proper if, and only if, in the normal form, the isolated blocks are primitive. For a regular matrix the number of isolated blocks is unity.

We note that if all the entries of W are positive, we have a primitive matrix and the theorem on stochastic primitive matrices applies, both LIP and LAP exist. LIP and LAP are the same and are given by the solution of the eigenvalue problem $W\mathbf{w} = \mathbf{w}$. Actually $\mathbf{w}$ is any column of $\lim W^k$. The same result is true if W is a primitive matrix.

In general the nonnegative matrix W may have some zeros. In that case it is either an irreducible or a reducible matrix. If it is irreducible then it is either primitive in which case the above discussion applies, or it is imprimitive. In the latter case it has a number c of eigenvalues (called the index of imprimitivity) that are not equal to unity whose moduli are equal to unity. This number plays an important role in the solution of the general case from which we can also obtain the solution to this case. It is sufficient to point out here that $W, W^2, \ldots, W^{c-1}$ are all not proper and multiples of these matrices tend toward periodic repetition. The system is cyclic with cyclicity c.

Remark

The system is acyclic, cyclic, irreducible, reducible, depending on whether the corresponding matrix W is primitive, imprimitive, irreducible, reducible.

If W is *nonnegative* and reducible then it is reduced to the normal form. If the isolated blocks are primitive (they are said to correspond to inessential components). The system is by definition called proper and LIP and LAP exist [5].

Important remark

When our column stochastic matrix is reducible its essential components drive the system since they are "sources" or impact-priority-diffusing components as opposed to "sinks" or transition-probability-absorbing states of a Markov chain. In any diagram, except for loops, arrows initiate from and nonterminate at such components.

The solution for LIP is given by

$$W^{\infty} = \lim_{k\to\infty} W^k = \frac{(I - W)^{-1}\Psi(1)}{\Psi'(1)},$$

where $\Psi(\lambda)$ is the minimum polynomial of W and $\Psi'(\lambda)$ is its first derivative with respect to λ. Each column $\mathbf{w}$ is a characteristic vector of W corresponding to $\lambda_{\max} = 1$. If $\lambda_{\max} = 1$ is simple, i.e. W is regular, $\Psi(\lambda)$ may be replaced by $\Delta(\lambda)$ the characteristic polynomial of W. LAP is obtained as $\mathbf{w}^{\infty} = W^{\infty}\mathbf{w}^{(0)}$ if W is proper, and the eigenvector solution of $W\mathbf{w}^{\infty} = \mathbf{w}^{\infty}$ if W is regular.

Remark

One can show that the matrices of W corresponding to essential components are positive and those to priority impacts from essential to inessential components are also positive. Only impacts from inessential to inessential or from inessential to essential components are zero.

Finally, if not all isolated blocks are primitive then each has an index of imprimitivity as we pointed out earlier. We consider the least-common multiple of these, which is the cyclicity c of the system. Using the powers of W, LIP is given by

$$\begin{aligned}\tilde{W} &= (1/c)(I + W + \cdots W^{c-1})(W^{c})^{\infty}\\ &= (1/c)(I - W^{c})(1 - W)^{-1}(W^{c})^{\infty}\end{aligned}$$

and LAP is given by $\mathbf{w} = \tilde{W}\mathbf{w}^{(0)}$. Both $\tilde{W}$ and $\mathbf{w}$ are called the mean LIP and mean LAP, respectively. If there is a single isolated block, then the mean LAPs are independent of the initial priorities and are uniquely determined by the solution of $W\mathbf{w} = \mathbf{w}$.

This is precisely the case of an irreducible imprimitive system. Several applications have been made to calculate priorities in a system with feedback. The calculations are long but the foregoing theory has been found very useful for this purpose. Let us note, in closing, that the supermatrix of a hierarchy has the following form:

$$W = \begin{bmatrix} 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ W_{21} & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & W_{32} & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & W_{n-1,n-2} & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & W_{n,n-1} & I \end{bmatrix}.$$

This matrix has the following stable form for all $k \geqslant n - 1$:

$$W^{k} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ W_{n,n-1}W_{n-1,n-2}\ldots W_{32}W_{21} & W_{n,n-1}W_{n-1,n-2}\ldots W_{32} & \cdots & W_{n,n-1}W_{n-1,n-2} & W_{n,n-1} & I \end{bmatrix}.$$

Each coefficient in the last row gives the composite priority impact of the last component on each of the remaining components. Note that the principle of hierarchical composition appears in the $(n, 1)$ position as the impact of the nth component on the first. The nth component drives the hierarchy and is the counterpart of an absorbing state in a Markov chain. It is a component of elements which diffuse or are a source of priority impacts. The essence of the above is summarized by the "Principle of Hierarchical Composition": the composite vector of a hierarchy of n levels is the entry in the $(n, 1)$ position of W^{k-1}, $k \geqslant n - 1$.

This discussion shows that the composition process in a hierarchy which is additive conforms with the general composition process of a system with feedback obtained by an alternative approach well-known in classical mathematics.

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